Stabilization of Uncertain Negative-Imaginary Systems Using a Riccati Equation Approach

Mohamed A. Mabrok, Abhijit G. Kallapur, Ian R. Petersen, and Alexander Lanzon

Abstract— **In this paper, a stabilization procedure that forces an uncertain system to be stable and satisfy the negative imaginary property is presented. The controller synthesis procedure is based on the negative imaginary lemma. As a result, the closed-loop system can be guaranteed to be robustly stable against any strict negative imaginary uncertainty, such as in the case of unmodeled spill-over dynamics in a lightly damped flexible structure. A numerical example is presented to illustrate the usefulness of the proposed results.**

Index Terms— **Negative imaginary systems, lightly damped systems, Riccati equations, uncertain system.**

I. Introduction

Highly resonant structural modes in machines and robots, ground and aerospace vehicles, and precision instrumentation, such as atomic force microscopes and optical systems can limit the ability of control systems in achieving a desired level of performance [1]. This problem is simplified to some extent by using force actuators combined with collocated measurements of velocity, position, or acceleration.

The use of force actuators combined with velocity measurements has been studied using the well known positive real (PR) theory for linear time invariant (LTI) systems; e.g., see [2], [3].

Lanzon and Petersen introduce a new class of systems in [4] called negative imaginary (NI) systems, which has fewer restrictions on the relative degree of the system transfer function than in the PR case. In the SISO case, such systems are defined by considering the properties of the imaginary part of the frequency response $G(j\omega) = D + C(j\omega I - A)^{-1}B$, and requiring the condition $j(G(j\omega) - G(j\omega)^*) \geq 0$ for all $\omega \in (0, \infty)$.

NI systems theory has many engineering applications. Such classes of systems include DC machines [5], electrical active filter circuits [6], lightly damped structures [4], [7], [1], [8], [9]. When force actuators and position sensors (such as piezoelectric sensors) are collocated on a flexible structure, the input/output map is NI. Stability results for interconnected systems with an NI frequency response have been applied to the decentralized control of large vehicle platoons in [10]. In [11], the authors show how the class of linear systems having negative imaginary transfer matrices is a direct extension of the class of linear Hamiltonian input-output systems. Also, an extension for negative imaginary systems to infinite-dimensional systems has been studied in [12].

In general, NI systems are stable systems having a phase lag between 0 and $-\pi$ for all $\omega > 0$. That is, their Nyquist plot lies below the real axis when the frequency varies in the open interval $(0, \infty)$ (for strictly negative-imaginary systems, the Nyquist plot should not touch the real axis except at zero frequency and/or at infinity). This is similar to PR systems where the Nyquist plot is constrained to lie in the right half of the complex plane [2], [3]. However, in contrast to PR systems, transfer functions for NI systems can have relative degrees greater than unity.

In [13], a systematic method for designing a controller to force the closed-loop system to satisfy the NI property based on the Riccati equation approach was presented. However, the resulting closed-loop system always has a pole at the origin.

In this paper, we extend the work in [13] to derive controller that stabilize an uncertain systems when full state feedback is available. Unlike the results in [13], the resulting closed-loop system is asymptotically stable. Also, we recall the results of [13] and present their proofs which were not included in [13].

This paper is further organized as follows: Section II introduces the concept of negative imaginary systems and provides some technical results that will be used in deriving the main results in the paper. In section III, a negative imaginary and a strict negative imaginary lemma based on Riccati equations have been introduced and a controller synthesis procedure is addressed. Section IV provides a numerical example to support the results.

II. Preliminaries

In this section, we introduce the definition for NI systems and present the NI Lemma. Also, we introduce some technical results which will be used in deriving the main results of this paper.

To establish the main results of this paper, we consider a generalized definition for NI systems which allows for poles at the origin as follows:

Definition 1: [14] A square transfer function matrix $G(s)$ is NI if all the following conditions are satisfied:

1) $G(s)$ has no pole in $Re[s] > 0$.

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Mohamed Mabrok, Abhijit Kallapur and Ian Petersen are with the School of Engineering and Information Technology, University of New South Wales at the Australian Defence Force Academy Canberra ACT 2600, Australia abdallamath@gmail.com, abhijit.kallapur@gmail.com, i.r.petersen@gmail.com.

Alexander Lanzon is with the Control Systems Centre, School of Electrical and Electronic Engineering, University of Manchester, Manchester M13 9PL, United Kingdom Alexander.Lanzon@manchester.ac.uk.

2) For all $\omega \geq 0$ such that $j\omega$ is not a pole of $G(s)$, $j(G(j\omega)-G(j\omega)^*) \geq 0$.
3) If $s=j\omega_0$, $\omega_0>0$ is a pole of $G(s)$ then it is a simple pole. Furthermore, if $s=j\omega_0$, $\omega_0>0$ is a pole of $G(s)$, the residual matrix $K=\lim_{s\longrightarrow j\omega_0}(s-j\omega_0)jG(s)$ is positive semidefinite Hermitian. If $s=0$ is a pole of $G(s)$, then it is either a simple pole or a double pole. If it is double pole, then, $\lim_{s\longrightarrow 0} s^2G(s)\geq 0$.

Definition 2: A square transfer function matrix $G(s)$ is strictly negative imaginary (SNI) if the following conditions are satisfied:

1) $G(s)$ has no pole in $Re[s]\geq 0$.
2) For all $\omega>0$, $j(G(j\omega)-G(j\omega)^*)>0$.

Consider the following LTI system,

$$\dot{x}(t)=Ax(t)+Bu(t), \tag{1}$$
$$y(t)=Cx(t)+Du(t), \tag{2}$$

where, $A\in\mathbb{R}^{n\times n}, B\in\mathbb{R}^{n\times m}, C\in\mathbb{R}^{m\times n}$, and $D\in\mathbb{R}^{m\times m}$.

Lemma 1: [15] Let $\left[\begin{array}{c|c} A & B \\ \hline C & D\end{array}\right]$ be a minimal realization of the transfer function matrix $G(s)$ for the system in (1)-(2). Then, $G(s)$ is NI if and only if $D=D^T$ and there exist matrices $P=P^T\geq 0$, $W\in\mathbb{R}^{m\times m}$, and $L\in\mathbb{R}^{m\times n}$ such that the following linear matrix inequality (LMI) is satisfied:

$$\begin{bmatrix} PA+A^TP & PB-A^TC^T \\ B^TP-CA & -(CB+B^TC^T)\end{bmatrix} = \begin{bmatrix} -L^TL & -L^TW \\ -W^TL & -W^TW\end{bmatrix}\leq 0. \tag{3}$$

The following lemma gives spectral conditions for a transfer functions which will be used in deriving the SNI lemma.

Lemma 2: Let $G(s)=\left[\begin{array}{c|c} A & B \\ \hline C & D\end{array}\right]$ be a minimal realization. Given A which has no pure imaginary eigenvalues, $\omega_0>0$ and $\lambda\in\mathbb{C}$ which is not an eigenvalue of $\frac{CB+B^TC^T}{2}>0$, then, λ is an eigenvalue of $H(j\omega_0)=\frac{1}{2}j\omega_0(G(j\omega_0)-G(j\omega_0)^*)$ if and only if $j\omega_0$ is an eigenvalue of the matrix

$$N_\lambda=\begin{bmatrix} A+BR_\lambda^{-1}CA & BR_\lambda^{-1}B^T \\ -A^TC^TR_\lambda^{-1}CA & -A^T-A^TC^TR_\lambda^{-1}B^T\end{bmatrix},$$

where $R_\lambda=2\lambda I-CB-B^TC^T$.

Proof: Since $G(s)=\left[\begin{array}{c|c} A & B \\ \hline C & D\end{array}\right]$, then $s(G(s)-D)=\left[\begin{array}{c|c} A & B \\ \hline CA & CB\end{array}\right]$.

Now λ is an eigenvalue of $H(j\omega_0)$, where $\omega>0$ if and only if there exists a vector $u\neq 0$, where $u\in\mathbb{C}$ such that

$$\begin{aligned} & H(j\omega_0)u= \ \lambda u, \\ & \Leftrightarrow \frac{1}{2}j\omega_0(G(j\omega_0)-G(j\omega_0)^*)u= \ \lambda u, \\ & \Leftrightarrow (CA(j\omega_0I-A)^{-1}B+B^T(-j\omega_0I-A^T)^{-1}A^TC^T \\ & \quad +CB+B^TC^T)u=2\ \lambda u, \\ & \Leftrightarrow (CAr+B^Ts)+(CB+B^TC^T)u=2\ \lambda u, \end{aligned} \tag{4}$$

where

$$r=(j\omega_0I-A)^{-1}Bu, \text{and} \tag{5}$$
$$s=(-j\omega_0I-A^T)^{-1}A^TC^Tu. \tag{6}$$

It follows from (4) that

$$\begin{aligned} & \begin{bmatrix} CA & B^T\end{bmatrix}\begin{bmatrix} r \\ s\end{bmatrix}=R_\lambda u, \\ & \Leftrightarrow R_\lambda^{-1}\begin{bmatrix} CA & B^T\end{bmatrix}\begin{bmatrix} r \\ s\end{bmatrix}=u, \end{aligned} \tag{7}$$

where $R_\lambda=2\ \lambda I-CB-B^TC^T$. Also, (5) and (6) can be rewritten as:

$$Bu=(j\omega_0I-A)r,$$
$$A^TC^Tu=(-j\omega_0I-A^T)s,$$

$$\Leftrightarrow \begin{bmatrix} B \\ -A^TC^T\end{bmatrix}u=j\omega_0\begin{bmatrix} r \\ s\end{bmatrix}+\begin{bmatrix} -A & 0 \\ 0 & A^T\end{bmatrix}\begin{bmatrix} r \\ s\end{bmatrix}, \tag{8}$$

substituting (7) in (8), we get

$$\begin{aligned} & \begin{bmatrix} B \\ -A^TC^T\end{bmatrix}R_\lambda^{-1}\begin{bmatrix} CA & B^T\end{bmatrix}\begin{bmatrix} r \\ s\end{bmatrix} \\ & = j\omega_0\begin{bmatrix} r \\ s\end{bmatrix}+\begin{bmatrix} -A & 0 \\ 0 & A^T\end{bmatrix}\begin{bmatrix} r \\ s\end{bmatrix}, \end{aligned} \tag{9}$$

$$\begin{aligned} \Leftrightarrow & \begin{bmatrix} B \\ -A^TC^T\end{bmatrix}R_\lambda^{-1}\begin{bmatrix} CA & B^T\end{bmatrix}\begin{bmatrix} r \\ s\end{bmatrix} \\ & +\begin{bmatrix} A & 0 \\ 0 & -A^T\end{bmatrix}\begin{bmatrix} r \\ s\end{bmatrix}=j\omega_0\begin{bmatrix} r \\ s\end{bmatrix}, \end{aligned} \tag{10}$$

$$\Leftrightarrow N_\lambda\begin{bmatrix} r \\ s\end{bmatrix}=j\omega_0\begin{bmatrix} r \\ s\end{bmatrix}, \tag{11}$$

where

$$N_\lambda=\begin{bmatrix} A+BR_\lambda^{-1}CA & BR_\lambda^{-1}B^T \\ -A^TC^TR_\lambda^{-1}CA & -A^T-A^TC^TR_\lambda^{-1}B^T\end{bmatrix}.$$

■

Now, consider the following theorem that defines an SNI system based on the spectrum of its corresponding Hamiltonian matrix.

Theorem 1: Let $G(s)=\left[\begin{array}{c|c} A & B \\ \hline C & D\end{array}\right]$ be a minimal realization and $CB+B^TC^T>0$. Then, $G(s)$ is SNI if and only if

1) A is a Hurwitz matrix, and $D=D^T$,
2) the Hamiltonian matrix $N_0=$
$$\begin{bmatrix} A+BQ^{-1}CA & BQ^{-1}B^T \\ -A^TC^TQ^{-1}CA & -A^T-A^TC^TQ^{-1}B^T\end{bmatrix}$$
has no positive pure imaginary eigenvalues. Here, $Q=-(CB+B^TC^T)$.

Proof: Let

$$H(j\omega)=\frac{1}{2}j\omega(G(j\omega)-G(j\omega)^*).$$

Suppose that $G(s)$ is SNI, it follows that A is a Hurwitz matrix. Also, $j(G(j\omega)-G(j\omega)^*)>0$ for all $\omega>0$, which implies that $H(j\omega)>0$ for all $\omega>0$. Then, $\lambda=0$ is not an eigenvalue of $H(j\omega)$ for any $\omega>0$. It follows from Lemma

2 that N_0 has no positive pure imaginary eigenvalues for any $\omega > 0$.

On the other hand, suppose that N_0 has no positive pure imaginary eigenvalues, it follows from Lemma 2 that $\lambda = 0$ is not an eigenvalue of $H(j\omega)$ for all $\omega > 0$. Since the eigenvalues of $H(j\omega)$ are continuous functions in ω and using the fact $CB + B^TC^T > 0$, it follows that $H(j\omega) > 0$, which implies that $j(G(j\omega) - G(j\omega)^*) > 0$. Hence $G(s)$ is SNI, since A is a Hurwitz matrix. ■

III. Main results

In this section, we use algebraic Riccati equations to give a new representation for the NI and SNI lemmas. Then, we will use the NI lemma to derive a static controller such that the closed-loop system stable and satisfies the NI property.

Theorem 2: Let $G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ be a minimal realization with $CB + B^TC^T > 0$. Then $G(s)$ is NI if and only if $D = D^T$ and there exists a matrix $P \geq 0$ such that P is a solution to the following algebraic Riccati equation

$$PA_0 + A_0^TP + PBR^{-1}B^TP + Q = 0, \qquad (12)$$

where

$$\begin{aligned} A_0 &= A - BR^{-1}CA, \\ R &= CB + B^TC^T, \ \text{and} \\ Q &= A^TC^TR^{-1}CA. \end{aligned}$$

Proof: From Lemma 1, $G(s)$ is NI if and only if there exist matrices $P = P^T \geq 0$, $W \in \mathbb{R}^{m\times m}$, and $L \in \mathbb{R}^{m\times n}$ such that

$$\begin{aligned} PA + A^TP &= -L^TL, & (13) \\ PB - A^TC^T &= -L^TW, & (14) \\ CB + B^TC^T &= W^TW. & (15) \end{aligned}$$

Since $CB + B^TC^T > 0$, it follows from (15) that (14) is equivalent to

$$L^T = (A^TC^T - PB)W^{-1}. \qquad (16)$$

Now, substituting (16) into (13) gives the equivalent condition

$$\begin{aligned} &PA + A^TP = \\ &\ -(A^TC^T - PB)(CB + B^TC^T)^{-1}(CA - B^TP), \\ &\Leftrightarrow PA + A^TP + (A^TC^T - PB)R^{-1}(CA - B^TP) = 0, \\ &\Leftrightarrow PA_0 + A_0^TP + PBR^{-1}B^TP + Q = 0. \end{aligned}$$

This completes the proof. ■

Theorem 3: Let $G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ be a minimal realization and $CB + B^TC^T > 0$. Then $G(s)$ is SNI if and only if

1) A has no $j\omega$-axis eigenvalues and $D = D^T$,
2) there exists a matrix $P > 0$ such that P is a solution to the following algebraic Riccati equation

$$PA_0 + A_0^TP + PBR^{-1}B^TP + Q = 0, \qquad (17)$$

where all the eigenvalues of the matrix $A_0 + BR^{-1}B^TP$ lie in the open left half of the complex plane or at the origin.

Proof: Suppose that $G(s)$ is SNI, it follows that A is a Hurwitz matrix. Also, there exists a matrix $P > 0$ such that the LMIs (13)-(15) are satisfied[4]. This implies that $P > 0$ is a solution to (17). Since (17) has a solution $P > 0$, it follows from Theorem 2.1 in [16] that (17) has a strong solution. Using Theorem 1, it follows that the corresponding Hamiltonian matrix for $G(s)$ has no eigenvalues on the jω-axis except at the origin. It follows from Theorem 2.6 in [17] that the matrix $A_0 + BR^{-1}B^TP$ has no eigenvalues on the imaginary axis except the origin.

On the other hand, suppose that (17) has a solution $P > 0$. It follows from Theorem 2 that $G(s)$ is NI. Now, if the matrix $A_0 + BR^{-1}B^TP$ has no eigenvalues on the jω-axis except at the origin, this implies that the corresponding Hamiltonian matrix of $G(s)$ has no eigenvalues on the jω-axis except the origin. Now, if A is a Hurwitz matrix, it follows from Theorem 1 that $G(s)$ is SNI. ■

A. A Synthesis Result

In order to present a synthesis result, consider the following state space representation for a linear uncertain system

$$\begin{aligned} \dot{x} &= Ax + B_1w + B_2u, \\ z &= C_1x, \\ w &= \Delta(s)z, & (18) \end{aligned}$$

where, $A \in \mathbb{R}^{n\times n}, B_1 \in \mathbb{R}^{n\times m}, B_2 \in \mathbb{R}^{n\times r}, C_1 \in \mathbb{R}^{m\times n}$, and $\Delta(s)$ represents the uncertainty matrix. Also, suppose that K is a static controller such that $u = Kx$. Then the closed-loop interconnection of the system (18) with the static controller K is given by;

$$\begin{aligned} \dot{x} &= (A + B_2K)x + B_1w, \\ z &= C_1x, \\ w &= \Delta(s)z. & (19) \end{aligned}$$

Our aim is to construct the controller K such that the corresponding closed-loop system (19) satisfies the NI property.

Consider the following transformation (Schur transformation)

$$A_f = U^T(A - B_2(C_1B_2)^{-1}C_1A)U = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, (20)$$

$$B_f = U^T(B_2(C_1B_2)^{-1} - B_1R^{-1}) = \begin{bmatrix} B_{f1} \\ B_{f2} \end{bmatrix}, \qquad (21)$$

$$\tilde{B}_1 = U^TB_1 = \begin{bmatrix} B_{11} \\ B_{22} \end{bmatrix}. \qquad (22)$$

The transformation (20) can be constricted such that A_{11} is a stable matrix (with a zero eigenvalue) and A_{22} is an anti-stable matrix.

Theorem 4: Consider an uncertain system model as in (18) with C_1B_2 invertible and $R = C_1B_1 + B_1^TC_1^T > 0$. Then there exists a controller K such that the closed-loop system in (19) is NI if there exist matrices $T \geq 0$ and $S \geq 0$ such that

$$-A_{22}T - TA_{22}^T + B_{f2}RB_{f2}^T = 0, \quad (23)$$
$$-A_{22}S - SA_{22}^T + B_{22}R^{-1}B_{22}^T = 0 \quad (24)$$

and $S - T < 0$. Here, A_{22} is the anti-stable block of the matrix A_f defined in (20). Furthermore, the controller gain matrix is

$$K = (C_1B_2)^{-1}(B_1^TP - C_1A - R(B_2^TC_1^T)^{-1}B_2^TP), \quad (25)$$

where $P = UP_fU^T$ and P_f is a solution to the algebraic Riccati equation

$$P_fA_f + A_f^TP_f - P_fB_fRB_f^TP_f + P_f\tilde{B}_1R^{-1}\tilde{B}_1^TP_f = 0. \quad (26)$$

Proof: Suppose there exist matrices $T \geq 0$ and $S \geq 0$ such that

$$-A_{22}T - TA_{22}^T + B_{f2}RB_{f2}^T = 0, \quad (27)$$
$$-A_{22}S - SA_{22}^T + B_{22}R^{-1}B_{22}^T = 0. \quad (28)$$

Subtracting (27) from (28) we get

$$-A_{22}X - XA_{22}^T - B_{f2}RB_{f2}^T + B_{22}R^{-1}B_{22}^T = 0, \quad (29)$$

where $X = S - T$. This means that $X < 0$ is a solution to (29) where A_{22} is an anti-stable matrix. Pre-multiplying and post-multiplying (29) by X^{-1} we get

$$-X^{-1}A_{22} - A_{22}^TX^{-1} - X^{-1}B_{f2}RB_{f2}^TX^{-1} + X^{-1}B_{22}R^{-1}B_{22}^TX^{-1} = 0. \quad (30)$$

Now let $P_1 = -X^{-1} > 0$, then (30) becomes

$$P_1A_{22} + A_{22}^TP_1 - P_1B_{f2}RB_{f2}^TP_1 + P_1B_{22}R^{-1}B_{22}^TP_1 = 0, \quad (31)$$

$$\Rightarrow P_fA_f + A_f^TP_f - P_fB_fRB_f^TP_f + P_f\tilde{B}_1R^{-1}\tilde{B}_1^TP_f = 0 \quad (32)$$

has a solution $P_f = \begin{bmatrix} 0 & 0 \\ 0 & P_1 \end{bmatrix} \geq 0$, where A_f, B_f and $\tilde{B}_1$ are given by (20)-(22).

Also, (32) can be written as

$$P_fA_f + A_f^TP_f + (K^TM^T + N^T)(MK + N) - P_fB_fRB_f^TP_f + P_f\tilde{B}_1R^{-1}\tilde{B}_1^TP_f = 0, \quad (33)$$

where

$$N = R^{-\frac{1}{2}}C_1A - R^{-\frac{1}{2}}B_1^TP + R^{\frac{1}{2}}(B_2^TC_1^T)^{-1}B_2^TP,$$
$$M = R^{-\frac{1}{2}}C_1B_2.$$

With some algebraic manipulation, (33) can be written as

$$P\tilde{A} + \tilde{A}^TP + PB_1R^{-1}B_1^TP + Q = 0, \quad (34)$$

where,

$$\begin{aligned}\tilde{A} &= A - B_1R^{-1}C_1A + (I - B_1R^{-1}C_1)B_2K, \\ &= A + B_2K - B_1R^{-1}C_1(A + B_2K), \\ &= A_{cl} - B_1R^{-1}C_1A_{cl}, \\ R &= C_1B_1 + B_1^TC_1^T, \\ Q &= A^TC_1^TR^{-1}C_1A + A^TC_1^TR^{-1}C_1B_2K \\ &\quad + K^TB_2^TC_1^TR^{-1}C_1B_2K + K^TB_2^TC_1^TR^{-1}C_1A, \\ &= (A + B_2K)^TC_1^TR^{-1}C_1(A + B_2K), \\ &= A_{cl}^TC_1^TR^{-1}C_1A_{cl}.\end{aligned}$$

Here, A_{cl} is the A matrix of the closed-loop system in (19). This implies that $P = U\begin{bmatrix} 0 & 0 \\ 0 & P_1 \end{bmatrix}U^T \geq 0$ is a solution to (34). It follows from Theorem 2 that the closed-loop system in (19) is NI. ■

Remark 1: In the case of no anti-stable modes in the matrix $A - B_2(C_1B_2)^{-1}C_1A$, we can choose

$$K = -(C_1B_2)^{-1}C_1A.$$

It can be proved that the corresponding closed-loop system is NI in a similar fashion to the proof of Theorem 4.

B. A Procedure for Designing a Sterilizing Controller

In this section, a step-wise approach is presented to designing a controller that makes the closed-loop is stable and satisfy the NI property .

Usung Theorem 4, the closed-loop system will always have a pole at the origin. This is because $rank(A_f) < n$. To avoid this problem, we can first perturb the plant matrix by adding εI to the matrix A for some $\varepsilon > 0$. The new perturbed matrix $A_\varepsilon = A + \varepsilon I$ will have its eigenvalues moved by ε to right of the complex plane.

Now, we can use the perturbed matrix A_ε as our plant matrix in the uncertain system (18). Then, apply the synthesis procedure in Theorem 4 to obtain the static controller K. According to Theorem 4, the closed-loop system will have a pole at the origin. However, the actual closed-loop system will have all the poles on the left half of the complex plane.

One farther advantage of introducing the parameter ε, is that it can be used to control the performance of the closed-loop. This can be done by calculating the closed-loop value of the cost function depending on that parameter. For example, the parameter ε can be chosen to minimize the closed-loop value of the quadratic cost function $J = \int_0^\infty (x^T(t)Qx(t) + u^T(t)Ru(t))$.

Also, choosing the parameter ε has to be done such that the closed-loop still satisfies the NI property, which is easy to check using Theorem 1.

IV. Illustrative Example

In this section, we provide a numerical example in order to validate our results.

Consider the following uncertain system of the general form (18) where,

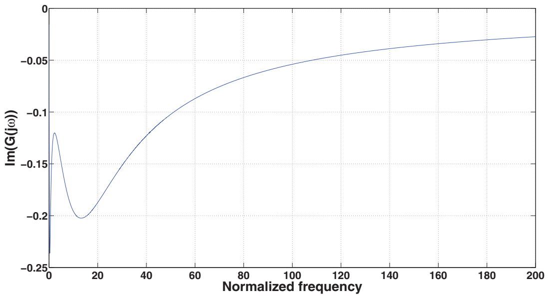

Fig. 1. Plot of the imaginary part of the closed-loop system.

$$A = \begin{bmatrix} -1 & 0 & -1 \\ 1 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix}; B_1 = \begin{bmatrix} 2 \\ 1 \\ .5 \end{bmatrix}; \quad (35)$$

$$B_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}; C_1 = \begin{bmatrix} 2 & 2 & -1 \end{bmatrix}. \quad (36)$$

Using the design procedure presented in the Subsection III-B and choosing $\varepsilon = .3$, the new A matrix will be $A_\varepsilon = A + 0.3I$. The synthesis procedure of Theorem 4 is applied to this system with the new A_ε. Applying the Schur decomposition to the matrix $(A_\varepsilon - B_2(C_1B_2)^{-1}C_1A)$ in (20) gives

$$A_f = \begin{bmatrix} -0.7 & 0 & -1 \\ 0.7 & 0.5 & 1.65 \\ 0 & 1 & 1.3 \end{bmatrix}. \quad (37)$$

The solution to Lyapunov equations (27) and (28)gives $T = 0.0566$ and $S = 0.0300$ which implies that $X = -0.0266$. It follows that $P_f = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 37.5603 \end{bmatrix} \geq 0$. Then, $P = UP_fU^T \geq 0$ is a solution to (34), where $U = \begin{bmatrix} 0.7447 & 0.6553 & 0.1262 \\ -0.6141 & 0.5989 & 0.5140 \\ 0.2612 & -0.4603 & 0.8485 \end{bmatrix}$ is the Schur transformation matrix. This implies that the controller gain matrix (25) is given by $K = \begin{bmatrix} -4.1782 & -16.5942 & -23.4228 \end{bmatrix}$. According to Theorem 4, the closed-loop feedback system (19) from w to z is NI with a pole at the origin, however the actual closed-loop is stable since actual the closed-loop poles are shifted by an amount of $-\varepsilon$ to $(-13.9434, -1.3508, -0.3000)$. Also, the closed-loop system is SNI, to illustrate this, we plot the imaginary part of the transfer function matrix of the closed-loop system from w to z in Fig. 1.

V. Conclusion

In this paper, the algebraic Riccati equation approach was used to derive a negative imaginary (NI) lemma and a strict negative imaginary (SNI) lemma. The NI lemma was employed to solve a negative imaginary controller synthesis problem for an uncertain system. A static controller was chosen to force the plant to be stable and satisfy the negative imaginary property under certain assumptions. This controller can be used to guarantee the robustness stability of the closed-loop system with strict negative imaginary uncertainty.

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