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Characterisations of the "Mixed" Small Gain and Passivity Property for Linear Systems in Discrete Time

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Abstract— **Characterisations of "mixed" systems are presented in a discrete-time setting. First, a feedback stability result based on the Nyquist stability theorem is presented. Second, an eigenvalue-based characterisation of "mixed" systems based on their state-space data is derived. The results are analogous to previous results presented for the continuous-time case and provide a foundation for further study concerning the discretisation of "mixed" systems.**

I. INTRODUCTION

The passivity theorem [1], [2] is a well-established stability result for engineering systems, used in a wide range of application areas such as circuit network theory [3], signal processing systems [4], mechanical networks [5] and robotics [6], [7]. The result guarantees the stability of a feedback interconnection of two stable systems if, for instance, both of the systems are passive, and one of the systems is input strictly passive with finite gain [8]. Passivity has an energy-based interpretation: passive systems are systems that consume, but do not produce, energy (eg: [2]). Related to passivity include the notions of positivity [1] and strict positive realness (SPRness) [2].

Problems can arise from using purely traditional passivity-based techniques for real-world applications. For example, unmodelled dynamics can destroy assumed or nominal passivity over certain frequency bandwidths [9], [10]; and meeting passivity criteria can conflict with system performance requirements [11]. The concept of finite frequency positive realness (ie: positive realness only over a certain frequency band) [12] or "restricted passivity" [13] thus provides engineers with a tool for potentially dealing with a number of these issues.

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Building on the idea of finite frequency positive realness, "mixed" systems were introduced in [14]–[17] as systems that combine notions of passivity and small gain type behaviour in a certain manner, eg: a "mixed" system has small gain behaviours over frequency bands where positivity is violated. "Mixed" systems were intended to aid in the formalisation and extension of the well-known engineering notion that keeping feedback-loop gain small at high frequencies where passivity might be violated avoids destabilisation of high frequency dynamics; see also [18], [19]. The stability of large-scale interconnections of "mixed" systems was considered in [17], and an eigenvalue-based characterisation for "mixed" systems in continuous-time was presented in [16].

While the study of systems with finite frequency positive realness (eg: "mixed" systems [16], [17]; see also [20], [21]) has seen much progress over the past number of years, many basic questions remain. For instance, engineers rarely work with continuous-time systems exclusively. For simulation purposes, or for the purpose of control design, or in order to implement a controller, at some stage a discrete-time representation of the system must be considered. Thus, it is critical to establish whether discrete-time systems inherit fundamental properties of the continuous-time systems from which they are derived. System discretisation has currently become an issue of importance once again, and several papers [22]–[27] have recently appeared on this topic, particularly in the switched systems community. The purpose of this paper is to lay the foundation for future studies concerning the discretisation of "mixed" systems by fully characterising "mixed" systems in the discrete-time setting. In Section II of the paper, "mixed" systems in discrete-time are defined. In Section III, a feedback stability result based on the Nyquist stability theorem is presented. An eigenvalue-based characterisation of "mixed" systems based on their state-space description is derived in Section IV. Directions for future research are presented in Section V.

II. MATHEMATICAL PRELIMINARIES

Before presenting the main results of the paper, some mathematical preliminaries are first established.

A. Notation

Let $\Re[\cdot]$ and $\rho(\cdot)$ denote the real part of a complex number and the spectral radius of a matrix, respectively. The conjugate of a complex number $z = re^{j\theta}$, where r is the magnitude of z, θ is the phase of z and $j^2 = -1$, will be denoted by $\bar{z}$.

B. Definitions

The following definitions are required.

Definition 1: [28, Section 10.1.3] A discrete-time system with proper, real-rational transfer function matrix $G(z)$ is said to be input-output stable if all of the poles of $G(z)$ lie inside the unit circle on the complex plane.

Suppose that $\theta := \omega T$, where T denotes a fixed sampling interval in seconds, and ω denotes any signal frequency in rad/s such that $\theta \in [-\pi, \pi]$. Suppose that $0 \leq \bar{a} \leq \bar{b} \leq \pi$, where $\bar{a}$ and $\bar{b}$ are in radians.

Definition 2: An input-output stable, discrete-time system with square, proper, real-rational transfer function matrix $M(z)$ is said to be input and output strictly positive over $[-\bar{b}, -\bar{a}] \cup [\bar{a}, \bar{b}]$ if there exist real numbers $k, l > 0$ such that

$$-kM^*(e^{j\theta})M(e^{j\theta}) + M^*(e^{j\theta}) + M(e^{j\theta}) - lI \geq 0$$

for all $\theta \in [-\bar{b}, -\bar{a}] \cup [\bar{a}, \bar{b}]$.

A system is said to be input strictly positive over $[-\bar{b}, -\bar{a}] \cup [\bar{a}, \bar{b}]$ if Definition 2 is satisfied with $k = 0$; output strictly positive over $[-\bar{b}, -\bar{a}] \cup [\bar{a}, \bar{b}]$ if the definition is satisfied with $l = 0$; and positive over $[-\bar{b}, -\bar{a}] \cup [\bar{a}, \bar{b}]$ if it is satisfied with $k = l = 0$.

Definition 3: For an input-output stable, discrete-time system with proper, real-rational transfer function matrix $M(z)$, define the system gain over $[-\bar{b}, -\bar{a}] \cup [\bar{a}, \bar{b}]$ as

$$\varepsilon := \min\{\bar{\varepsilon} \in \mathbb{R}_+ : -M^*(e^{j\theta})M(e^{j\theta}) + \bar{\varepsilon}^2 I \geq 0 \text{ for all } \theta \in [-\bar{b}, -\bar{a}] \cup [\bar{a}, \bar{b}]\}.$$

The system is said to have a gain of less than one over $[-\bar{b}, -\bar{a}] \cup [\bar{a}, \bar{b}]$ if $\varepsilon < 1$.

A "mixed" discrete-time system, analogous to the description of a "mixed" continuous-time system provided in [16], [17], is now defined.

Definition 4: An input-output stable, discrete-time system with square, proper, real-rational transfer function matrix $M(z)$ is said to be "mixed" if, for each $\theta \in [-\pi, \pi]$, either of the following hold:

(i) there exist $k, l > 0$ such that $-kM^*(e^{j\theta})M(e^{j\theta}) + M^*(e^{j\theta}) + M(e^{j\theta}) - lI \geq 0$;
(ii) there exists $\varepsilon < 1$ such that $-M^*(e^{j\theta})M(e^{j\theta}) + \varepsilon^2 I \geq 0$.

The following example further illustrates Definition 4.

Example 1: Suppose that

$$A_c = \begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix}, \; B_c = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \; C = \begin{bmatrix} 0 & 1.5 \end{bmatrix}, \; D = -0.2$$

from which the transfer function

$$m(s) = \frac{-0.2(s^2 + 3s - 13)}{(s+1)(s+2)}$$

is obtained. The Nyquist diagram of $m(s)$ is illustrated in Fig. 1. From [17, Definition 3], this continuous-time system is classified as "mixed."

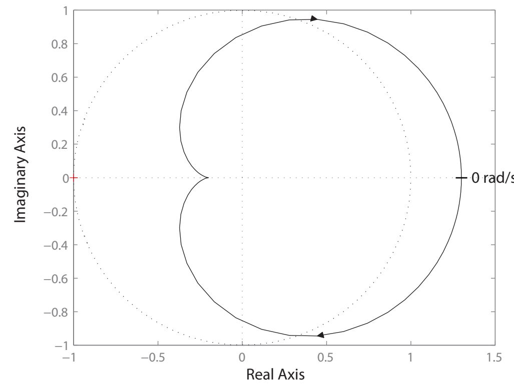

Fig. 1. Nyquist diagram of $m(s)$.

Next, consider the zero-order hold discretisation method described in [29, Chapter 13]; that is, for the nth-order, continuous-time plant described by

$$\begin{aligned} \dot{x}(t) &= A_c x(t) + B_c u(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned}$$

its discrete-time model is given by

$$\begin{aligned} x[(k+1)T] &= Ax(kT) + Bu(kT) \\ y(kT) &= Cx(kT) + Du(kT) \end{aligned}$$

where

$$\begin{aligned} A &:= e^{A_c T} = I + A_c T + \frac{A_c^2 T^2}{2!} + \frac{A_c^3 T^3}{3!} + \dots \\ B &:= \left[IT + \frac{A_c T^2}{2!} + \frac{A_c^2 T^3}{3!} + \dots \right] B_c \end{aligned}$$

or $B = A_c^{-1}[e^{A_c T} - I]B_c = [e^{A_c T} - I]A_c^{-1}B_c$ when A_c is non-singular.

Using the *expm* function in MATLAB Version 7.13.0.564 (R2011b) to compute $e^{A_c T}$ for a sampling interval of $T = 0.2s$ gives

$$e^{A_c 0.2} = \begin{bmatrix} 0.5219 & -0.2968 \\ 0.1484 & 0.9671 \end{bmatrix}.$$

Then

$$A = \begin{bmatrix} 0.5219 & -0.2968 \\ 0.1484 & 0.9671 \end{bmatrix}, \; B = \begin{bmatrix} 0.2968 \\ 0.03286 \end{bmatrix}$$

and

$$m(z) = \frac{-0.2z^2 + 0.3471z - 0.06941}{z^2 - 1.489z + 0.5488}.$$

The Nyquist diagram of $m(z)$ is shown in Fig. 2. From the Nyquist diagram, it is clear that there exists a $\theta_0 = \omega_0 T$ such that, over $[-\theta_0, \theta_0]$, Property (i) of Definition 4 holds and, over $[-\pi, -\theta_0]$ and $[\theta_0, \pi]$, Property (ii) of the definition is satisfied, noting that $\Re[m(e^{j\omega T})] = \frac{1}{2}[m^*(e^{j\omega T}) + m(e^{j\omega T})]$ and $|m(e^{j\omega T})|^2 = m^*(e^{j\omega T})m(e^{j\omega T})$. Hence, this discrete-time model of the continuous-time system, obtained when $T = 0.2s$, is "mixed."

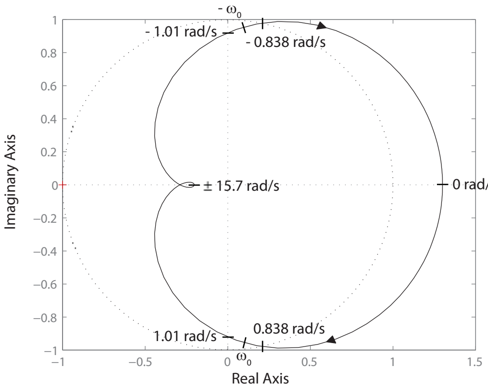

Fig. 2. Nyquist diagram of $m(z)$ with $T = 0.2$s.

On the other hand, computing e^{A_cT} for a sampling interval of $T = 1s$ yields

$$e^{A_c} = \begin{bmatrix} -0.09721 & -0.4651 \\ 0.2325 & 0.6004 \end{bmatrix}.$$

Then

$$A = \begin{bmatrix} -0.09721 & -0.4651 \\ 0.2325 & 0.6004 \end{bmatrix}, \; B = \begin{bmatrix} 0.4651 \\ 0.3996 \end{bmatrix}$$

and

$$m(z) = \frac{-0.2z^2 + 0.7z + 0.2105}{z^2 - 0.5032z + 0.04979}.$$

In this case, the Nyquist diagram of $m(z)$ is given in Fig. 3. Note that there exists on the Nyquist diagram a range of ω from 0.757 rad/s to 0.952 rad/s (and another range from -0.952 rad/s to -0.757 rad/s) over which neither Property (i) nor Property (ii) of Definition 4 holds. The discretisation procedure thus fails on this occasion to preserve the property of "mixedness." That is, this discrete-time model of the continuous-time system, obtained when $T = 1s$, is not "mixed."

C. Preliminary Results

A feedback stability result for "mixed" discrete-time systems is presented later (in Section III). The proof of the result is based on classical Nyquist techniques. Hence, a discrete-time version of the well-known Nyquist stability theorem is recalled, as follows.

Theorem 1: [30, page 74] [31, Section 3.2] Consider the feedback-loop depicted in Fig. 4. Suppose that $G(z)$ is a strictly proper, real-rational transfer function of a stable discrete-time system. Then the feedback-loop is stable if and only if the Nyquist plot of $1 + G(e^{j\theta})$ for $-\pi \le \theta \le \pi$ does not make any encirclements of the origin.

In the above theorem, stability is defined in the sense of [30, Section 3.7]. Note, also, the following observations concerning the Nyquist plot of $1 + G(e^{j\theta})$ for $-\pi \le \theta \le \pi$.

Observation 1: The Nyquist plot of $1 + G(e^{j\theta})$ belongs to a family of Nyquist plots of $1 + \frac{1}{\kappa}G(e^{j\theta})$, where $\kappa \in [1, \infty)$.

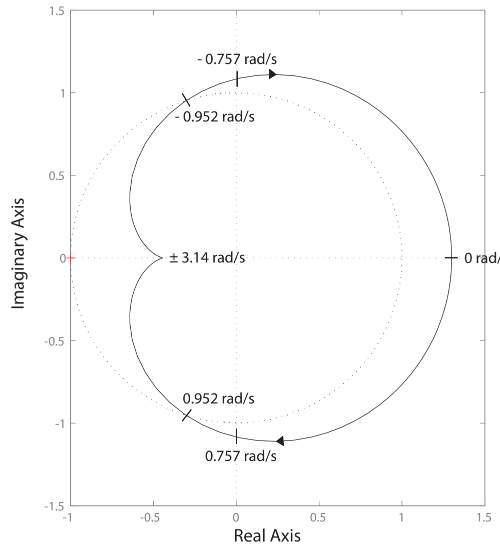

Fig. 3. Nyquist diagram of $m(z)$ with $T = 1$s.

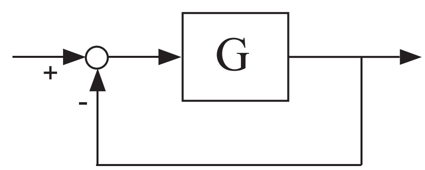

Fig. 4. A negative feedback-loop.

Observation 2: Each Nyquist plot of $1 + \frac{1}{\kappa}G(e^{j\theta})$ is symmetrical about the real axis of the complex plane, where $\kappa \in [1, \infty)$.[1]

Observation 3: As κ and θ vary continuously, the point in the complex plane on which the Nyquist plot of $1 + \frac{1}{\kappa}G(e^{j\theta})$ lies varies continuously.

Observation 4: As $\kappa \to \infty$, $1 + \frac{1}{\kappa}G(e^{j\theta}) \to 1$.

Observation 5: Suppose that κ is very large such that $1 + \frac{1}{\kappa}G(e^{j\theta})$ is almost equal to 1 for all $\theta \in [-\pi, \pi]$. Then suppose that κ is continuously decreased towards 1. Suppose that the Nyquist plot of $1 + G(e^{j\theta})$ encircles the origin at least once. Then there must exist at least one κ_0 and one θ_0 for which $1 + \frac{1}{\kappa_0}G(e^{j\theta_0}) = 0$.

The following corollary has thus been established.

Corollary 2: Adopt the hypotheses of Theorem 1. Then a sufficient condition for the Nyquist plot of $1 + G(e^{j\theta})$ to make no encirclements of the origin is that, for all $\kappa \in [1, \infty)$ and all $\theta \in [-\pi, \pi]$, $1 + \frac{1}{\kappa}G(e^{j\theta}) \neq 0$.

The next result is also required.

Lemma 3: Let $G_1(z)$ and $G_2(z)$ be square, proper, real-rational transfer function matrices with no poles on or outside of the unit circle in the complex plane. Suppose that $G_1^*(e^{j\theta_0}) + G_1(e^{j\theta_0}) > 0$ and $G_2^*(e^{j\theta_0}) + G_2(e^{j\theta_0}) \ge 0$ for some $\theta_0 \in [-\pi, \pi]$. Then $\det[I + G_1(e^{j\theta_0})G_2(e^{j\theta_0})] \neq 0$.

[1] Since $1 + \frac{1}{\kappa}G(e^{-j\omega T}) = \overline{1 + \frac{1}{\kappa}G(e^{j\omega T})}$.

Proof: Since $G_1^*(e^{j\theta_0}) + G_1(e^{j\theta_0}) > 0$, $\Re[\lambda_i[G_1(e^{j\theta_0})]] > 0 \ \forall i$ (where $\lambda_i[\cdot]$ denotes the ith eigenvalue) [32, Theorem 1 of Section 13.1] and so $G_1(e^{j\theta_0})$ is nonsingular. Then $G_1^{-*}(e^{j\theta_0}) + G_1^{-1}(e^{j\theta_0}) > 0$ since $G_1^*(e^{j\theta_0}) + G_1(e^{j\theta_0})$ and $G_1^{-1}(e^{j\theta_0}) + G_1^{-*}(e^{j\theta_0})$ are Hermitian congruent [33, page 415]. Then $G_1^{-*}(e^{j\theta_0}) + G_2^*(e^{j\theta_0}) + G_1^{-1}(e^{j\theta_0}) + G_2(e^{j\theta_0}) > 0$. Hence $\Re[\lambda_i[G_1^{-1}(e^{j\theta_0}) + G_2(e^{j\theta_0})]] > 0 \ \forall i$ and so $\det[G_1^{-1}(e^{j\theta_0}) + G_2(e^{j\theta_0})] \neq 0$. Then $\det[I + G_1(e^{j\theta_0})G_2(e^{j\theta_0})] \neq 0$ since $\det[I + G_1(e^{j\theta_0})G_2(e^{j\theta_0})] = \det[G_1(e^{j\theta_0})]\det[G_1^{-1}(e^{j\theta_0}) + G_2(e^{j\theta_0})]$ and $G_1(e^{j\theta_0})$ is nonsingular. ■

III. Feedback Stability

As demonstrated in Example 1, the assumption that, upon discretisation, systems retain certain properties, such as "mixedness" or passivity, is not always a valid one. This issue with system discretisation is well-known of in the case of passivity [34]. The following result shows that, if "mixedness" has been established in discrete-time, then a feedback stability result holds. (A test for determining whether a system is "mixed" in discrete-time is the subject of Section IV.)

Theorem 4 is analogous to the feedback stability result presented in [17] for "mixed" continuous-time systems. A simpler version of Theorem 4 was proposed in [35, Proposition 4].

Theorem 4: Suppose that $M_1(z)$ and $M_2(z)$ denote the transfer functions of "mixed" discrete-time systems, interconnected as depicted in Fig. 5, where one of these transfer functions is strictly proper. Suppose that there exist two closed sets of θ: (a) a set denoted by Θ_p that consists of $\theta \in [-\pi, \pi]$ over which **both** $M_1(e^{j\theta})$ and $M_2(e^{j\theta})$ have associated with them Property (i) as given in Definition 4; and (b) a set denoted by Θ_s that consists of $\theta \in [-\pi, \pi]$ over which **both** $M_1(e^{j\theta})$ and $M_2(e^{j\theta})$ have associated with them Property (ii) as given in Definition 4. Finally, suppose that $\Theta_p \cup \Theta_s = \{\theta \in \mathbb{R} : -\pi \leq \theta \leq \pi\}$. Under these assumptions, the feedback-loop in Fig. 5 is stable.

Proof: The goal is to show that, for all $\kappa \in [1, \infty)$ and all $\theta \in [-\pi, \pi]$, $1 + \frac{1}{\kappa}M_1(e^{j\theta})M_2(e^{j\theta}) \neq 0$. From Corollary 2, this is a sufficient condition for stability of the feedback-loop. Subsequently, the proof is split into two parts: first, it is shown that $1 + \frac{1}{\kappa}M_1(e^{j\theta})M_2(e^{j\theta}) \neq 0$ for all $\kappa \in [1, \infty)$ and all $\theta \in \Theta_s$; and second, it is shown that $1 + \frac{1}{\kappa}M_1(e^{j\theta})M_2(e^{j\theta}) \neq 0$ for all $\kappa \in [1, \infty)$ and all $\theta \in \Theta_p$.

Part I: for any $\theta \in \Theta_s$. From Property (ii) of Definition 4, $|M_i(e^{j\theta})| < 1$ for $i = 1, 2$, and hence $|M_1(e^{j\theta})M_2(e^{j\theta})| < 1$. Then $\frac{1}{\kappa}|M_1(e^{j\theta})M_2(e^{j\theta})| < \frac{1}{\kappa} \leq 1$ for any $\kappa \geq 1$; ie: $|\frac{1}{\kappa}M_1(e^{j\theta})M_2(e^{j\theta})| < 1$ since $\frac{1}{\kappa}|M_1(e^{j\theta})M_2(e^{j\theta})| = |\frac{1}{\kappa}M_1(e^{j\theta})M_2(e^{j\theta})|$. So $\frac{1}{\kappa}M_1(e^{j\theta})M_2(e^{j\theta}) \neq -1$ for any $\kappa \in [1, \infty)$.

Part II: for any $\theta \in \Theta_p$. From Property (i) of Definition 4, $M_i^*(e^{j\theta}) + M_i(e^{j\theta}) > 0$ for $i = 1, 2$. Observe that $M_i^*(e^{j\theta}) + M_i(e^{j\theta}) > 0$ if and only if $\frac{1}{\sqrt{\kappa}}M_i^*(e^{j\theta}) + \frac{1}{\sqrt{\kappa}}M_i(e^{j\theta}) > 0$, where $\kappa > 0$. Then, from Lemma 3, $1 + \frac{1}{\kappa}M_1(e^{j\theta})M_2(e^{j\theta}) \neq 0$ for any $\kappa > 0$, and hence for any $\kappa \geq 1$. ■

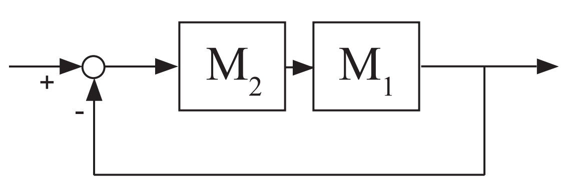

Fig. 5. A negative feedback interconnection of "mixed" systems.

IV. Eigenvalue-Based Characterisation

A procedure for testing whether a discrete-time system is "mixed" is now provided. Consider an arbitrary, causal, linear, shift-invariant system, described by the equations

$$x(k+1) = Ax(k) + Bu(k), \ x(0) = x_0,$$
$$y(k) = Cx(k) + Du(k),$$

where $x(k) \in \mathbb{R}^n$, $u(k) \in \mathbb{R}^m$, $y(k) \in \mathbb{R}^m$, $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{m \times n}$ and $D \in \mathbb{R}^{m \times m}$ with A stable.[2] Furthermore, suppose that A is nonsingular. Denoting $M(z) := C(zI - A)^{-1}B + D$ and $M^*(z) := [M(z^{-1})]^T$ gives

$$M^*(z) = \left[\begin{array}{c|c} A^{-T} & -A^{-T}C^T \\ \hline B^TA^{-T} & D^T - B^TA^{-T}C^T \end{array}\right], \qquad (1)$$

from [36, Section 21.4].[3] Let $G_1(e^{j\theta}) := -kM^*(e^{j\theta})M(e^{j\theta}) + M^*(e^{j\theta}) + M(e^{j\theta}) - lI$ and $G_2(e^{j\theta}) := -M^*(e^{j\theta})M(e^{j\theta}) + \varepsilon^2 I$. Consider the following two results.

Lemma 5: Suppose that $k, l \in \mathbb{R}$ and consider $G_1(e^{j\theta})$ as defined above. Let $Y := I - kD$ and suppose that $X_1 := -kD^TD + D^T + D - lI$ and $\widetilde{X}_1 := X_1 - B^TA^{-T}C^TY$ are invertible. For some $\theta_0 \in [-\pi, \pi]$, the matrix $G_1(e^{j\theta_0})$ has a zero eigenvalue if and only if the simplectic matrix S_1 has an eigenvalue on the unit circle at the point $e^{j\theta_0}$, where

$$S_1 := \begin{pmatrix} E_1 + U_1E_1^{-T}V_1 & -U_1E_1^{-T} \\ -E_1^{-T}V_1 & E_1^{-T} \end{pmatrix}$$

and $E_1 := A - BX_1^{-1}Y^TC$, $U_1 := -BX_1^{-1}B^T$, $V_1 := kC^TC + C^TYX_1^{-1}Y^TC$.

Proof: Given that

$$\begin{bmatrix} (e^{j\theta_0}I - A)^{-1} & 0 \\ -k(e^{j\theta_0}I - A^{-T})^{-1}A^{-T}C^TC(e^{j\theta_0}I - A)^{-1} & (e^{j\theta_0}I - A^{-T})^{-1} \end{bmatrix}$$
$$= \left[e^{j\theta_0}I - \begin{pmatrix} A & 0 \\ -kA^{-T}C^TC & A^{-T} \end{pmatrix}\right]^{-1}, \qquad (2)$$

note that $G_1(e^{j\theta_0}) = -k[-B^TA^{-T}(e^{j\theta_0}I - A^{-T})^{-1}A^{-T}C^T + D^T - B^TA^{-T}C^T][C(e^{j\theta_0}I - A)^{-1}B + D] - B^TA^{-T}(e^{j\theta_0}I - A^{-T})^{-1}A^{-T}C^T + D^T - B^TA^{-T}C^T + C(e^{j\theta_0}I - A)^{-1}B + D - lI = \bar{C}(e^{j\theta_0}I - \bar{A})^{-1}\bar{B} + \widetilde{X}_1$, where

$$\bar{A} := \begin{pmatrix} A & 0 \\ -kA^{-T}C^TC & A^{-T} \end{pmatrix}, \quad \bar{B} := \begin{pmatrix} B \\ A^{-T}C^TY \end{pmatrix}$$

and

$$\bar{C} := \begin{pmatrix} Y^TC + kB^TA^{-T}C^TC & -B^TA^{-T} \end{pmatrix},$$

[2] Ie: $\rho(A) < 1$ [36, Section 21.1].

[3] The notation on the right-hand side of (1) denotes a state-space realisation.

using [37, Lemma 3]. Then, in the manner of [38, Lemma 1],

$$\begin{aligned}
&\det(G_1(e^{j\theta_0}))\\
&=\det(\bar{C}(e^{j\theta_0}I-\bar{A})^{-1}\bar{B}+\widetilde{X}_1)\\
&=\det(\widetilde{X}_1)\det(I+\widetilde{X}_1^{-1}\bar{C}(e^{j\theta_0}I-\bar{A})^{-1}\bar{B})\\
&=\det(\widetilde{X}_1)\det(I+(e^{j\theta_0}I-\bar{A})^{-1}\bar{B}\widetilde{X}_1^{-1}\bar{C}) \quad \text{(Sylvester's Determinant Theorem)}\\
&=\det(\widetilde{X}_1)\det((e^{j\theta_0}I-\bar{A})^{-1})\det(e^{j\theta_0}I-\bar{A}+\bar{B}\widetilde{X}_1^{-1}\bar{C})\\
&=\det(\widetilde{X}_1)\det((e^{j\theta_0}I-A)^{-1})\det((e^{j\theta_0}I-A^{-T})^{-1})\\
&\quad\det(e^{j\theta_0}I-\widetilde{H}_1),
\end{aligned}$$

where $\widetilde{H}_1:=\bar{A}-\bar{B}\widetilde{X}_1^{-1}\bar{C}$. Since A is stable, then $\det(e^{j\theta_0}I-A)\neq 0$ for any $\theta_0\in\mathbb{R}$; and $e^{j\theta_0}I-A$ is invertible and so $\det((e^{j\theta_0}I-A)^{-1})\neq 0$. Similarly, $\det((e^{j\theta_0}I-A^{-T})^{-1})\neq 0$ noting that $(-1)^n\det(e^{j\theta_0}I)\det(e^{-j\theta_0}I-A)\det(A^{-1})=\det(e^{j\theta_0}I-A^{-1})=\det(e^{j\theta_0}I-A^{-T})$ from [39, Equation 6.1.4]. Thus, $G_1(e^{j\theta_0})$ has a zero eigenvalue if and only if $\det(e^{j\theta_0}I-\widetilde{H}_1)=0$, ie: $\widetilde{H}_1$ has an eigenvalue on the unit circle at the point $e^{j\theta_0}$. Finally, $\widetilde{H}_1=S_1$ via matrix inversion identities [36, Section 2.3]. ■

Lemma 6: Suppose that $\varepsilon\in\mathbb{R}\setminus\{0\}$ and consider $G_2(e^{j\theta})$ as defined at the beginning of the section. Suppose that $-DD^T+\varepsilon^2I$, $X_2:=-D^TD+\varepsilon^2I$ and $\widetilde{X}_2:=X_2+B^TA^{-T}C^TD$ are invertible. For some $\theta_0\in[-\pi,\pi]$, the matrix $G_2(e^{j\theta_0})$ has a zero eigenvalue if and only if the simplectic matrix S_2 has an eigenvalue on the unit circle at the point $e^{j\theta_0}$, where

$$S_2:=\begin{pmatrix} E_2+U_2E_2^{-T}V_2 & -U_2E_2^{-T}\\ -E_2^{-T}V_2 & E_2^{-T}\end{pmatrix}$$

and $E_2:=A+BX_2^{-1}D^TC$, $U_2:=-BX_2^{-1}B^T$, $V_2:=\varepsilon^2C^T(-DD^T+\varepsilon^2I)^{-1}C$.

Proof: Given (2) with $k=1$, note that $G_2(e^{j\theta_0})=-[-B^TA^{-T}(e^{j\theta_0}I-A^{-T})^{-1}A^{-T}C^T+D^T-B^TA^{-T}C^T][C(e^{j\theta_0}I-A)^{-1}B+D]+\varepsilon^2I=\bar{C}(e^{j\theta_0}I-\bar{A})^{-1}\bar{B}+\widetilde{X}_2$, where

$$\bar{A}:=\begin{pmatrix} A & 0\\ -A^{-T}C^TC & A^{-T}\end{pmatrix},\quad \bar{B}:=\begin{pmatrix} B\\ -A^{-T}C^TD\end{pmatrix}$$

and

$$\bar{C}:=\begin{pmatrix}-D^TC+B^TA^{-T}C^TC & -B^TA^{-T}\end{pmatrix},$$

from [37, Lemma 3]. Then, in the manner of [38, Lemma 1] and similar to the proof of Lemma 5, $\det(G_2(e^{j\theta_0}))=\det(\widetilde{X}_2)\det((e^{j\theta_0}I-A)^{-1})\det((e^{j\theta_0}I-A^{-T})^{-1})\det(e^{j\theta_0}I-\widetilde{H}_2)$, where $\widetilde{H}_2:=\bar{A}-\bar{B}\widetilde{X}_2^{-1}\bar{C}$. The remainder of this proof follows in the manner of the proof to Lemma 5. ■

Lemmas 5 and 6 can be utilised for testing whether a system of the form given at the beginning of the section is "mixed" in the following manner.

Let $\widetilde{G}_1(e^{j\theta})$ denote $G_1(e^{j\theta})$, where $k=l=0$. Similarly, let $\widetilde{G}_2(e^{j\theta})$ denote $G_2(e^{j\theta})$, where $\varepsilon=1$. Upon applying Lemmas 5 and 6 to $\widetilde{G}_1(e^{j\theta})$ and $\widetilde{G}_2(e^{j\theta})$, respectively, set

$$\Psi_p:=\{\theta\in[-\pi,\pi]: S_1 \text{ has an eigenvalue on the unit circle at } e^{j\theta}\}$$

and

$$\Psi_s:=\{\theta\in[-\pi,\pi]: S_2 \text{ has an eigenvalue on the unit circle at } e^{j\theta}\}.$$

Remark 1: It has been assumed that the system does not have a strictly proper transfer function in order to facilitate the application of Lemma 5 to $\widetilde{G}_1(e^{j\theta})$.

Next, divide two intervals of $-\pi$ to π up into smaller intervals, where any elements of Ψ_p and Ψ_s are set as open interval endpoints, as follows:

Division Group 1 :=

$$[-\pi,\theta_{p_1}),(\theta_{p_1},\theta_{p_2}),\ldots,(\theta_{p_{\bar{n}-1}},\theta_{p_{\bar{n}}}),(\theta_{p_{\bar{n}}},\pi]$$

Division Group 2 :=

$$[-\pi,\theta_{s_1}),(\theta_{s_1},\theta_{s_2}),\ldots,(\theta_{s_{\bar{m}-1}},\theta_{s_{\bar{m}}}),(\theta_{s_{\bar{m}}},\pi]$$

where $\bar{n}$ = number of elements in Ψ_p; $\bar{m}$ = number of elements in Ψ_s; $\theta_{p_1},\theta_{p_2},\ldots,\theta_{p_{\bar{n}}}$ denote the elements of Ψ_p listed in increasing order; and $\theta_{s_1},\theta_{s_2},\ldots,\theta_{s_{\bar{m}}}$ denote the elements of Ψ_s listed in increasing order. If Ψ_p is empty, then $\bar{n}=0$ and Division Group 1 consists of the single interval $[-\pi,\pi]$; similarly, if Ψ_s is empty, then $\bar{m}=0$ and Division Group 2 consists of the single interval $[-\pi,\pi]$. If $\theta_{p_1}=-\pi$ and $\theta_{p_{\bar{n}}}=\pi$, then Division Group 1 becomes $(-\pi,\theta_{p_2}),(\theta_{p_2},\theta_{p_3}),\ldots,(\theta_{p_{\bar{n}-1}},\pi)$. Similarly, if $\theta_{s_1}=-\pi$ and $\theta_{s_{\bar{m}}}=\pi$, then Division Group 2 becomes $(-\pi,\theta_{s_2}),(\theta_{s_2},\theta_{s_3}),\ldots,(\theta_{s_{\bar{m}-1}},\pi)$.

Finally, identify the sign definiteness of $\widetilde{G}_1(e^{j\theta})$ over each of the individual intervals in Division Group 1, and the sign definiteness of $\widetilde{G}_2(e^{j\theta})$ over each of the individual intervals in Division Group 2. Determining the sign definiteness over any of these intervals can be achieved by checking the sign definiteness at a single θ from within the interval, eg: at the interval midpoint. Let $I_{\widetilde{G}_1}$ denote the set of θ belonging to those intervals over which $\widetilde{G}_1(e^{j\theta})>0$, and $I_{\widetilde{G}_2}$ denote the set of θ belonging to those intervals over which $\widetilde{G}_2(e^{j\theta})>0$. Then, implement the following result.

Theorem 7: The following two statements are equivalent.

(a) A discrete-time system, as described at the beginning of the section, is "mixed."
(b) $I_{\widetilde{G}_1}\cup I_{\widetilde{G}_2}=\{\theta\in\mathbb{R}:-\pi\leq\theta\leq\pi\}$

Sketch of Proof: Recall, from Definition 4, that an input-output stable, discrete-time system with square, proper, real-rational transfer function matrix $M(z)$ is "mixed" if, for each $\theta\in[-\pi,\pi]$, Property (i) and/or Property (ii) hold. For θ continuously varying over some small interval, over which a "mixed" system is alternating between exhibiting only Property (i) and only Property (ii), there exists at least one common θ in that range at which both Property (i) and Property (ii) hold due to continuity. In general, over intervals with open endpoints, the existence of $k,l>0$ such that $-kM^*(e^{j\theta})M(e^{j\theta})+M^*(e^{j\theta})+M(e^{j\theta})-lI\geq 0$ implies that $M^*(e^{j\theta})+M(e^{j\theta})>0$, and the existence of $\varepsilon<1$ such that $-M^*(e^{j\theta})M(e^{j\theta})+\varepsilon^2I\geq 0$ implies that $-M^*(e^{j\theta})M(e^{j\theta})+I>0$; while the converse (ie: $M^*(e^{j\theta})+M(e^{j\theta})>0$ implying the existence of $k,l>0$ such that $-kM^*(e^{j\theta})M(e^{j\theta})+$

$M^*(e^{j\theta})+M(e^{j\theta})-lI\geq 0$, and $-M^*(e^{j\theta})M(e^{j\theta})+I>0$ implying the existence of $\varepsilon<1$ such that $-M^*(e^{j\theta})M(e^{j\theta})+\varepsilon^2 I\geq 0$) is not necessarily true. However, any overlap of open intervals from Division Group 1 and Division Group 2 such that $I_{\widetilde{G}_1}\cup I_{\widetilde{G}_2}=\{\theta\in\mathbb{R}:-\pi\leq\theta\leq\pi\}$ implies the existence of common θ at which both $\widetilde{G}_1(e^{j\theta})>0$ and $\widetilde{G}_2(e^{j\theta})>0$, and these common θ can be taken as closed endpoints of subintervals existing within the open intervals over which $\widetilde{G}_1(e^{j\theta})>0$ or $\widetilde{G}_2(e^{j\theta})>0$. For closed interval endpoints, the implication directions concerning the matrix inequalities go both ways, and hence the equivalence in the theorem statement holds. ■

V. Conclusions

In this paper, "mixed" systems were characterised in a discrete-time setting. The purpose of doing so was to provide a foundation for future studies concerning discretisation procedures that preserve "mixedness." A discussion on systems with strictly proper transfer functions in relation to Section IV will also follow at a later date.

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