Stabilization of Uncertain Negative-Imaginary Systems via State-Feedback Control

Ian R. Petersen Alexander Lanzon Zhuoyue Song

Abstract— **The controller synthesis problem of uncertain negative-imaginary systems has important engineering applications in, for example, lightly damped flexible structures with colocated position sensors and force actuators. This paper provides a systematic controller synthesis procedure via an LMI approach to construct a state-feedback internally stabilizing controller such that the nominal closed-loop system satisfies negative-imaginary properties and a DC gain condition. As a result of this, the closed-loop system can then be guaranteed to be robustly stable against uncertainties that are stable strictly negative-imaginary (e.g. unmodeled spill-over dynamics in a lightly damped flexible structure). An numerical example is given to show the usefulness of the proposed results.**

Index Terms— **Positive real, bounded real, negative-imaginary systems, lightly damped systems, $\mathcal{H}_\infty$ control, small-gain theorem, passivity.**

I. INTRODUCTION

This paper is concerned with the robust stabilization problem of uncertain negative-imaginary systems when full state feedback is available. The definition of negative-imaginary systems implies in the SISO case that such systems have a phase lag between 0 and π for all $\omega > 0$. Hence such systems have a Nyquist plot which lies below the real axis for all $\omega > 0$. This frequency response property allows stability results to be established for a positive feedback interconnection of negative-imaginary systems in a similar way to negative feedback interconnections of positive-real systems [2]. More specifically, a positive feedback interconnection of one stable negative-imaginary system and one stable strictly negative-imaginary system is internally stable if and only if the DC loop gain less than unity under some simple strict-properness conditions [3]. This result provides a useful and natural framework for the analysis of robust stability for lightly damped systems with unmodeled spill-over dynamics. It also captures in a systematic rigorous framework, graphical design methods adopted in the 1980s by practical engineers related to positive position feedback control [1] [4] and similar techniques.

Corresponding author is Zhuoyue Song. Tel: +44-161-306-2821, Fax: +44-161-306-4647. Email: Zhuoyue.Song@postgrad.manchester.ac.uk.

Ian R. Petersen is with the School of Information Technology and Electrical Engineering, University of New South Wales at the Australian Defence Force Academy, Canberra, ACT 2006 Australia. Email: i.r.petersen@gmail.com.

Alexander Lanzon and Zhuoyue Song are with the Control Systems Centre, School of Electrical and Electronic Engineering, University of Manchester, Manchester M60 1QD, UK. Alexander Lanzon's Email is: a.lanzon@ieee.org.

Negative-imaginary systems arise for example, when the transfer function corresponding to a structure with colocated position sensors and force actuation is considered. These systems have important engineering applications to lightly damped flexible structures [3] [5]. These structures with colocated position sensors and force actuation typically give rise to a sum of an infinite number of second order transfer functions. Normally, only a small number of modes are considered for controller synthesis and unmodeled high-frequency dynamics can easily destabilize the system unless due care is exercised. Since these systems are highly resonant, synthesis based on the small-gain theorem tends to be conservative, and since the relative degree of these systems is greater than unity, positive-real and passivity-based synthesis methods are not applicable to this problem. In this situation, the theory of negative-imaginary systems may be used. As the unmodeled spill-over dynamics often have stable strictly negative-imaginary properties [5], it is desirable to design a internally stabilizing controller so that the nominal closed-loop system is stable negative-imaginary and a DC gain condition is satisfied. Then the resulting closed-loop uncertain system is guaranteed to be robustly stable. Preliminary results concerning the synthesis of dynamic output feedback controllers for uncertain negative-imaginary systems are presented in [5]. This paper will present sufficient results for the synthesis of state feedback controllers using an LMI approach.

Our main result here is a simple derivation of the LMI conditions that allow state feedback static controller synthesis such that the closed-loop systems satisfy stable negative-imaginary properties and a DC gain condition. This derivation involves applications of a complete state-space characterization of negative-imaginary systems and a trick to convert the DC gain condition to an $\mathcal{H}_\infty$-norm constraint so that the Bounded Real Lemma can be used to drive a corresponding LMI condition.

II. PRELIMINARIES

In this section, definitions and results are collected from earlier sources for ease of reference. First, two sets of stable negative-imaginary systems are defined formally as in [2]:

Definition 1: Let the set of square stable negative-imaginary transfer functions be defined by

$$\mathcal{I} := \{X \in \mathcal{RH}_\infty^{n\times n} : \\ j[X(j\omega) - X(j\omega)^*] \geq 0 \ \forall \omega \in (0,\infty)\}. \quad (1)$$

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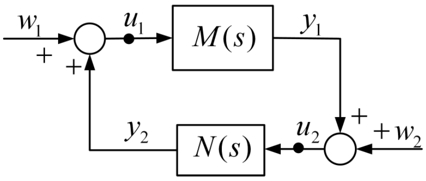

Fig. 1. A positive feedback interconnection

and let the set of square stable strictly negative-imaginary transfer functions be defined by

$$\mathcal{I} := \{X \in \mathcal{RH}_\infty^{n\times n} : \\ j[X(j\omega) - X(j\omega)^*] > 0 \ \forall\omega \in (0,\infty)\} \subset \mathcal{I}. \quad (2)$$

Here $\mathcal{RH}_\infty^{n\times n}$ denotes real-rational stable transfer functions of dimensions $n \times n$, and $X(j\omega)^*$ denotes the complex conjugate transpose of the matrix $X(j\omega)$.

The following theorem, given in [3], gives a complete state-space characterization of stable negative-imaginary systems. The theorem has a similar form to the well-known Positive Real Lemma [6] and hence, is referred to as the *Negative Imaginary Lemma*. This theorem will be used in the next section to develop LMI conditions for state-feedback controller synthesis for negative-imaginary systems.

Theorem 1: [3] Consider a system $M(s)$ with minimal state-space realization

$$\begin{aligned} \dot{x} &= Ax + Bu, \\ y &= Cx + Du \end{aligned} \quad (3)$$

and transfer function matrix $M(s) = C(sI - A)^{-1}B + D$. Then the system is stable negative-imaginary if and only if $\det(j\omega I - A) \neq 0 \ \forall\omega \in \mathbb{R}$, $D = D^T$ and there exists a real symmetric matrix $Y > 0$ such that

$$AY + YA^T \leq 0 \quad \text{and} \quad B + AYC^T = 0,$$

where $\mathbb{R}$ denotes the field of real numbers.

The next theorem, which was also given in [3], is the main analysis result that establishes the internal stability of positive feedback interconnections of negative-imaginary systems. It states that a positive feedback interconnection of one stable negative-imaginary system and one stable strictly negative-imaginary system, is internally stable if and only if the DC loop gain is less than unity. This stability analysis result motivates the controller synthesis method for the negative-imaginary systems given in this paper.

Theorem 2: [3] Given $M(s) \in \mathcal{I}$ and $N(s) \in \mathcal{I}_s$, and suppose that $M(\infty)N(\infty) = 0$ and $N(\infty) \geq 0$. Then, the positive-feedback interconnection of these two systems illustrated in Fig. 1 is internally stable if and only if the maximum eigenvalue of the matrix $M(0)N(0)$, denoted by $\overline{\lambda}(M(0)N(0))$, satisfies

$$\overline{\lambda}(M(0)N(0)) < 1. \quad (4)$$

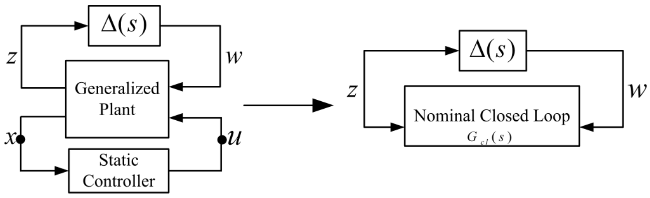

Fig. 2. A feedback control system where the plant uncertainty $\Delta(s)$ is stable strictly negative-imaginary, and satisfies $\Delta(\infty) \geq 0$ and a DC gain condition $\overline{\lambda}(\Delta(0)) < \mu$. If the controller is chosen so that the nominal closed-loop transfer function matrix $G_{cl}(s)$ is strictly proper, stable negative-imaginary and satisfies the DC condition $\overline{\lambda}(G_{cl}(0)) \leq \frac{1}{\mu}$, then the closed-loop system is robustly stable.

III. STATE-FEEDBACK STATIC CONTROLLER SYNTHESIS

The main result of this paper will be given in this section. We will formulate the problem of finding a static state-feedback internally stabilizing controller such that the closed-loop system satisfies a negative-imaginary property and a DC loop gain condition.

Consider a robust feedback control system of the form shown in Fig. 2 in the case that full state feedback is available. In this case, Theorem 1 can be used to synthesize a static state-feedback control law such that the resulting closed-loop system is stable negative-imaginary. Indeed, suppose the uncertain system shown in Fig. 2 is described by the state equations

$$\begin{aligned} \dot{x} &= Ax + B_1 w + B_2 u, \\ z &= C_1 x, \\ w &= \Delta(s)z, \\ u &= Kx, \end{aligned} \quad (5)$$

where the uncertainty transfer function matrix $\Delta(s)$ is assumed to be stable strictly negative-imaginary with $\overline{\lambda}(\Delta(0)) < 1$ and $\Delta(\infty) \geq 0$. If a static state-feedback control law $u = Kx$ is applied to this system, the resulting closed-loop uncertain system is described by the state equations

$$\begin{aligned} \dot{x} &= (A + B_2 K)x + B_1 w, \\ z &= C_1 x, \\ w &= \Delta(s)z. \end{aligned} \quad (6)$$

The corresponding nominal closed-loop transfer function matrix is

$$G_{cl}(s) = C_1(sI - A - B_2K)^{-1}B_1.$$

In order to construct the static gain K so that this system is stable negative-imaginary according to Theorem 1, let

$$K = \bar{K}Y^{-1},$$

for an arbitrary matrix $Y > 0$ and a matrix $\bar{K}$ to be determined. Then according to Theorem 1, the system (6) is stable negative-imaginary if and only if there exist matrices

$\bar{K}$ and $Y > 0$ such that $A + B_2K = A + B_2\bar{K}Y^{-1}$ has no eigenvalues on the imaginary axis and

$$\begin{aligned}(A + B_2K)Y &+ Y(A + B_2K)^T \\ &= AY + YA^T + B_2\bar{K} + \bar{K}^T B_2^T \leq 0, \\ B_1 + (A + B_2K)YC_1^T &= B_1 + AYC_1^T + B_2\bar{K}C_1^T = 0.\end{aligned}$$

Furthermore, the requirement that $A+B_2K$ has no eigenvalues on the imaginary axis can be guaranteed if the inequality

$$(A + B_2K)Y + Y(A + B_2K)^T \leq 0$$

is replaced by an inequality of the form

$$(A + B_2K)Y + Y(A + B_2K)^T + \epsilon I \leq 0,$$

for any $\epsilon > 0$. This inequality leads to a set of LMIs, which provide a sufficient condition for the existence of a static state-feedback control law such that the closed-loop nominal system is stable negative-imaginary

$$\begin{bmatrix} AY + YA^T + B_2\bar{K} + \bar{K}^T B_2^T + \epsilon I & B_1 + AYC_1^T + B_2\bar{K}C_1^T \\ B_1^T + C_1YA^T + C_1\bar{K}^T B_2^T & 0 \end{bmatrix} \leq 0, \quad \epsilon > 0, \quad Y > 0. \tag{7}$$

Here $\epsilon > 0$ is a parameter chosen to be sufficiently small. If there exists a solution to these LMIs, then the corresponding state feedback control law is given by $u = \bar{K}Y^{-1}x$. Note that in order to use Theorem 2 to guarantee that the closed-loop uncertain system is stable, it is required that

$$\Delta(\infty)G_{cl}(\infty) = 0, \ \Delta(\infty) \geq 0, \text{ and } \overline{\lambda}(\Delta(0)G_{cl}(0)) < 1.$$

However, $\Delta(\infty)G_{cl}(\infty) = 0$ holds automatically since $G_{cl}(s)$ is strictly proper. Also, the assumptions on $\Delta(s)$ in (5) imply that $\Delta(\infty) \geq 0$ and $\overline{\lambda}(\Delta(0)) < 1$. From these conditions, it follows that the condition $\overline{\lambda}(G_{cl}(0)) \leq 1$ ensures $\overline{\lambda}(\Delta(0)G_{cl}(0)) < 1$. However, since $G_{cl}(s)$ is assumed to be stable negative-imaginary, it follows that $\overline{\lambda}(G_{cl}(0)) = \sigma_{max}(G_{cl}(0))$, the maximum singular value of $G_{cl}(0)$. This fact follows using Lemma 2 of [3], which implies $G_{cl}(0) \geq G_{cl}(\infty) = 0$. From this result, it follows that the condition $\overline{\lambda}(\Delta(0)G_{cl}(0)) < 1$ is satisfied if

$$\sigma_{max}(G_{cl}(0)) \leq 1.$$

A sufficient condition for this inequality to hold is that the closed-loop weighted H_∞-norm bound

$$\|\frac{\alpha}{s+\alpha}G_{cl}(s)\|_\infty \leq 1 \tag{8}$$

is satisfied for some sufficiently small $\alpha > 0$. In order to incorporate this requirement into the static state-feedback design, the closed-loop nominal system (6) is augmented with the transfer function matrix $\frac{\alpha}{s+\alpha}I$ to obtain the system

$$\begin{aligned}\dot{x} &= \begin{bmatrix} A + B_2K & 0 \\ C_1 & -\alpha I \end{bmatrix} x + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} w, \\ z &= \begin{bmatrix} 0 & \alpha I \end{bmatrix} x.\end{aligned} \tag{9}$$

As in [8], a transfer function matrix of the form $H_{cl}(s) = C_{cl}\left(sI - A_{cl}\right)^{-1}B_{cl}$ satisfies the H_∞-norm

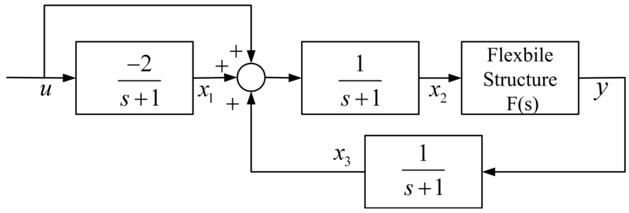

Fig. 3. Block diagram of a system to be controlled using a state-feedback LMI approach to robust controller design.

bound $\|H_{cl}(s)\|_\infty \leq 1$ if and only if there exists a matrix $Q > 0$ such that

$$\begin{bmatrix} A_{cl}Q + QA_{cl}^T + B_{cl}B_{cl}^T & QC_{cl}^T \\ C_{cl}Q & -I \end{bmatrix} \leq 0.$$

Applying this result to the system (9) with $K = \bar{K}Y^{-1}$, and restricting the matrix $Q > 0$ to have the form

$$Q = \begin{bmatrix} Y & 0 \\ 0 & Z \end{bmatrix}$$

leads to the LMI

$$\begin{bmatrix} AY + YA^T + B_2\bar{K} + \bar{K}^T B_2^T + B_1B_1^T & YC_1^T & 0 \\ C_1Y & -2\alpha Z & \alpha Z \\ 0 & \alpha Z & -I \end{bmatrix} \leq 0.$$

Combining this LMI with the LMI (7) leads to the following sufficient condition for the existence of a static state-feedback controller that guarantees closed-loop robust stability for the uncertain system (5).

Theorem 3: Suppose there exist a real number $\epsilon > 0$ and matrices $Y > 0, \bar{Z} > 0$ and $\bar{K}$ such that the following LMIs are satisfied:

$$\begin{bmatrix} AY + YA^T + B_2\bar{K} + \bar{K}^T B_2^T + \epsilon I & B_1 + AYC_1^T + B_2\bar{K}C_1^T \\ B_1^T + C_1YA^T + C_1\bar{K}^T B_2^T & 0 \end{bmatrix} \leq 0, \tag{10}$$

$$\begin{bmatrix} AY + YA^T + B_2\bar{K} + \bar{K}^T B_2^T + B_1B_1^T & YC_1^T & 0 \\ C_1Y & -2\bar{Z} & \bar{Z} \\ 0 & \bar{Z} & -I \end{bmatrix} \leq 0. \tag{11}$$

Then the static state-feedback control law $u = \bar{K}Y^{-1}x$ is robustly stabilizing for the uncertain system (5).

IV. ILLUSTRATIVE EXAMPLE

As an illustration of the application of Theorem 3, consider the system shown in Fig. 3. This system includes a flexible structure. The force applied to the flexible structure is denoted x_2, and the deflection of the structure at the same location is denoted y. The transfer function from x_2 to y is denoted $F(s)$. The flexible structure has a colocated force actuator and position sensor and as a result, the transfer function $F(s)$ is stable strictly negative-imaginary. It is desired to construct a static state feedback controller for this system, which is robust against unmodeled flexible dynamics. Indeed, in order to apply the method of Theorem 3 to this example, the transfer function $F(s)$ is replaced by a constant unity gain, and the resulting error is the stable strictly negative-imaginary transfer function $\Delta(s) = F(s) - 1$. The transfer

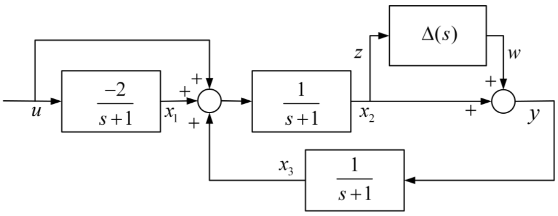

Fig. 4. Block diagram of an uncertain system to be controlled by means of a state-feedback controller to be constructed using an LMI approach.

function $\Delta(s)$ is treated as an uncertainty in the system as shown in Fig. 4. A state-space realization of this uncertain system is as follows

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} w + \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} u,$$
$$z = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix},$$
$$w = \Delta(s)z.$$

That is, Theorem 3 can be applied with

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix},$$
$$C_1 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}.$$

Then the LMIs (10) - (11) are solved using standard LMI solving software to find the unknown matrices Y, M and $\bar{Z}$. Indeed, using such software, the solution

$$Y = \begin{bmatrix} 3.1095 \times 10^9 & -1.9990 & -3.1095 \times 10^9 \\ -1.9990 & 0.40282 & 1.4019 \\ -3.1095 \times 10^9 & 1.4019 & 3.1095 \times 10^9 \end{bmatrix} > 0,$$
$$\bar{K} = \begin{bmatrix} -4.9362 & 0.9996 & 4.4478 \end{bmatrix}$$
$$\bar{Z} = 1.1868 > 0, \ \epsilon = 10^{-6} \qquad ,$$

is obtained. Therefore using Theorem 3, the required state feedback gain matrix K can be constructed according to the formula

$$K = \bar{K}Y^{-1} = \begin{bmatrix} 152.2434 & 228.1875 & 152.2434 \end{bmatrix}.$$

The open-loop response and closed-loop response of the nominal system under three randomly chosen nonzero initial states are shown in Fig. 5. A Bode plot of the corresponding closed-loop transfer function from w to z,

$$G_{cl}(s) = C_1(sI - A - B_2K)^{-1}B_1,$$

is shown in Fig. 6. From this Bode plot, it can be seen that $G_{cl}(s)$ is indeed stable strictly negative-imaginary since

$$\angle G_{cl}(j\omega) \in (-\pi, 0) \text{ for all } \omega \in (0, \infty).$$

Also, it can be seen that the DC gain of $G_{cl}(s)$ is less than unity. Since, the uncertainty transfer function $\Delta(s)$ in this example is stable strictly negative-imaginary, then it follows from Theorem 2 that the true closed-loop system is stable provided that $\Delta(0) < 1$.

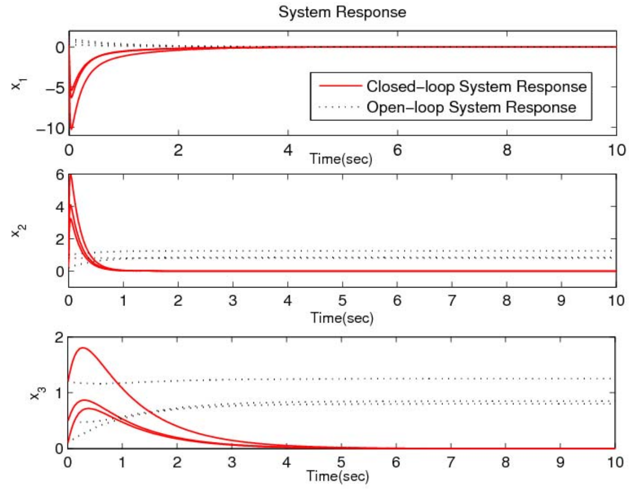

Fig. 5. Open-loop response and closed-loop response of the nominal system under three randomly chosen nonzero initial states

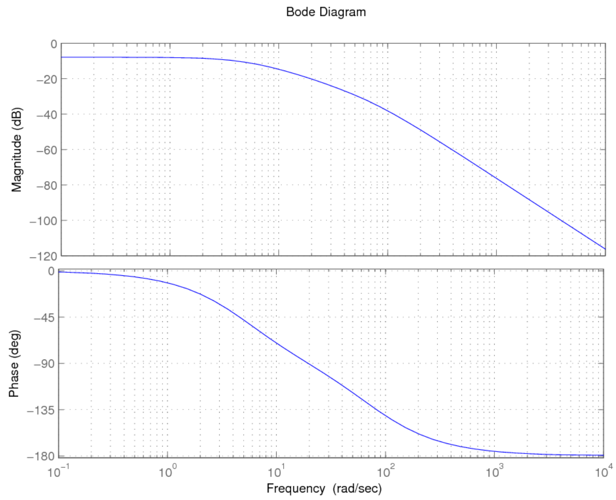

Fig. 6. Bode plot of the closed-loop transfer function $G_{cl}(s)$ from the uncertainty input w to the uncertainty output z.

Note that in the above example, the nominal system is obtained by replacing the flexible structure transfer function $F(s)$ by a fixed unity gain. This fixed gain can be regarded as an approximation of the flexible structure transfer function DC gain $F(0)$. If the flexible structure transfer function DC gain is known to be exactly unity, then it follows that $\Delta(0) = F(0) - 1 = 0$. In this case, the DC gain condition in Theorem 2 is automatically satisfied, and there is no need to require the extra LMI (11) in constructing the static state-feedback controller. However, the current approach means that the DC gain of the flexible structure transfer function does not have to be known exactly, and the control system has a level of robustness against uncertainty in $F(0)$.

V. CONCLUSIONS

This paper provides an explicit state feedback controller synthesis method to achieve robust stability in the face of negative-imaginary uncertainties. LMI-based conditions are developed to design a static state feedback internally stabilizing controller with constant gain such that the closed-loop system satisfies negative-imaginary properties and a DC gain condition. This result assists engineers to design controllers that robust stabilize systems against stable strictly negative-imaginary uncertainties which arise for example in unmodeled spill-over dynamics in lightly damped structures. Results on dynamic output feedback controller synthesis for negative-imaginary systems are currently also being pursued.

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