μ Analysis for Interconnections of Systems with Negative Imaginary Frequency Response

Sönke Engelken, Alexander Lanzon and Ian R. Petersen

Abstract— **Many physical systems are subject to real parametric uncertainty. A recently developed method gives exact parametric stability regions for systems with so called negative imaginary frequency response. This method is contrasted in this paper with μ analysis for the same system class. The two methods are shown to yield similar results for some simple SISO cases. Limitations of both methods are pointed out through a numerical example.**

I. Introduction

This paper will compare a recently derived robust stability result for a particular class of linear systems [1], those with negative imaginary frequency response, to results that can be computed using the well known μ analysis technique, described e.g. in [2]. An intuitive definition of systems with negative imaginary frequency response for the SISO case follows immediately from the description: The imaginary part of the frequency response is negative (or rather non-positive) for all $\omega > 0$; i.e., the positive frequency branch of the Nyquist plot of any such system is located on or below the real axis. Formally, the set of stable systems with negative imaginary frequency response is defined as:

$$\begin{aligned}\mathscr{C} \quad := \quad & \left\{R(s) \in \mathscr{RH}_\infty^{n\times n} : \right. \\ & \left. j\left[R(j\omega) - R(j\omega)^*\right] \geq 0 \ \forall \omega \in (0,\infty)\right\}. \quad (1)\end{aligned}$$

Stable systems with strictly negative imaginary frequency response form a second set:

$$\begin{aligned}\mathscr{C}_{\mathrm{s}} \quad := \quad & \left\{R(s) \in \mathscr{RH}_\infty^{n\times n} : \right. \\ & \left. j\left[R(j\omega) - R(j\omega)^*\right] > 0 \ \forall \omega \in (0,\infty)\right\} \subset \mathscr{C} (2)\end{aligned}$$

where $\mathscr{RH}_\infty^{n\times n}$ denotes the set of real-rational stable transfer function matrices of dimension $n \times n$, and $R(j\omega)^*$ denotes the complex conjugate transpose of $R(j\omega)$. These set definitions are taken from [1], where the following necessary and sufficient stability condition for positive-feedback interconnections of negative imaginary systems was derived:

Theorem 1.1: Given $M(s) \in \mathscr{C}_{\mathrm{s}}$ and $\Delta(s) \in \mathscr{C}$ that also satisfy $M(\infty)\Delta(\infty) = 0$ and $M(\infty) \geq 0$. Then, $[M(s), \Delta(s)]$ is internally stable if and only if $\overline{\lambda}\left(M(0)\Delta(0)\right) < 1$, where $[.,.]$ denotes the positive feedback interconnection of two systems as shown in Fig. 1, and $\overline{\lambda}(.)$ denotes the largest eigenvalue of a matrix.

Thus, under certain conditions the stability of the feedback interconnection can be checked by computing the steady state gain of the two transfer functions. For parametric uncertainty, the exact stability regions for all uncertain parameters follow immediately from the theorem. The negative-imaginary systems property is analogous to passivity and typically arises in mechanical structures when considering transfer functions from force actuators to collocated position sensors (see e.g. [3] for details).

In this paper, real μ analysis will be applied to the abovementioned positive feedback interconnection of negative imaginary systems. The real structured singular value $\mu_{\mathrm{R}}(.)$ of a transfer function matrix $M(s) \in \mathbb{C}^{n\times n}$ with respect to an uncertainty matrix Δ has been defined, e.g. in [4], as

$$\begin{aligned}&\mu_{\mathrm{R}}(M) := \\ &\operatorname*{ess\,sup}_{\omega\geq 0} \left\{ \min_{\Delta\in\mathbf{\Delta}} \overline{\sigma}(\Delta) : \ \det\left(I - \Delta M(j\omega)\right) = 0 \right\}^{-1}. \quad (3)\end{aligned}$$

For real μ, the uncertainty is restricted to real numbers and is a member of the following set:

$$\mathbf{\Delta} = \left\{\mathrm{blockdiag}\left[\delta_1 I_{k_1}, \ldots, \delta_m I_{k_m}\right] : \ \delta_i \in \mathbb{R}\right\} \quad (4)$$

where $\sum_{i=1}^{m} k_i = n$. The restriction to real μ reflects the focus on real parametric uncertainty. A general solution to the problem of computing the real structured singular value or the essentially equivalent real stability radius has been given in [5].

The remainder of this paper is structured as follows: Sections II and III show that the stability results obtained through Theorem 1.1 and μ analysis are very similar in the case of real additive uncertainty and uncertainty in the steady state coefficients of $\Delta(s)$, respectively. Section IV discusses limitations of the two methods which will have become apparent in the previous comparison, and illustrates these with an example. Finally, Section V concludes the paper.

II. Case 1: Real Additive Uncertainty

A. Transformation of the feedback structure

Consider the two SISO systems,

$$\begin{aligned}&M(s) \in \mathscr{C}_{\mathrm{s}} \\ &\Delta(s) = \frac{sn(s)}{d(s)} + k_0 \in \mathscr{C} \quad (5)\end{aligned}$$

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ISBN 978-963-311-369-1

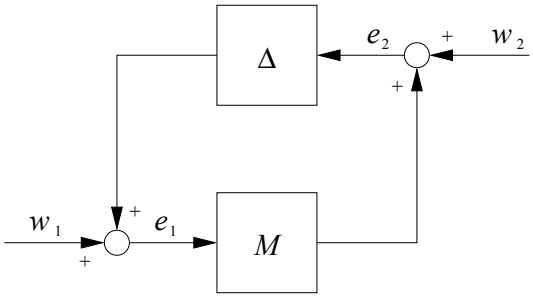

Fig. 1. Positive feedback interconnection

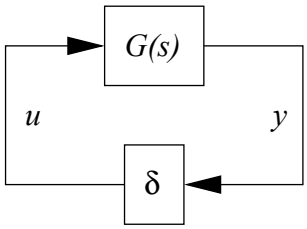

Fig. 2. Equivalent structure for μ analysis

with $n(s) = \sum_{i=0}^{p-1} n_i s^i$, $d(s) = \sum_{i=0}^{p} d_i s^i$. Any SISO negative imaginary system can be expressed as in Eqn. 5, by isolating the fixed real part. Now, let $k_0 = \bar{k}_0 + \delta$, with $\delta \in \mathbb{R}$ an additive uncertainty in the nominal real parameter $\bar{k}_0$. As $\bar{k}_0 + \delta$ does not influence the imaginary part of the system, we also have

$$\frac{sn(s)}{d(s)} \in \mathscr{C}.$$

Theorem 1.1 assumes a positive feedback system interconnection as displayed in Fig. 1. We here assume that $\Delta(s)$ contains one real additive uncertainty δ. Later it will become apparent that parametric uncertainties not influencing the steady state gain of the system do not influence stability as long as the property of a negative imaginary frequency response is maintained.

To carry out μ analysis, the uncertainty δ must be extracted and the resulting transfer function $G(s)$ between the output and the input of the scalar uncertainty block must be computed using the remainder of $\Delta(s)$ and $M(s)$. This system structure for μ analysis is displayed in Fig. 2. For additive uncertainty as considered above, this transformation can easily be carried out by observing the structure of $\Delta(s)$, as displayed in Fig. 3.

As illustrated in Fig. 2, $G(s)$ is the transfer function from the output of the uncertainty u to its input y. From Fig. 3:

$$\begin{aligned} & y = M(s)\left(u + \frac{sn(s)}{d(s)} + \bar{k}_0\right) y \\ \Leftrightarrow \quad & G(s) = \frac{M(s)}{1 - M(s)\left(\frac{sn(s)}{d(s)} + \bar{k}_0\right)} \end{aligned} \tag{6}$$

B. Comparing the stability conditions

This section will show that both Theorem 1.1 and the real structured singular value (equivalent to a small gain type condition for SISO systems) give similar stability results, subject to certain limitations, for the case of real additive uncertainty.

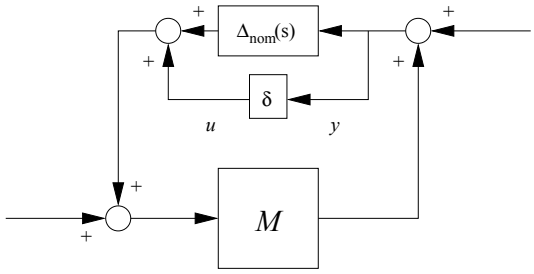

Fig. 3. Feedback structure in the case of additive uncertainty in $\Delta(s)$ (using $\Delta_{\text{nom}}(s) = \Delta(s)|_{\delta=0}$).

At first, the result of Theorem 1.1 for this case of uncertainty will be obtained. Consider $M(s) \in \mathscr{C}_{\text{s}}$ with $M(\infty) = 0$, and an uncertain system

$$\Delta(s) = \frac{sn(s)}{d(s)} + \bar{k}_0 + \delta \in \mathscr{C}, \ \delta \in \mathbb{R}. \tag{7}$$

Then the positive feedback interconnection shown in Fig. 1 is internally stable if and only if

$$\begin{aligned} & \Delta(0)M(0) < 1 \\ \Leftrightarrow \quad & \delta < \tfrac{1}{M(0)} - \bar{k}_0. \end{aligned} \tag{8}$$

Note that $M(0) > M(\infty) = 0$ via Lemma 2 in [1].

The resulting stability region for the uncertain parameter δ is one-sided and non-conservative (i.e. exact), as $\Delta(s)$ will remain in the class of systems with negative imaginary frequency response for all $\delta \in \mathbb{R}$.

Now μ analysis will be applied to obtain the corresponding stability criterion. For $M(s)$ and $\Delta(s)$ as defined above, the nominal system description $G(s)$ is given by Eqn. (6).

The equation we wish to solve to obtain $\mu_\delta(G)$ as defined in Eqn. (3) is

$$1 - \delta G(j\omega) = 0. \tag{9}$$

As described earlier, the real structured singular value is equal to the inverse of the smallest absolute value of a solution $\delta \in \mathbb{R}$ to this equation. In the following, it will be shown that under the conditions imposed on the imaginary part of the frequency response of $M(s)$ and $\Delta(s)$, this equation will only have a solution at $\omega = 0$.

As $\delta \in \mathbb{R}$, we require that $G(j\omega)$ be real, too, for a solution of Eqn. (9) to exist. At $\omega = 0$,

$$G(0) = \frac{M(0)}{1 - M(0)\bar{k}_0} \in \mathbb{R}$$

For any $\omega > 0$, we have the condition:

$$\begin{aligned} & \Im\{G(j\omega)\} = 0 \\ \Leftrightarrow \quad & \Im\left\{\frac{M(j\omega)}{1-\left(\frac{j\omega n(j\omega)}{d(j\omega)}+\bar{k}_0\right)M(j\omega)}\right\} = 0 \\ \Leftrightarrow \quad & \Im\left\{\frac{1}{M(j\omega)} - \frac{j\omega n(j\omega)}{d(j\omega)} - \bar{k}_0\right\} = 0 \\ \Leftrightarrow \quad & \Im\left\{\frac{1}{M(j\omega)}\right\} = \Im\left\{\frac{j\omega n(j\omega)}{d(j\omega)}\right\} \end{aligned} \tag{10}$$

Note that $M(j\omega) \neq 0 \ \forall \omega \in (0,\infty)$ because $M(s) \in \mathscr{C}_{\mathrm{s}}$. The last line, Eqn. (10), is a contradiction, as

$$\begin{aligned} \Delta(s) \in \mathscr{C} \quad &\Rightarrow \quad \Im\left\{\frac{j\omega n(j\omega)}{d(j\omega)}\right\} \leq 0 \\ M(s) \in \mathscr{C}_{\mathrm{s}} \quad &\Rightarrow \quad \Im\left\{\frac{1}{M(j\omega)}\right\} > 0 \end{aligned}$$

This proves our claim that Eqn. (9) only has a solution at $\omega = 0$. Therefore,

$$\begin{aligned} \left\{\mu_\delta^{\mathrm{r}}(G)\right\}^{-1} &= \{|\delta| : \delta \in \mathbb{R},\ 1-\delta G(0) = 0\} \\ &= \left|\frac{1}{G(0)}\right| \\ &= \left|\frac{1}{M(0)} - \bar{k}_0\right|. \end{aligned} \tag{11}$$

This stability result, though similar to that obtained via application of Theorem 1.1 and given in Eqn. (8), is conservative due to the absolute value that is involved in Eqn. (11). This conservativeness is inherent to the defintion of the structured singular value, since it yields a "ball" of allowable uncertainty. Hence for a single uncertain parameter one always obtains a two-sided interval, restricted by the smallest destabilizing uncertainty.

III. Case 2: A single real parametric uncertainty

A. Transformation of the feedback structure

In this second case, $\Delta(s)$ contains a single parametric uncertainty, either in one of the numerator or denominator coefficients. Even though all transfer functions may be written in the form of Eqn. (5), an uncertainty in the polynomial coefficients has different effects on the transfer function than an uncertainty in the additive part. The case of an uncertainty in the numerator is considered first, where $\Delta(s)$ is given by

$$\Delta(s) = \frac{n(s) + \delta s^i}{d(s)},\ i \in [0,\ldots,p-1]$$

with $n(s) = \sum_{i=1}^{p-1} n_i s^i$ and $d(s) = s^p + \sum_{i=1}^{p-1} d_i s^i$. The first step is to obtain a state space model of $\Delta(s)$. The Kalman controllable canonical form is chosen as a suitable state-space realization. Then,

$$\Delta(s) \hat{=} \left[\begin{array}{c|c} A & B \\ \hline C + \delta\hat{C}_i & D \end{array}\right]$$

is the state space representation of $\Delta(s)$, s.t. $A \in \mathbb{R}^{p\times p}$, $B \in \mathbb{R}^{p\times 1}$, $C \in \mathbb{R}^{1\times p}$, $D = 0$ and

$$\Delta_{\mathrm{nom}}(s) := \Delta(s)|_{\delta=0} = C(sI-A)^{-1}B + D$$

is the nominal transfer function when the uncertainty is zero. The effect of δ on the state space model is represented by the vector

$$\hat{C}_i = [\underbrace{0,\ldots,0}_{p-i-1}, 1, \underbrace{0,\ldots,0}_{i}],$$

corresponding to an uncertainty in the i-th order numerator coefficient. From this state space model, a 2×2 transfer function matrix $\Sigma(s)$ can be deduced (see [6], p. 261-264 for details) which fulfills

$$\Delta(s) = \mathscr{F}_{\mathrm{l}}(\Sigma(s), \delta).$$

In the above equation, $\mathscr{F}_{\mathrm{l}}(.,.)$ denotes a lower linear fractional transformation (LFT). For the current setup of a single uncertainty δ in the i-th numerator coefficient, the elements of $\Sigma(s)$ can be derived as:

$$\Sigma(s) = \begin{bmatrix} \Delta_{\mathrm{nom}}(s) & 1 \\ \frac{s^i}{d(s)} & 0 \end{bmatrix}.$$

The last step of the transformation procedure is to compute $G(s)$ from $M(s)$ and $\Sigma(s)$:

$$\begin{aligned} G(s) &= \mathscr{F}_{\mathrm{u}}(\Sigma(s), M(s)) \\ &= \frac{\frac{s^i}{d(s)}M(s)}{1-\Delta_{\mathrm{nom}}(s)M(s)}. \end{aligned} \tag{12}$$

A similar procedure can be carried out for an uncertainty in the denominator of $\Delta(s)$, i.e.

$$\Delta(s) = \frac{n(s)}{d(s) + \delta s^i},\ i \in [0,\ldots,p-1].$$

Its state space realization is now given by

$$\Delta(s) \hat{=} \left[\begin{array}{c|c} A + \delta\hat{A}_i & B \\ \hline C & D \end{array}\right]$$

with

$$\hat{A}_i = \begin{bmatrix} \overbrace{0\ldots0}^{p-i-1} & -1 & \overbrace{0\ldots0}^{i} \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 \end{bmatrix}.$$

Again employing the procedure from [6], one obtains

$$\Sigma(s) = \begin{bmatrix} \Delta_{\mathrm{nom}}(s) & \Delta_{\mathrm{nom}}(s) \\ -\frac{s^i}{d(s)} & -\frac{s^i}{d(s)} \end{bmatrix}.$$

It follows that for an uncertainty δ in the i-th coefficient of the denominator,

$$\begin{aligned} G(s) &= -\frac{s^i}{d(s)}\left(1 + \frac{M(s)\Delta_{\mathrm{nom}}(s)}{1-\Delta_{\mathrm{nom}}(s)M(s)}\right) \\ &= \frac{-s^i}{d(s) - n(s)M(s)}. \end{aligned} \tag{13}$$

Note that this procedure for transfer function transformation may also be applied to the case of additive uncertainty, where it results in a non-zero D matrix in the state space representation, and yields exactly the $G(s)$ that was derived in Section II.

B. Comparing the stability conditions

Two simplifying assumptions will now be made: Firstly, it is assumed that the uncertainty occurs in the zero-order coefficient of either the numerator or the denominator. From Theorem 1.1, it follows that only uncertainties in the steady state gain of the system influence stability, and therefore this assumption seems justified. Uncertainties in the higher-order coefficients may occur in addition to zero-order parametric uncertainties, as long as the negative imaginary property remains valid. The second assumption is that $n(s) = n_0 > 0$ is entirely real. This restricts the order of the denominator to $p \leq 2$ and entails $\frac{1}{d(s)} \in \mathscr{C}$.

Uncertainty in the numerator of $\Delta(s)$ is considered first. Given $M(s) \in \mathscr{C}_\mathrm{s}$, with $M(\infty) = 0$ and

$$\Delta(s) = \frac{n_0 + \delta}{d(s)}$$

with $\delta > -n_0$ s.t. $\Delta(s) \in \mathscr{C}$. Then, according to Theorem 1.1, their positive feedback interconnection is stable iff

$$\begin{aligned} & M(0)\tfrac{n_0+\delta}{d_0} < 1 \\ \Leftrightarrow \quad & \delta < \tfrac{d_0 - n_0 M(0)}{M(0)}. \end{aligned} \tag{14}$$

Following a similar outline as in Section II for the case of additive uncertainty, it will now be shown that the real structured singular value is determined by the steady state value of $G(s)$. To solve Eqn. (9) for an $\omega_0 \in [0, \infty)$, we require

$$\begin{aligned} & \Im\{G(j\omega_0)\} = 0 \\ \Leftrightarrow \quad & \Im\left\{\frac{M(j\omega_0)}{d(j\omega_0) - n_0 M(j\omega_0)}\right\} = 0 \\ \Leftrightarrow \quad & \Im\left\{\frac{d(j\omega_0)}{M(j\omega_0)} - n_0\right\} = 0 \\ \Leftrightarrow \quad & \angle d(j\omega_0) - \angle M(j\omega_0) = k\pi,\ k \in \mathbb{Z}. \end{aligned} \tag{15}$$

Note that by Lemma 2 in [1], $M(0) > M(\infty) = 0$. Eqn. (15) is satisfied for $\omega_0 = 0$, where we find

$$\angle d_0 = \angle M(0)$$

and hence $k = 0$. For any $\omega_0 > 0$, consider that

$$\begin{aligned} M(s) \in \mathscr{C}_\mathrm{s} \quad &\Rightarrow \quad \angle M(j\omega) \in (-\pi, 0)\ \forall \omega > 0 \\ \frac{1}{d(s)} \in \mathscr{C} \quad &\Rightarrow \quad \angle d(j\omega) \in [0, \pi]\ \forall \omega > 0 \end{aligned}$$

and therefore

$$\angle d(j\omega) - \angle M(j\omega) \in (0, 2\pi)\ \forall \omega > 0.$$

Thus we can restrict k in Eqn. (15) to

$$k = \begin{cases} 0, & \omega_0 = 0 \\ 1, & \omega_0 > 0 \end{cases}.$$

We will now consider the values of $G(j\omega)$ at frequencies ω_0 at which the function is entirely real. At $\omega_0 = 0$ we find

$$G(0) = \frac{M(0)}{d_0 - n_0 M(0)}.$$

At any $\omega_0 > 0$ (denoting $\angle M(j\omega_0) = \angle d(j\omega_0) - \pi =: \alpha$),

$$\begin{aligned} G(j\omega_0) &= \frac{|M(j\omega_0)| \exp j\alpha}{|d(j\omega_0)| \exp j(\alpha+\pi) - n_0 |M(j\omega_0)| \exp j\alpha} \\ &= \frac{|M(j\omega_0)| \exp j\alpha}{(|d(j\omega_0)| + n_0 |M(j\omega_0)|) \exp j(\alpha+\pi)} \\ &= (-1)\frac{|M(j\omega_0)|}{|d(j\omega_0)| + n_0 |M(j\omega_0)|}. \end{aligned}$$

This expression is bounded by

$$-\frac{1}{n_0} < G(j\omega_0) < 0$$

since $M(s) \in \mathscr{C}_\mathrm{s}$. If $|G(0)| \geq |G(j\omega_0)|$ for all ω_0 s.t. $G(j\omega_0) \in \mathbb{R}$,

$$\left\{\mu^\mathrm{r}_\delta(G)\right\}^{-1} = \left|\frac{1}{G(0)}\right| \tag{16}$$

which is similar to the result derived through Theorem 1.1 (and given in Eqn. (14), though this result in Eqn. (16) has an absolute value). Otherwise, we have to assume that there exists an $\omega_0 > 0$ s.t. $|G(j\omega_0)| > |G(0)|$, resulting in a smallest destabilizing uncertainty given by:

$$\begin{aligned} \delta &= \frac{1}{G(j\omega_0)} \\ &= (-1)\frac{|d(j\omega_0)| + n_0 |M(j\omega_0)|}{|M(j\omega_0)|} \\ &< -n_0 \end{aligned}$$

This, however, would violate the assumption that $\Delta(s)$ is a system with negative imaginary frequency response, which requires $\delta > -n_0$. Thus we have a contradiction, which proves our claim that $|G(0)| \geq |G(j\omega_0)|\ \ \forall \omega_0 : G(j\omega_0) \in \mathbb{R}$.

For the second case, an uncertainty in the zero-order coefficient of the denominator, we have

$$\Delta(s) = \frac{n_0}{d(s) + \delta}.$$

Theorem 1.1 states that given $M(s) \in \mathscr{C}_\mathrm{s}$, with $M(\infty) = 0$, and $\delta > -d_0$ s.t. $\Delta(s) \in \mathscr{C}$, the positive feedback interconnection of $M(s)$ and $\Delta(s)$ is stable iff

$$\begin{aligned} & M(0)\frac{n_0}{d_0 + \delta} < 1 \\ \Leftrightarrow \quad & \delta > n_0 M(0) - d_0. \end{aligned} \tag{17}$$

Note that according to this result, the interconnection is stable for $\delta = 0$ iff $n_0 M(0) - d_0 < 0$. The μ approach requires that $G(s)$ be stable, and therefore this condition is assumed in the following derivations. An equivalent result can again be obtained through a μ approach. When isolating δ, the transfer function $G(s)$ given in Eqn. (13) can easily computed for the case $i = 0$ and $n(s) = n_0$. A solution $\delta \in \mathbb{R}$ to Eqn. (9) requires

$$\begin{aligned} & \Im\{G(j\omega_0)\} = 0 \\ \Leftrightarrow \quad & \Im\{d(j\omega_0)\} = \Im\{M(j\omega_0\} \end{aligned}$$

for some $\omega_0 \in [0,\infty)$. This only holds at $\omega_0 = 0$, since

$$M(s) \in \mathscr{C}_s \;\Rightarrow\; \Im\{M(j\omega)\} < 0 \;\forall \omega > 0$$
$$\frac{1}{d(s)} \in \mathscr{C} \;\Rightarrow\; \Im\{d(j\omega)\} \geq 0 \;\forall \omega > 0.$$

Thus the smallest destabilizing uncertainty is given by

$$\delta = \frac{1}{G(0)} = n_0 M(0) - d_0.$$

The resulting structured singular value is again a two-sided bounded interval with one side identical to the stability result obtained through Theorem 1.1 (and given in Eqn. (17)):

$$\{\mu_\delta(G)\}^{-1} = \left|\frac{1}{n_0 M(0) - d_0}\right|.$$

IV. Limitations of the methods via a numerical example

Both of the tested methods for robust stability analysis have their limitations. Theorem 1.1 is only applicable when the real parametric uncertainty preserves the negative-imaginary property. On the other hand, the simple μ-based approach used above only yields two-sided stability boundaries, which can often be conservative. Furthermore, it requires the stability region to include the nominal case, i.e. the value $\delta = 0$. These characteristic limitations of both methods are illustrated through a simple example. Consider a positive feedback interconnection of the following two functions:

$$M(s) = \frac{s+3}{s^3+3s^2+19s+55};$$
$$\Delta(s) = \frac{3+\delta}{s^2+2s+2}.$$

To apply Theorem 1.1, we require $\delta > -3$ s.t. $M(s), \Delta(s) \in \mathscr{C}_s$. As $M(\infty) = \Delta(\infty) = 0$, these functions then fulfill all assumptions of Theorem 1.1. We can deduce that the positive feedback interconnection of $M(s)$ and $\Delta(s)$ is stable iff

$$M(0)\Delta(0) < 1$$
$$\Leftrightarrow \quad \delta < \frac{101}{3} \simeq 33.67. \tag{18}$$

For stability analysis using μ, the transfer function with isolated uncertainty is derived. By Eqn. (12),

$$G(s) = \frac{M(s)}{d(s) - n_0 M(s)}. \tag{19}$$

This is a fifth-order function with a relative degree of four. Its Bode plot is depicted in Fig. 4. The phase diagram shows that only two finite frequencies make $G(s)$ entirely real, $\omega_{0,1} = 0\ \frac{\text{rad}}{\text{sec}}$ and $\omega_{0,2} \simeq 4.26\ \frac{\text{rad}}{\text{sec}}$. The real value at the limit of $\omega \to \infty$ is never reached for any finite frequency and is not of consequence to stability since $M(\infty) = \Delta(\infty) = 0$. At each of the two frequencies, a smallest destabilizing δ can be computed.

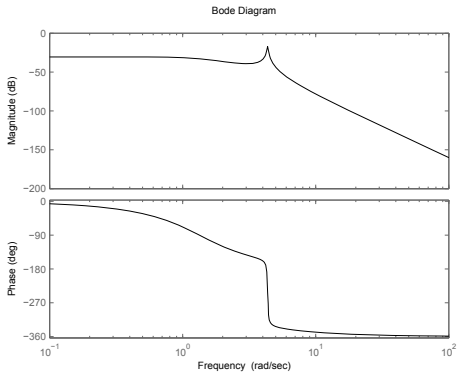

Fig. 4. Bode diagram of G(s)

$$1 - \delta_1 G(j\omega_{0,1}) = 0$$
$$\Leftrightarrow \quad \delta_1 = \frac{1}{G(0)} = \frac{101}{3} \simeq 33.67$$
$$1 - \delta_2 G(j\omega_{0,2}) = 0$$
$$\Leftrightarrow \quad \delta_2 = \frac{1}{G(j4.26)} \simeq -15.85.$$

If the analysis is restricted to systems with negative imaginary frequency response, i.e. if $\delta > -3$ is guaranteed, then the instability at $\omega_{0,2}$ can not occur, and we have

$$\{\mu_\delta(G)\}^{-1} = |\delta_1| \simeq 33.67.$$

This result is equivalent to the one derived through Theorem 1.1. If, however, the restriction to negative imaginary systems cannot be guaranteed, it can be seen that the actual smallest destabilizing uncertainty is much closer to the nominal value:

$$\{\mu_\delta(G)\}^{-1} = |\delta_2| \simeq 15.85.$$

In this case it is closer by a factor of two, but much larger differences are possible. In such cases, the conservativeness of the μ method will restrict the guaranteed stability regions around the nominal value, whereas Theorem 1.1 restricts the stability regions to areas where the system has a negative imaginary frequency response. Three stability regions obtained through application of Theorem 1.1, μ analysis with restriction to negative-imaginary systems, and normal unrestricted μ analysis are illustrated in Fig. 5.

V. Conclusion

The analysis carried out in the previous sections has compared robust stability criteria for positive feedback interconnections of two systems with negative imaginary frequency response. The systems were restricted to SISO with a single real uncertainty. As had been shown by [1], it suffices to compute the steady state gain of the systems to derive necessary and sufficient robust stability conditions in terms of the uncertain parameters. This method has been compared to the computation of the real structured singular value for the abovementioned system interconnections. In the SISO

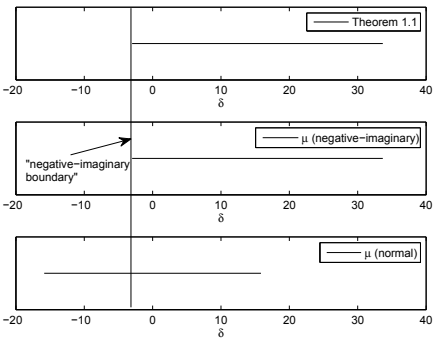

Fig. 5. Stability regions obtained through Theorem 1.1, μ analysis restricted to negative imaginary systems, and normal μ analysis (from top to bottom). The system is negative imaginary for $\delta > -3$.

case, the computation of the real structured singular value reduces to checking the Nyquist criterion for the nominal system description $G(s)$, with only real gains permitted. For the uncertainties considered here, it could be shown that a permissible solution to the main stability equation can only be found at $\omega = 0$, and that the resulting stability condition is similar to that given in [1].

Some limitations of both of the considered methods haven been illustrated through a numerical example: Due to its definition, the structured singular value may often yield conservative stability results. In contrast, the method in [1] gives exact stability regions (which need not be symmetric and are usually one-sided), but depends on the systems being restricted to having a negative imaginary frequency response.

REFERENCES

[1] A. Lanzon and I. R. Petersen, "Stability robustness of a feedback interconnection of systems with negative imaginary frequency response," *Automatic Control, IEEE Transactions on*, vol. 53, no. 4, pp. 1042–1046, 2008.

[2] A. Packard and J. Doyle, "The complex structured singular value," *Automatica*, vol. 29, no. 1, pp. 71–109, January 1993.

[3] J. L. Fanson and T. Caughey, "Positive position feedback control for large space structures," *AIAA Journal*, vol. 28, no. 4, pp. 717–724, April 1990.

[4] P. M. Young and J. C. Doyle, "Computation of μ with real and complex uncertainties," in *Decision and Control, 1990., Proceedings of the 29th IEEE Conference on*, 1990, pp. 1230–1235 vol.3.

[5] L. Qiu, B. Bernhardsson, A. Rantzer, E. J. Davison, P. M. Young, and J. C. Doyle, "A formula for computation of the real stability radius," *Automatica*, vol. 31, no. 6, pp. 879–890, June 1995.

[6] K. Zhou, J. C. Doyle, and K. Glover, *Robust and Optimal Control*. Upper Saddle River: Prentice Hall, August 1995.