All solutions to a class of two-block μ-synthesis problems

George Papageorgiou Alexander Lanzon

***Abstract*— In this paper, we solve a class of two-block μ-synthesis problems by showing that these problems can be converted to $\mathscr{H}_\infty$ synthesis problems. The μ-synthesis problems considered correspond to controller design problems where plant model uncertainty and closed-loop performance are captured using the same type of dynamic uncertainty. We show that we can solve two-block μ-synthesis problems when both blocks correspond to additive, inverse additive or coprime factor uncertainty (left and right).**

***Index Terms*— μ-synthesis, $\mathscr{H}_\infty$ control, additive uncertainty, inverse additive uncertainty, coprime factor uncertainty.**

I. INTRODUCTION

Controller design using the structured singular value (μ) was proposed in the mid eighties by Doyle [1]. Unlike $\mathscr{H}_\infty$ synthesis though, μ-synthesis (as it is known) has not yet been solved. In practise, μ-synthesis is approximately solved by iteratively minimizing the upper bound of μ by so-called $D-K$ iteration. This approximate solution has yielded good performance controllers for many applications, e.g. see the booster rocket design example by Enns [2].

In this paper, we solve a class of two-block μ-synthesis problems by converting them to $\mathscr{H}_\infty$ synthesis problems. By "solve" we mean that we can compute the minimum value of μ and that we can parameterize all controllers that achieve the minimum value of μ.

Since both blocks correspond to the same type of dynamic uncertainty that can be additive, inverse additive or coprime factor (left and right), these results, though interesting, are of limited practical value. They are however, to the best of our knowledge, the only μ-synthesis problems that have been solved to date. Smith and Packard [3], [4] have also solved two-block μ-synthesis problems but for constant matrices (i.e. non-dynamic systems). In their work, they consider the problem of designing an optimal static controller for a linear static system with static uncertainty modeled as a single induced Euclidean norm bounded perturbation. They show that this static design problem is convex and that it can be formulated as a sequence of generalized eigenvalue problems.

Corresponding author is George Papageorgiou. Tel: +420-545-501-843, Fax: +420-545-501-800, Email: george.papageorgiou@honeywell.com. This work has been supported by ARC Discovery-Projects Grant DP0342683, by National ICT Australia Ltd. and by Honeywell Inc. Internal Research and Development funds.

George Papageorgiou is at Honeywell, Turanka 96/1236, 627 00 Brno, Czech Republic.

Alexander Lanzon is at the Control Systems Centre, School of Electrical and Electronic Engineering, University of Manchester, Sackville Street, Manchester M60 1QD, United Kingdom.

II. MOTIVATION

Let

$$\mathbf{\Delta} = \left\{ \mathrm{diag}(\Delta_1, \Delta_2) : \Delta_i \in \mathbb{C}^{p\times m} \right\}.$$

For $M \in \mathbb{C}^{2m\times 2p}$, define the structured singular value of M as (see for example [1])

$$\mu_{\mathbf{\Delta}}(M) := \frac{1}{\inf\left\{ \overline{\sigma}(\Delta) : \Delta \in \mathbf{\Delta}, \det(I - M\Delta) = 0 \right\}}.$$

For the purposes of motivation and illustration, consider the following robust control design problem. With respect to Fig. 1, $P \in \mathscr{R}^{p\times m}$ is the nominal plant model (where $\mathscr{R}^{p\times m}$

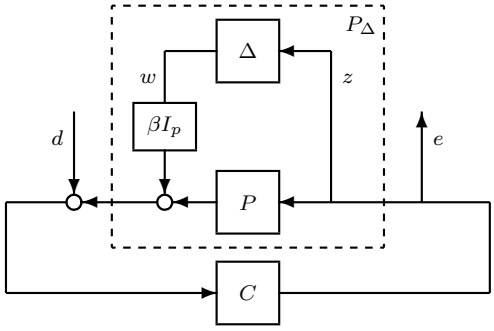

Fig. 1. Additive plant model uncertainty with additive closed-loop performance.

denotes the set of proper real-rational transfer function matrices of dimension $p \times m$), $\Delta \in \mathscr{RL}_\infty^{p\times m}$ is additive plant model uncertainty, $\beta \in \mathscr{R}$ is a single-input single-output transfer function, and $C \in \mathscr{R}^{m\times p}$ is the controller to be designed. The control objective is to minimize the $\mathscr{H}_\infty$-norm of the transfer function from disturbances (d) to errors (e) over all perturbed plant models (P_Δ) that belong to the model set

$$\left\{ P + \beta\Delta : \Delta \in \mathscr{RL}_\infty^{p\times m}, \|\Delta\|_\infty \le 1 \right\}.$$

This controller design problem can be cast as a two-block μ-synthesis problem with plant model uncertainty and closed-loop performance each being captured by an additive uncertainty block. Note that the transfer function from d to e is related to (using, for example, the small gain theorem [1]) robust stability with respect to additive perturbations to P_Δ. From Fig. 1 and after some algebra,

$$\begin{bmatrix} e \\ z \end{bmatrix} = \begin{bmatrix} I_m \\ I_m \end{bmatrix} \mathcal{F}_l\left(\begin{bmatrix} 0_{m\times p} & I_m \\ I_p & P \end{bmatrix}, C \right) \begin{bmatrix} I_p & \beta I_p \end{bmatrix} \begin{bmatrix} d \\ w \end{bmatrix}.$$

Therefore, the μ-synthesis problem that we would like to solve is

$$\inf_C \sup_\omega \mu_{\mathbf{\Delta}}\left(\begin{bmatrix} I_m \\ I_m \end{bmatrix} \mathcal{F}_l\left(\begin{bmatrix} 0_{m\times p} & I_m \\ I_p & P \end{bmatrix}, C \right) \begin{bmatrix} I_p & \beta I_p \end{bmatrix} (j\omega) \right). \quad (1)$$

III. MAIN RESULT

We will now state and prove the main result of this paper that will be used to solve the above and similar μ-synthesis problems.

Lemma 1: *Given* $G = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \in \mathscr{R}^{(m+p)\times(p+m)}$, $C \in \mathscr{R}^{m\times p}$ *and* $\beta \in \mathscr{R}$, *then*

$$\inf_C \sup_\omega \mu_{\mathbf{\Delta}} \left(\begin{bmatrix} I_m \\ I_m \end{bmatrix} \mathcal{F}_l(G,C) \begin{bmatrix} I_p & \beta I_p \end{bmatrix} (j\omega) \right) = \inf_C \|w\, \mathcal{F}_l(G,C)\|_\infty , \quad (2)$$

where $|w(j\omega)| = 1 + |\beta(j\omega)|$ *for all* ω *and with the infimum taken over all internally stabilizing* C.

PROOF: Let $M = \begin{bmatrix} I_m \\ I_m \end{bmatrix} X \begin{bmatrix} I_p & \beta I_p \end{bmatrix}$ where $X = \mathcal{F}_l(G,C)$. Then

$$\begin{aligned}
&\inf_C \sup_\omega \mu_{\mathbf{\Delta}}(M(j\omega)) \\
&= \inf_C \sup_\omega \inf_{d_\omega \in \mathbb{R}_+} \overline{\sigma} \left(\begin{bmatrix} d_\omega I_m & \\ & I_m \end{bmatrix} M(j\omega) \begin{bmatrix} d_\omega^{-1} I_p & \\ & I_p \end{bmatrix} \right) \\
&\quad \text{(from the properties of } \mu \text{ for two full blocks [1])} \\
&= \inf_C \sup_\omega \inf_{d_\omega \in \mathbb{R}_+} \overline{\sigma} \left(\begin{bmatrix} d_\omega I_m \\ I_m \end{bmatrix} X(j\omega) \begin{bmatrix} d_\omega^{-1} I_p & \beta(j\omega) I_p \end{bmatrix} \right) \\
&= \inf_C \sup_\omega \inf_{d_\omega \in \mathbb{R}_+} \overline{\sigma} \left(\sqrt{d_\omega^2 + 1}\, X(j\omega) \sqrt{d_\omega^{-2} + |\beta(j\omega)|^2} \right) \\
&\quad \left(\text{since } \overline{\sigma} \left(\begin{bmatrix} d_\omega I_m \\ I_m \end{bmatrix} Y \right) = \overline{\sigma} \left(\sqrt{d_\omega^2 + 1}\, Y \right) \right) \\
&= \inf_C \sup_\omega \inf_{d_\omega \in \mathbb{R}_+} \sqrt{1 + d_\omega^2 |\beta(j\omega)|^2 + d_\omega^{-2} + |\beta(j\omega)|^2} \\
&\qquad \times \overline{\sigma}(X(j\omega)) \\
&\quad \left(d^2_{\omega,\text{opt}} = \frac{1}{|\beta(j\omega)|} \right) \\
&= \inf_C \sup_\omega \left(1 + |\beta(j\omega)|\right) \overline{\sigma}(X(j\omega)) \\
&= \inf_C \|wX\|_\infty . \qquad \blacksquare
\end{aligned}$$

This lemma presents a solution to a class of two-block μ-synthesis problems by converting them to $\mathscr{H}_\infty$ synthesis problems. Using $\mathscr{H}_\infty$ theory [1], it is now straightforward to parameterize all controllers that solve the μ-synthesis problem stated in (2).

What follows are three remarks on Lemma 1.

Remark 1: *If* $\beta \in \mathbb{R}_+$, *then the right-hand side of* (2) *becomes*

$$(1+\beta) \inf_C \|\mathcal{F}_l(G,C)\|_\infty .$$

Remark 2: *Via a similar technique to the above, we can solve the following larger class of two-block* μ*-synthesis problems*

$$\inf_C \sup_\omega \mu_{\mathbf{\Delta}} \left(\begin{bmatrix} \gamma I_m \\ \delta I_m \end{bmatrix} W_2\, \mathcal{F}_l(G,C)\, W_1 \begin{bmatrix} \alpha I_p & \beta I_p \end{bmatrix} (j\omega) \right), \quad (3)$$

where α, β, γ *and* δ *are any single-input single-output transfer functions, and* W_1 *and* W_2 *are multi-input multi-output transfer functions, as it is easy to find the optimal* d_ω *at each frequency (see proof of Lemma 1) for this more general case. Note that the additional transfer functions enable the designer to trade off robustness and performance.*

Remark 3: *The identity matrices in* (3) *can also be replaced by all-pass transfer functions, though it is unclear how one could use in practice this mathematical possibility.*

Returning to our example, since (1) belongs to the class of problems treated in Lemma 1, the two-block μ-synthesis problem with both blocks corresponding to additive uncertainty can now be solved.

IV. FINAL REMARKS

We will now state three more two-block μ-synthesis problems that can be solved via Lemma 1. Note that for the sake of brevity, no figures are provided, that no weighting functions are considered and that all dimensions are dropped, though of course all these can be easily incorporated. The first is inverse additive plant model uncertainty with inverse additive closed-loop performance:

$$\inf_C \sup_\omega \mu_{\mathbf{\Delta}} \left(\begin{bmatrix} I \\ I \end{bmatrix} \mathcal{F}_l \left(\begin{bmatrix} P & P \\ P & P \end{bmatrix}, C \right) \begin{bmatrix} I & I \end{bmatrix} (j\omega) \right).$$

The second is left coprime factor plant model uncertainty with left coprime factor closed-loop performance:

$$\inf_C \sup_\omega \mu_{\mathbf{\Delta}} \left(\begin{bmatrix} I \\ I \end{bmatrix} \mathcal{F}_l \left(\left[\begin{array}{c|c} \tilde{M}^{-1} & P \\ \hline 0 & I \\ \hline \tilde{M}^{-1} & P \end{array}\right], C \right) \begin{bmatrix} I & I \end{bmatrix} (j\omega) \right),$$

where $\{\tilde{N}, \tilde{M}\}$ is a left coprime factorization of P. And the third is right coprime factor plant model uncertainty with right coprime factor closed-loop performance:

$$\inf_C \sup_\omega \mu_{\mathbf{\Delta}} \left(\begin{bmatrix} I \\ I \end{bmatrix} \mathcal{F}_l \left(\left[\begin{array}{c|c} M^{-1} & 0 \\ \hline P & I \\ \hline M^{-1} & P \end{array}\right], C \right) \begin{bmatrix} I & I \end{bmatrix} (j\omega) \right),$$

where $\{N, M\}$ is a right coprime factorization of P.

Preliminary evidence also suggests that two-block μ-synthesis problems with left (or right) four-block plant model uncertainty and left (or right) four-block closed-loop performance are also solvable. Since four-block uncertainty is related to normalized coprime factor uncertainty and the ν-gap metric [5], and four-block performance is related to $\mathscr{H}_\infty$ loop shaping and the generalized stability margin [1], these two cases are of practical interest.

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