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A modified positive-real type stability condition

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***Abstract*— Lightly damped structures with colocated (position) sensors and (force) actuators can typically be modeled by a (possibly infinite) sum of second order transfer functions. Since the relative degree of these systems is more than unity, they are not directly amenable to standard positive real analysis and since these systems are highly resonant, application of the small gain theorem would be usually very conservative. However, a condition for the internal stability of a positive feedback interconnection of two such systems has been available in the theory of positive position control. This result has important implications on stability robustness due to spill-over dynamics. In this paper, we propose a new stability analysis result which generalizes the positive position control stability result to the feedback interconnection of systems satisfying a certain modified positive-real type condition. Broadly speaking, the result states that a necessary and sufficient condition for the internal stability of the feedback interconnection of such systems is that the DC loop gain is less than unity.**

***Index Terms*— positive position feedback, positive-real systems, bounded-real systems, small gain theorem, passivity, dissipativity.**

I. INTRODUCTION

Positive Position Feedback (PPF) is an approach to the control of flexible structures with co-located actuators and sensors which was introduced by Caughey and his co-authors in [1] and [2]. Positive position feedback was introduced as an alternative to the direct velocity feedback (e.g., see [3]) approach to the control of flexible structures in order to overcome instability problems in direct velocity feedback which may arise as a result of the actuator dynamics; e.g., see [1].

In both direct velocity feedback and positive position feedback control, the flexible structure is modeled by a (possibly infinite) sum of second order transfer functions. Also, in positive position feedback, the controller transfer function from the position measurement to the applied force is also assumed to have the form of a finite sum of second order transfer functions. In the stability results presented in [1], [2], [4], it is shown that in the case of positive position control, the feedback interconnection of the flexible structure plant and the controller is guaranteed to be stable irrespective of the order of the plant model and the values of the plant model resonant frequency and damping parameters provided that an extra condition related to the DC loop gain is satisfied. This means that positive position controllers will have very good robustness properties with respect to spillover dynamics and uncertainties in the plant parameters. These robustness properties were maintained even in the case in which the plant model included the actuator dynamics.

The direct velocity feedback method, which preceded the positive position control method involved the use of feedback control in which the control force is directly proportional to the measured velocity. In the absence of actuator dynamics, this method also has guaranteed robustness properties. Indeed, when there are no actuator dynamics, the plant transfer function from the applied forces to the measured velocities is guaranteed to be positive real. Then, the standard passivity theorem (e.g., see [5]) will guarantee closed loop stability and robustness against a class of spillover dynamics and plant parameter uncertainty.

The main result of this paper is a new passivity type theorem which can be applied directly to prove closed loop stability and robustness against spillover dynamics and plant parameter uncertainty for positive position control. This result introduces new passivity type properties of "rotated positive real" and "rotated strictly positive real" transfer functions. The plant and controller transfer functions arising in positive position control are shown to have these new passivity type properties. Our main result shows that the feedback interconnection of a rotated positive real system and a rotated strictly positive real system will be stable if and only if a certain small gain condition on the DC gains is satisfied. Thus, our new passivity type result enables the robustness properties of positive position control with respect to spillover dynamics and uncertainties in the plant parameters to be established. The results in this paper are derived for Multiple-Input Multiple-Output (MIMO) Linear Time-Invariant (LTI) systems, and so in this sense, also extend the Single-Input Single-Output (SISO) work of [1] and [2]. It is worth also noting that a closely connected idea on counterclockwise input-output dynamics for SISO nonlinear systems has also been recently derived in [6]. This idea captures some of the SISO results of [1] and [2] when restricted to the same setting.

An important feature of our new result is that the class of controllers which have the rotated positive real property is larger than the class of transfer functions considered in positive position control which are only sums of second order systems. Thus, we are able to conclude that a larger class of controller transfer functions will lead to, at least, the same stability and robustness properties of positive position control. This means that there is scope for greater flexibility in the controller design which can be used to achieve improved performance of the control system as well as improved

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robustness in terms of standard robustness measures such as gain and phase margins.

One drawback of positive position control is that there is not really a systematic procedure for designing the controller transfer functions. In [1], [2], graphical procedures were used to construct the controller transfer functions to damp specified modes in the structure. Also, in [4], nonlinear non-convex optimization techniques were used to construct the controller parameters to minimize a performance index, and so no guarantees could be given regarding convergence to a global solution. Only the controller feasibility set was shown to be convex in that paper. Although, controller synthesis is beyond the scope of this paper, it is hoped that because of the system theoretic nature of our rotated positive real condition, it may be possible to obtain systematic controller synthesis techniques by extending methods from, for example, H^∞ control theory.

The remainder of the paper proceeds as follows: In Section II, we show that the class of systems considered in this paper captures and goes beyond PPF systems. In Section III, we provide two technical lemmas that underpin the main theorem that is presented in Section IV. This main result is a new modified positive-real type theorem applicable to systems such as the positive position feedback control of flexible structures. In Section V, we present an illustrative example in which we show that our result predicts stability of the feedback loop. Finally, Section VI outlines two important directions for future extensions of this research.

II. PRELIMINARY OBSERVATIONS

In this section, we provide three very simple remarks that show the connection of this work to previous PPF analysis methods. In particular, we show that PPF systems are a strict subset of the class of systems considered in this paper.

The first two remarks trivially show that SISO and MIMO PPF systems belong to the class of systems considered in this paper.

Remark 1: *The scalar system $R : u \to y$ defined via*

$$\ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2 y = ku \tag{1}$$

with $\omega_n, \zeta, k > 0$ has transfer function $R(s)$ which satisfies $R(s) \in \mathscr{RH}_\infty$, $R(\infty) = 0$ and $j[R(j\omega) - R(j\omega)^*] > 0\ \forall\omega \in (0, \infty)$. Here $(\cdot)^*$ denotes the operation of complex conjugate (or complex conjugate transpose in the case of matrices).

Proof: The stability and strict properness of $R(s)$ are trivial to show. The frequency domain condition, which is equivalent in the scalar case to $\Im(R(j\omega)) < 0\ \forall\omega \in (0, \infty)$, easily follows from a Nyquist plot of $R(s)$. Here $\Im(R(j\omega))$ denotes the imaginary part of the complex number $R(j\omega)$. ■

Remark 2: *Given strictly positive diagonal matrices $D, \Omega \in \mathbb{R}^{m\times m}$ and a matrix $K \in \mathbb{R}^{m\times n}$ with $n, m \in \mathbb{Z}_{>0}$. The n-dimensional[1] multivariable system $R : u \to y$ defined via*

$$y = K^* x \quad and \quad \ddot{x} + D\dot{x} + \Omega x = Ku \tag{2}$$

has transfer function matrix $R(s)$ which satisfies $R(s) \in \mathscr{RH}_\infty^{n\times n}$, $R(\infty) = 0$ and $j[R(j\omega) - R(j\omega)^*] \geq 0\ \forall\omega \in (0, \infty)$.

Also if K has full column rank (only possible when $m \geq n$), then $j[R(j\omega) - R(j\omega)^*] > 0\ \forall\omega \in (0, \infty)$.

Proof: The stability and strict properness of $R(s)$ are trivial to show. The frequency domain condition follows on noting that $R(s) = K^*[s^2 I + Ds + \Omega]^{-1}K$ and since the central s-dependent object is completely diagonal, we can invoke the scalar result of Remark 1 on each diagonal element to complete the proof. ■

The next remark shows that the class of systems considered in this paper captures more than just PPF systems.

Remark 3: *Remarks 1 and 2 state that systems of the form* (1) *and* (2) *are contained in the class*

$$\mathscr{C} := \big\{R(s) \in \mathscr{RH}_\infty^{n\times n} : R(\infty) \geq 0,\ j[R(j\omega) - R(j\omega)^*] \geq 0\ \forall\omega \in (0, \infty)\big\}.$$

However, there exists $R(s) \in \mathscr{C}$ that cannot be expressed in form (1) *or* (2).

Proof: As two examples, note that the scalar systems

$$R(s) = \frac{k}{s^2 + 2\zeta\omega_n s + \omega_n^2} \cdot \frac{s + a}{s + b}$$

and

$$R(s) = \frac{k(s + a)}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

cannot be expressed in form (1) (due to a different McMillan degree and a different relative degree respectively) but can belong to $\mathscr{C}$ for appropriate choices of constants a, b, ζ, ω_n. ■

III. SOME TECHNICAL RESULTS

In this section, we generate the technical machinery that will enable us to prove the main result in the next section. The first lemma given below is well known and given here only for completeness.

Lemma 1: *Given $A \in \mathbb{C}^{n\times n}$ with $A + A^* \geq 0$ and $B \in \mathbb{C}^{n\times n}$ with $B + B^* > 0$. Then,*

$$\det(I + AB) \neq 0.$$

Proof: First note that $B + B^* > 0$ implies B is nonsingular. Therefore,

$$\begin{aligned} & B + B^* > 0 \\ \Leftrightarrow\quad & B^{-1} + B^{-*} > 0. \end{aligned}$$

Now,

$$\begin{aligned} & A + A^* \geq 0 \text{ and } B^{-1} + B^{-*} > 0 \\ \Rightarrow\quad & (A + B^{-1}) + (A + B^{-1})^* > 0 \\ \Rightarrow\quad & (A + B^{-1}) \text{ is nonsingular.} \end{aligned}$$

[1] $u, y \in \mathbb{R}^n$ and $x \in \mathbb{R}^m$

Then, $\det(I + AB) = \det(A + B^{-1})\det(B) \neq 0$. ■

The following lemma underpins the main result in the next section and will be used in its proof. It is presented here in order to simplify the flow of ideas.

Lemma 2: *Given* $R(s) \in \mathscr{RH}_\infty^{n\times n}$ *with* $j[R(j\omega) - R(j\omega)^*] \geq 0$ (*resp.* > 0) $\forall \omega \in (0,\infty)$. *Then,*

$$R(\infty) = R(\infty)^* \text{ and } R(0) - R(\infty) \geq 0 \text{ (resp. } > 0).$$

Proof: First note that

$$\begin{aligned}
& j[R(j\omega) - R(j\omega)^*] \geq 0 \text{ (or } > 0) \quad \forall \omega \in (0,\infty) \\
\Rightarrow \quad & j[R(\infty) - R(\infty)^*] \geq 0 \qquad (3)\\
\Leftrightarrow \quad & \left(j[R(\infty) - R(\infty)^*]\right)^T \geq 0 \\
\Leftrightarrow \quad & j[R(\infty)^* - R(\infty)] \geq 0 \text{ (as } R(\infty) \in \mathbb{R}^{n\times n}) \\
\Leftrightarrow \quad & j[R(\infty) - R(\infty)^*] \leq 0. \qquad (4)
\end{aligned}$$

Inequalities (3) and (4) together imply that $R(\infty) = R(\infty)^*$.

Second, let $R(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ be a minimal realization with $D = D^*$ and let $\hat{R}(s) = R(s) - D$. It is evident that $R(s) \in \mathscr{RH}_\infty^{n\times n} \Leftrightarrow \hat{R}(s) \in \mathscr{RH}_\infty^{n\times n}$ and $j[R(j\omega) - R(j\omega)^*] = j[\hat{R}(j\omega) - \hat{R}(j\omega)^*]$. The problem hence reduces to showing that given $\hat{R}(s) \in \mathscr{RH}_\infty^{n\times n}$ with $j[\hat{R}(j\omega) - \hat{R}(j\omega)^*] \geq 0$ (resp. > 0) $\forall\omega \in (0,\infty)$, we get $\hat{R}(0) \geq 0$ (resp. > 0).

Since $\hat{R}(s) = \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right]$ is a minimal realization, $\hat{R}(s) \in \mathscr{RH}_\infty^{n\times n} \Leftrightarrow A$ is Hurwitz. Now define

$$\begin{aligned}
F(s) &= s\hat{R}(s) \\
&= C(I - A/s)^{-1}B \\
&= C[I + A/s(I - A/s)^{-1}]B \\
&= \left[\begin{array}{c|c} A & B \\ \hline CA & CB \end{array}\right].
\end{aligned}$$

It can be easily shown that this realization is also minimal, as observability of the above realization follows on noticing that (C, A) is observable and 0 is not in the spectrum of A because A is Hurwitz.

Also,

$$\begin{aligned}
& j[\hat{R}(j\omega) - \hat{R}(j\omega)^*] \geq 0 \quad \forall\omega \in (0,\infty) \\
\Leftrightarrow \quad & [j\omega\hat{R}(j\omega)] + [j\omega\hat{R}(j\omega)]^* \geq 0 \quad \forall\omega \in (0,\infty) \\
\Leftrightarrow \quad & F(j\omega) + F(j\omega)^* \geq 0 \quad \forall\omega \in \mathbb{R}.
\end{aligned}$$

Consequently, under the assumption that $R(s) \in \mathscr{RH}_\infty^{n\times n}$ has a minimal realization $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ with $D = D^*$, we have shown that $F(s) = \left[\begin{array}{c|c} A & B \\ \hline CA & CB \end{array}\right]$ is a minimal realization with A Hurwitz (therefore $F(s) \in \mathscr{RH}_\infty^{n\times n}$) and $F(j\omega) + F(j\omega)^* \geq 0 \ \ \forall\omega \in \mathbb{R}$. We then also have following chain of equivalent reformulations (under the same assumption):

(a) $F(s) = \left[\begin{array}{c|c} A & B \\ \hline CA & CB \end{array}\right] \in \mathscr{RH}_\infty^{n\times n}$ and $F(j\omega) + F(j\omega)^* \geq 0 \ \forall\omega \in \mathbb{R}$

(b) $\exists X \geq 0, L, W$ such that

$$\begin{aligned}
XA + A^*X &= -L^*L, \\
B^*X + W^*L &= CA, \\
CB + (CB)^* &= W^*W.
\end{aligned}$$

This equivalence is via the Positive Real Lemma (see for example [7, Theorems 13.25,13.26]).

(c) $\exists X > 0, L, W$ such that

$$\begin{aligned}
XA + A^*X &= -L^*L, \\
B^*X + W^*L &= CA, \\
CB + (CB)^* &= W^*W.
\end{aligned}$$

This equivalence is due to the fact that A is Hurwitz and (L^*L, A) is observable (which can be seen via the above three equations).

(d) $\exists X > 0, L, W$ such that

$$\begin{aligned}
& XA + A^*X = -L^*L, \\
& B = X^{-1}(A^*C^* - L^*W), \\
& CX^{-1}A^*C^* + CAX^{-1}C^* \\
& \qquad = W^*W + CX^{-1}L^*W + W^*LX^{-1}C^*.
\end{aligned}$$

(e) $\exists X > 0, L, W$ such that

$$\begin{aligned}
& XA + A^*X = -L^*L, \\
& B = X^{-1}(A^*C^* - L^*W), \\
& (W + LX^{-1}C^*)^*(W + LX^{-1}C^*) = 0.
\end{aligned}$$

(f) $\exists X > 0, L, W$ such that

$$\begin{aligned}
& XA + A^*X = -L^*L, \\
& B = X^{-1}(A^*C^* - L^*W), \\
& W = -LX^{-1}C^*.
\end{aligned}$$

(g) $\exists X > 0, L$ such that

$$\begin{aligned}
& XA + A^*X = -L^*L, \\
& B = X^{-1}(A^*C^* + L^*LX^{-1}C^*).
\end{aligned}$$

(h) $\exists X > 0, L$ such that

$$\begin{aligned}
& XA + A^*X = -L^*L, \\
& B = -AX^{-1}C^*.
\end{aligned}$$

Thus,

$$\begin{aligned}
\hat{R}(0) &= -CA^{-1}B \\
&= CA^{-1}AX^{-1}C^* \\
&= CX^{-1}C^* \geq 0 \qquad (5)
\end{aligned}$$

which concludes the proof for the non-strict inequalities.

Now, we focus on the strict inequalities conditions and suppose there exists an $x \in \mathbb{R}^{n\times n}$ such that $\hat{R}(0)x = 0$. Then, it follows that $CX^{-1}C^*x = 0$ which implies that $C^*x = 0$ as $X > 0$. This then also gives that $Bx = 0$ via $B = -AX^{-1}C^*$ which yields

$$\hat{R}(j\omega)x = C(j\omega I - A)^{-1}Bx = 0 \quad \forall\omega \in \mathbb{R}.$$

But $j[\hat{R}(j\omega) - \hat{R}(j\omega)^*] > 0 \ \forall \omega \in (0, \infty)$ implies that $\hat{R}(j\omega)$ is nonsingular for all $\omega \in (0, \infty)$ and hence the only possible $x \in \mathbb{R}^{n\times n}$ such that $\hat{R}(0)x = 0$ is $x = 0$. This shows that $\hat{R}(0)$ is also nonsingular and thus $\hat{R}(0) > 0$. This concludes the proof. ■

IV. THE MAIN RESULT

The key result in this paper is Theorem 3 below. It is an analysis theorem that states that provided one system belongs to class $\mathscr{C}_1$, defined by

$$\mathscr{C}_1 := \{R(s) \in \mathscr{RH}_\infty^{n\times n} : j[R(j\omega) - R(j\omega)^*] \geq 0 \ \forall \omega \in (0, \infty)\}, \quad (6)$$

and the other system belongs to class $\mathscr{C}_2$, defined by

$$\mathscr{C}_2 := \{R(s) \in \mathscr{RH}_\infty^{n\times n} : R(\infty) \geq 0, j[R(j\omega) - R(j\omega)^*] > 0 \ \forall \omega \in (0, \infty)\}, \quad (7)$$

then a necessary and sufficient condition[2] for internal stability of a positive feedback interconnection of these two systems is to check that the DC loop gain is less than unity.

Theorem 3: *Given $P(s) \in \mathscr{C}_1$ and $Q(s) \in \mathscr{C}_2$ that also satisfy $P(\infty)Q(\infty) = 0$. Then,*

$$[P(s), Q(s)] \text{ is internally stable} \ \Leftrightarrow \ \overline{\lambda}(P(0)Q(0)) < 1.$$

Here the notation $[P(s), Q(s)]$ refers to the positive feedback interconnection between the systems $P(s)$ and $Q(s)$, and $\overline{\lambda}(\cdot)$ refers to the maximum eigenvalue of a matrix $(\cdot)$ which has only real eigenvalues.

Proof: [*Necessity*] Let $P(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ be a minimal realization. Then

$$P(s) \in \mathscr{RH}_\infty \ \Leftrightarrow \ A \text{ is Hurwitz} \ \Rightarrow \ A \text{ is nonsingular}.$$

Let $Q(s) = \left[\begin{array}{c|c} \bar{A} & \bar{B} \\ \hline \bar{C} & \bar{D} \end{array}\right]$ be a minimal realization. Then

$$Q(s) \in \mathscr{RH}_\infty \ \Leftrightarrow \ \bar{A} \text{ is Hurwitz} \ \Rightarrow \ \bar{A} \text{ is nonsingular}.$$

Now,

$$\begin{aligned}(I - P(s)Q(s))^{-1} &= \left[\begin{array}{cc|c} A & B\bar{C} & B\bar{D} \\ 0 & \bar{A} & \bar{B} \\ \hline -C & -D\bar{C} & I \end{array}\right]^{-1} \\ &= \left[\begin{array}{c|c} \begin{pmatrix} A & B\bar{C} \\ 0 & \bar{A}\end{pmatrix} + \begin{pmatrix} B\bar{D} \\ \bar{B}\end{pmatrix}\begin{pmatrix} C & D\bar{C}\end{pmatrix} & \begin{matrix} B\bar{D} \\ \bar{B}\end{matrix} \\ \hline \begin{matrix} C & \quad D\bar{C}\end{matrix} & I \end{array}\right].\end{aligned}$$

The last realization is stabilizable and detectable as its inverse realization is clearly stabilizable and detectable. Consequently, this realization has no unstable hidden modes. Therefore, $[P(s), Q(s)]$ is internally stable if and only if $(I - P(s)Q(s))^{-1} \in \mathscr{RH}_\infty$ which holds if and only if $\mathcal{A} := \begin{bmatrix} A & B\bar{C} \\ 0 & \bar{A}\end{bmatrix} + \begin{bmatrix} B\bar{D} \\ \bar{B}\end{bmatrix}\begin{bmatrix} C & D\bar{C}\end{bmatrix}$ is Hurwitz.

Also, via an argument similar to the proof in Lemma 2 (repeated on $P(s)$ and $Q(s)$), we have:

$$\begin{aligned}&(\text{for } P(s)) \quad \exists X > 0 : \ XA + A^*X \leq 0, \\ &\qquad\qquad\qquad\qquad\quad B = -AX^{-1}C^*; \\ &(\text{for } Q(s)) \quad \exists \bar{X} > 0 : \ \bar{X}\bar{A} + \bar{A}^*\bar{X} \leq 0, \\ &\qquad\qquad\qquad\qquad\quad \bar{B} = -\bar{A}\bar{X}^{-1}\bar{C}^*.\end{aligned}$$

Using these relations, we have

$$\begin{aligned}\mathcal{A} &= \begin{bmatrix} A & B\bar{C} \\ 0 & \bar{A}\end{bmatrix} + \begin{bmatrix} B\bar{D} \\ \bar{B}\end{bmatrix}\begin{bmatrix} C & D\bar{C}\end{bmatrix} \\ &= \begin{bmatrix} A & -AX^{-1}C^*\bar{C} \\ 0 & \bar{A}\end{bmatrix} - \begin{bmatrix} AX^{-1}C^*\bar{D} \\ \bar{A}\bar{X}^{-1}\bar{C}^*\end{bmatrix}\begin{bmatrix} C & D\bar{C}\end{bmatrix} \\ &= \begin{bmatrix} AX^{-1} & 0 \\ 0 & \bar{A}\bar{X}^{-1}\end{bmatrix}\begin{bmatrix} X - C^*\bar{D}C & -C^*\bar{C} \\ -\bar{C}^*C & \bar{X} - \bar{C}^*D\bar{C}\end{bmatrix} \quad (8)\end{aligned}$$

where the last equality follows after some algebra on noting that $\bar{D}D = 0$ because $D\bar{D} = 0$ and $D = D^*$ (via Lemma 2), $\bar{D} = \bar{D}^*$.

Since $\mathcal{A}$ is Hurwitz (as we are proving necessity, i.e. $[P(s), Q(s)]$ is assumed to be internally stable) and $\begin{bmatrix} AX^{-1} & 0 \\ 0 & \bar{A}\bar{X}^{-1}\end{bmatrix}$ is nonsingular, it follows from (8) that

$$T := \begin{bmatrix} X - C^*\bar{D}C & -C^*\bar{C} \\ -\bar{C}^*C & \bar{X} - \bar{C}^*D\bar{C}\end{bmatrix}$$

is nonsingular. Now,

$$\begin{aligned}& XA + A^*X \leq 0 \quad \text{and} \quad \bar{X}\bar{A} + \bar{A}^*\bar{X} \leq 0 \\ \Leftrightarrow \ & \begin{bmatrix} AX^{-1} & 0 \\ 0 & \bar{A}\bar{X}^{-1}\end{bmatrix} + \begin{bmatrix} X^{-1}A^* & 0 \\ 0 & \bar{X}^{-1}\bar{A}^*\end{bmatrix} \leq 0 \\ \Leftrightarrow \ & T\begin{bmatrix} AX^{-1} & 0 \\ 0 & \bar{A}\bar{X}^{-1}\end{bmatrix}T + T\begin{bmatrix} X^{-1}A^* & 0 \\ 0 & \bar{X}^{-1}\bar{A}^*\end{bmatrix}T \leq 0 \\ \Leftrightarrow \ & T\mathcal{A} + \mathcal{A}^*T \leq 0\end{aligned}$$

Therefore $T \geq 0$ because $\mathcal{A}$ is Hurwitz. But T is also nonsingular and hence $T > 0$. Hence,

$$T = \begin{bmatrix} X - C^*\bar{D}C & -C^*\bar{C} \\ -\bar{C}^*C & \bar{X} - \bar{C}^*D\bar{C}\end{bmatrix} > 0$$

$$\begin{aligned}\Leftrightarrow \ & \bar{X} - \bar{C}^*D\bar{C} > 0 \quad \text{and} \\ & (X - C^*\bar{D}C) - C^*\bar{C}(\bar{X} - \bar{C}^*D\bar{C})^{-1}\bar{C}^*C > 0 \\ \Leftrightarrow \ & \overline{\lambda}\left[\bar{X}^{-\frac{1}{2}}\bar{C}^*D\bar{C}\bar{X}^{-\frac{1}{2}}\right] < 1 \quad \text{and} \quad X - C^*\bar{D}C \\ & \quad - C^*(I - \bar{C}\bar{X}^{-1}\bar{C}^*D)^{-1}\bar{C}\bar{X}^{-1}\bar{C}^*C > 0 \\ \Leftrightarrow \ & \overline{\lambda}\left[D\bar{C}\bar{X}^{-1}\bar{C}^*\right] < 1 \quad \text{and} \quad X - C^*\bar{D}C \\ & \quad - C^*(I - Q(0)D)^{-1}(Q(0) - \bar{D})C > 0 \\ & [\text{as } Q(0) - \bar{D} = \bar{C}\bar{X}^{-1}\bar{C}^* \text{ via (5) and } \bar{D}D = 0]\end{aligned}$$

[2]Under the assumption that the loop gain at infinite frequency is zero.

$$\begin{aligned}
\Leftrightarrow\quad & \overline{\lambda}[DQ(0)] < 1 \quad \text{and} \\
& X - C^*(I - Q(0)D)^{-1}[\bar{D} + (Q(0) - \bar{D})]C > 0 \\
& [\text{as } Q(0) - \bar{D} = \bar{C}\bar{X}^{-1}\bar{C}^* \text{ via (5) and } D\bar{D} = 0] \\
\Leftrightarrow\quad & Q(0)^{-1} - D > 0 \quad \text{and} \\
& X - C^*(Q(0)^{-1} - D)^{-1}C > 0 \\
& [\text{as } Q(0) > \bar{D} \text{ via Lemma 2 and } \bar{D} \geq 0] \\
\Leftrightarrow\quad & \begin{bmatrix} X & C^* \\ C & Q(0)^{-1} - D \end{bmatrix} > 0 \\
\Leftrightarrow\quad & Q(0)^{-1} - D - CX^{-1}C^* > 0 \\
\Leftrightarrow\quad & Q(0)^{-1} - P(0) > 0 \\
& [\text{as } P(0) = D + CX^{-1}C^* \text{ via (5)}] \\
\Leftrightarrow\quad & \overline{\lambda}(P(0)Q(0)) < 1.
\end{aligned}$$

[*Sufficiency*] Since $P(0) = P(0)^*$ and $Q(0) > Q(\infty) \geq 0$ via Lemma 2 and the suppositions, it follows that all the eigenvalues of $P(0)Q(0)$ belong to $\mathbb{R}$. Then,

$$\begin{aligned}
& \overline{\lambda}(P(0)Q(0)) < 1 \\
\Leftrightarrow\quad & \det(\lambda I - P(0)Q(0)) \neq 0 \quad \forall \lambda \in [1, \infty) \\
\Leftrightarrow\quad & \det(I - \alpha P(0)Q(0)) \neq 0 \quad \forall \alpha \in [0, 1] \\
\Leftrightarrow\quad & \det(I - \alpha P(j\omega)Q(j\omega)) \neq 0 \quad \forall \omega \in \mathbb{R} \cup \{\infty\}, \\
& \qquad \forall \alpha \in [0, 1]. \qquad (9)
\end{aligned}$$

The last equivalence is due to the following three facts:

(a) It is easy to see that it is true at $\omega = 0$ (since the conditions are identical) and $\omega = \infty$ (since $P(\infty)Q(\infty) = 0$);

(b) For the frequency region $\omega \in (0, \infty)$,

$$\begin{aligned}
& \det(I - \alpha P(j\omega)Q(j\omega)) \\
& \qquad = \det(I + [\alpha j P(j\omega)][jQ(j\omega)])
\end{aligned}$$

where $[jQ(j\omega)] + [jQ(j\omega)]^* > 0 \ \forall \omega \in (0, \infty)$ and $[\alpha j P(j\omega)] + [\alpha j P(j\omega)]^* \geq 0 \ \forall \alpha \in [0, 1], \omega \in (0, \infty)$. Then, invoking Lemma 1, we see that $\det(I - \alpha P(j\omega)Q(j\omega)) \neq 0 \ \forall \alpha \in [0, 1], \omega \in (0, \infty)$.

(c) For the frequency region $\omega \in (-\infty, 0)$,

$$\begin{aligned}
& \det(I - \alpha P(j\omega)Q(j\omega)) \neq 0 \\
& \quad \Leftrightarrow \quad \overline{\det(I - \alpha P(j\omega)Q(j\omega))} \neq 0 \\
& \quad \Leftrightarrow \quad \det(I - \alpha P(-j\omega)Q(-j\omega)) \neq 0
\end{aligned}$$

which then follows as in (b).

Now, since $P(s), Q(s) \in \mathscr{RH}_\infty$, there exists $\gamma > 0$ such that $\|PQ\|_\infty < \gamma$. Consequently,

$$\det(I - \alpha P(s)Q(s)) \neq 0 \quad \forall \alpha \in [0, 1/\gamma], s \in \overline{\mathbb{C}}_+.$$

This condition states that all the transmission zeros of $(I - \alpha P(s)Q(s))$ are in $\mathbb{C}_-$ for all $\alpha \in [0, 1/\gamma]$.

The situation $1/\gamma \geq 1$ results in a trivial completion of the proof as in this case we easily have $\det(I - P(s)Q(s)) \neq 0 \ \forall s \in \overline{\mathbb{C}}_+$, which is of course equivalent to $(I - P(s)Q(s))^{-1} \in \mathscr{RH}_\infty$ or $[P(s), Q(s)]$ is internally stable.

The situation $1/\gamma < 1$ is hence the more interesting situation. In this case, as α varies continuously, the transmission zeros of $(I - \alpha P(s)Q(s))$ vary continuously in $\mathbb{C}$. We know that for all $\alpha \in [0, 1/\gamma]$, all transmission zeros are in $\mathbb{C}_-$ and from condition (9) (which is true because we are proving sufficiency) we also know that these transmission zeros can never cross the imaginary axis for all $\alpha \in [0, 1]$. Therefore, the transmission zeros of $(I - \alpha P(s)Q(s))$ must remain in $\mathbb{C}_-$ for all $\alpha \in [0, 1]$, which gives $(I - P(s)Q(s))^{-1} \in \mathscr{RH}_\infty$ or equivalently $[P(s), Q(s)]$ is internally stable. ■

It is worth pointing out that the condition $R(s) \in \mathscr{C}_1$ (and similarly $\mathscr{C}_2$) can be checked using the same techniques adopted in the computation of an $\mathscr{H}_\infty$-norm (via the application of a bilinear transformation), or using directly a complex version of the positive-real lemma.

V. ILLUSTRATIVE EXAMPLE

In this section, we present a simple illustrative example which demonstrates the usefulness of our analysis result compared to existing analysis results such as the small gain theorem and the passivity theorem.

We consider a plant with transfer function

$$\begin{aligned}
P(s) &= \frac{10}{(s + 0.1)^2} \, \frac{s^2 + 0.1s + 1}{s^2 + 0.02s + 2} \\
&= \frac{0.052s + 5.016}{s^2 + 0.02s + 2} - \frac{0.52}{s + 0.1} + \frac{4.98}{(s + 0.1)^2}. \qquad (10)
\end{aligned}$$

It is straightforward to verify that this system is not passive and also is not of the form required in Positive Position Control. Also, the infinity norm of this system is 500 attained at zero frequency. Consequently for the application of the small gain theorem, the Nyquist plot of the controller would need to be contained in a ball of radius smaller than $\frac{1}{500}$. We will show that our analysis result can be applied to this system and reduces the conservatism of the small gain theorem.

The Bode and Nyquist plots of this plant transfer function are shown in Figures 1 and 2. Note that $P(s) \in \mathscr{RH}_\infty$, $P(\infty) = 0$. Also, it can be seen from the plots that $j[P(j\omega) - P(j\omega)^*] = -2\Im(P(j\omega)) > 0 \ \forall \omega \in (0, \infty)$. Hence $P(s) \in \mathscr{C}_1$, with $\mathscr{C}_1$ defined in (6).

We now restrict attention to controllers that belong to the class $\mathscr{C}_2$, defined in (7). Our main result, Theorem 3, says that a necessary and sufficient condition for internal stability of the closed loop system is then $C(0) < \frac{1}{P(0)} = \frac{1}{500}$. Thus, in our case only the DC gain is restricted to be less than $\frac{1}{500}$, unlike the small gain theorem which would require that the controller gain at all frequencies is less than $\frac{1}{500}$.

For example, the simple second-order controller

$$C(s) = \frac{0.001\sqrt{2}}{s^2 + 0.003s + \sqrt{2}}$$

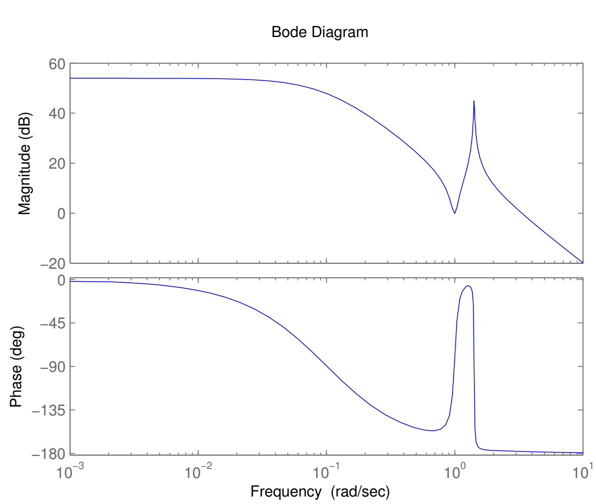

Fig. 1. Plant Bode Plot

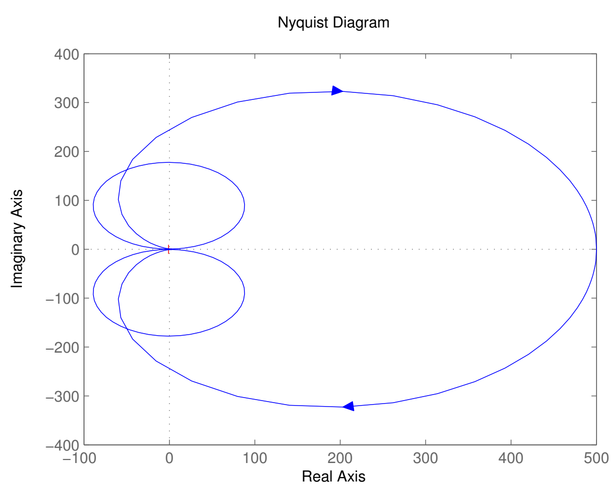

Fig. 2. Plant Nyquist Plot

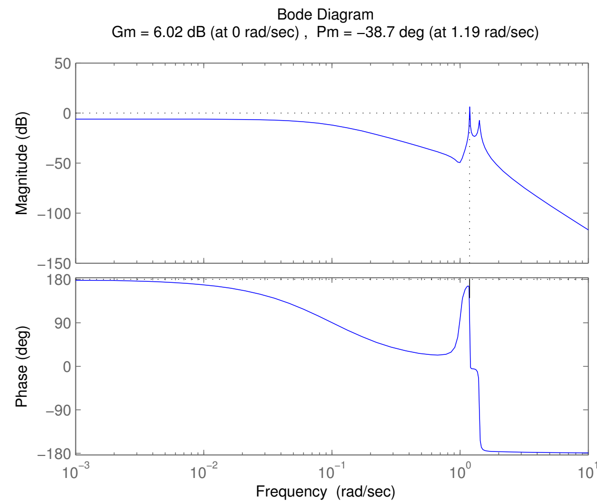

Fig. 3. Bode plot of $-P(j\omega)C(j\omega)$

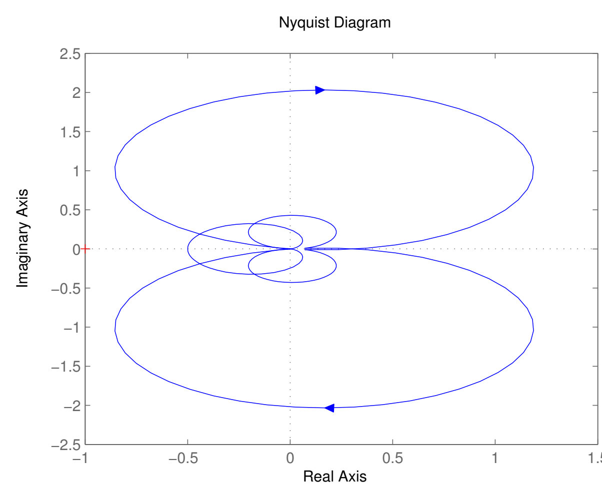

Fig. 4. Nyquist plot of $-P(j\omega)C(j\omega)$

clearly belongs to the class $\mathscr{C}_2$, has DC gain $C(0) = 0.001 < \frac{1}{500}$ and has infinity norm $\|C(s)\|_\infty = 0.3964 \gg \frac{1}{500}$. The Bode and Nyquist plots of the resulting loop gain frequency response function $-P(j\omega)C(j\omega)$, where the negative sign is introduced as a result of the positive feedback interconnection, are shown in Figures 3 and 4. The Nyquist plot in Figure 4 confirms that the closed-loop feedback interconnection will be stable as there are no clockwise encirclements of the $-1 + j0$ point, as predicted by Theorem 3.

It is also worth pointing out here that any additive perturbation $\Delta(s) \in \mathscr{C}_1$ to plant $P(s)$ (for example due to spill-over or unmodeled dynamics) is guaranteed to be robustly stabilized by the same controller $C(s)$ (via Theorem 3), provided the DC gain condition continues to be satisfied. This is because the perturbed plant $(P(s) + \Delta(s))$ would remain within the class $\mathscr{C}_1$.

VI. FUTURE DIRECTIONS

The next logical steps to further extend the applicabilty of this research are: (a) to devise a controller synthesis procedure that generates controllers that belong to either class $\mathscr{C}_1$ or $\mathscr{C}_2$; and (b) to generalize the analysis result given in this article to allow one (or possibly both) systems to be MIMO nonlinear and/or time-varying.

The authors have commenced exploring these ideas and preliminary calculations show that this will be possible. Readers interested in collaborating on these topics are invited to contact the authors.

REFERENCES

[1] C. Goh and T. Caughey, “On the stability problem caused by finite actuator dynamics in the collocated control of large space structures,” *International Journal of Control*, vol. 41, no. 3, pp. 787–802, 1985.

[2] J. Fanson and T. Caughey, “Positive position feedback control for large space structures,” *AIAA Journal*, vol. 28, no. 4, pp. 717–724, 1990.

[3] M. Balas, "Direct velocity feedback control of large space structures," *Journal of Guidance and Control*, vol. 2, no. 3, pp. 253–253, 1979.
[4] S. Moheimani, B. Vautier, and B. Bhikkaji, "Experimental implementation of extended multivariable PPF control of an active structure," *IEEE Transactions on Control Systems Technology*, vol. 14, no. 3, pp. 433–455, 2006.
[5] J. C. Willems, *The Analysis of Feedback Systems*. Cambridge, MA: MIT Press, 1971.
[6] D. Angeli, "Systems with counterclockwise input-output dynamics," *IEEE Transactions on Automatic Control*, vol. 51, no. 7, pp. 1130–1143, 2006.
[7] K. Zhou, J. Doyle, and K. Glover, *Robust and Optimal Control*. Upper Saddle River, NJ: Prentice-Hall, 1996.