

A Systematic Weight Synthesis Procedure for Performance Optimisation in \mathcal{H}_∞ Loop Shaping

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Abstract—This paper proposes a systematic method to synthesise \mathcal{H}_∞ loop shaping weights that would optimise performance while maintaining a pre-specified level of optimal robust stability margin. This objective is captured by two optimisation problems expressed in the form of Linear Matrix Inequalities (LMI), which are solved by means of an iterative algorithm. This scheme will aid the designer to gauge the achievable performance of a plant for a specified level of robustness and to adjust pre-designed weights to further enhance performance.

Index Terms— \mathcal{H}_∞ loop-shaping, weight synthesis, performance optimisation, robust stability margin, robust control.

I. INTRODUCTION

McFarlane and Glover in [3] proposed a loop shaping design procedure, which has been proven effective in many applications [6]. In this paper we aim to form a systematic method of optimising \mathcal{H}_∞ loop-shaping weights, which is part of the said procedure, building on the work by Lanzon [1][2]. First, we give a brief account of loop-shaping.

It is possible to translate closed loop performance requirements, which may be in time domain or frequency domain, to requirements on the (open) loop-gain, $L(j\omega)$ [3]. The goal of loop-shaping design is to design a controller C such that $L = PC$ meets the translated requirements.

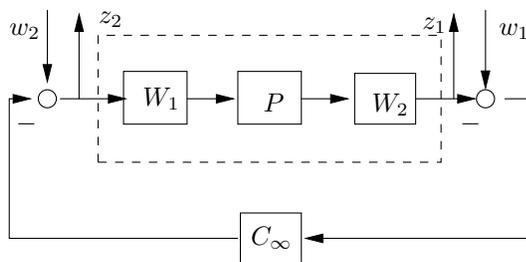


Fig. 1. Typical \mathcal{H}_∞ loop-shaping framework

The challenges involving controller design are twofold:

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a) *Achieve the performance specifications imposed on the open loop gain:* In \mathcal{H}_∞ loop shaping, analogous to classical loop-shaping, the performance specifications generally require that the loop gain is large in low frequencies, roll off not too steeply at crossover and roll off at a higher rate in high frequencies [6]. However, in \mathcal{H}_∞ loop shaping, unlike classical loop shaping, it is not necessary to shape the phase of P . The above shaping is done by designing weights W_1, W_2 such that the gain of the shaped plant $P_s = W_2PW_1$ has the aforementioned characteristics.

b) *Robust stability:* Subsequent to the design of weights, the loop has to be closed using a robust internally stabilising controller C_∞ . (See Figure 1.) C_∞ should ensure that this interconnection with the open loop gain $L = W_2PW_1C_\infty$ is robustly stable and that $W_2PW_1C_\infty$ does not deviate significantly from $P_s = W_2PW_1$ to preserve the specified performance. [3] showed that the extent that this is met by P_s and C_∞ is captured by the index *robust stability margin*, $b(P_s, C_\infty)$, introduced in [4]. The optimal value of $b(P_s, C_\infty)$ over the set of internally stabilising controllers is called $b_{opt}(P_s)$ and [4] also characterised all such controllers that achieve a sub-optimal robust stability margin.

At this point, if $b_{opt}(P_s)$ is not satisfactory the weights have to be redesigned, and this may involve several iterations before both challenges are met simultaneously. Once the requirements are met, the controller C , that will be implemented on the physical plant, is obtained by pulling over the weights to construct $C = W_1C_\infty W_2$. A more detailed treatment of this procedure is found in, for example, [6].

A. Notation

The same notation used in [1][2] is used in this paper and repeated here for convenience.

Let the feedback interconnection of P_s and C_∞ shown in Figure 1 be denoted by $[P_s, C_\infty]$. This interconnection is said to be internally stable if it is well-posed and each of the four transfer functions mapping disturbances to outputs

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} P_s \\ I \end{bmatrix} (I - C_\infty P_s)^{-1} \begin{bmatrix} -C_\infty & I \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

belongs to \mathcal{RH}_∞ . Furthermore, given a plant P_s and a controller C_∞ , the *robust stability margin* $b(P_s, C_\infty)$ is given by

$$b(P_s, C_\infty) := \left\| \begin{bmatrix} P_s \\ I \end{bmatrix} (I - C_\infty P_s)^{-1} \begin{bmatrix} -C_\infty & I \end{bmatrix} \right\|_\infty^{-1},$$

if $[P_s, C_\infty]$ is internally stable and by $b(P_s, C_\infty) := 0$ otherwise. Then, the largest value of the robust stability margin is defined by $b_{opt}(P_s) = \sup_{C_\infty} b(P_s, C_\infty)$. It is shown in [4] that $b_{opt}(P_s) \leq 1$ for any P_s .

II. PROBLEM BACKGROUND

\mathcal{H}_∞ loop-shaping has been a successful method of designing controllers due to its simplicity and the fact that it carries notions of classical loop-shaping through to MIMO controller design. However, the practice of designing loop-shaping weights for MIMO plants is not yet a systematic procedure and hence remains largely dependent on the engineer's intuition and experience, though initial works to alleviate this has been done by, for example, [1][2]. This is especially true when the plant has strong cross coupling and/or calls for the use of nondiagonal loop-shaping weights.

Lanzon's [1] method for synthesising weights involves optimising the robust stability margin whilst maintaining the level of performance within acceptable limits specified by means of an allowed loop-shape region.

This paper attempts to use the same framework to devise a systematic method to optimise the performance, in the sense as described in Section I, while maintaining the robust stability margin above an acceptable limit specified by the designer. If the robust stability margin is kept above an acceptable level, then this will guarantee a certain amount of generic robustness to uncertainty as interpreted classically by gain and phase margin [5]. Consequently, optimising the level of performance while retaining an acceptable level of robustness is a sensible paradigm. Furthermore, often the performance of a plant reaches unacceptable levels in the face of disturbances and uncertainty *before* instability becomes an issue [8, Section 11.3.2].

The proposed algorithm consists of two optimisation steps in cascade which attempts to achieve desired low frequency behaviour and high frequency behaviour respectively.

III. PROPOSED SOLUTION

Two optimisation problems are proposed which, used in conjunction with each other, can be used to improve performance of a plant (i.e. maximise low frequency gain and minimise high frequency gain) while maintaining a specified level of robust stability margin. First, we introduce two frequency dependant "adjustment weights",

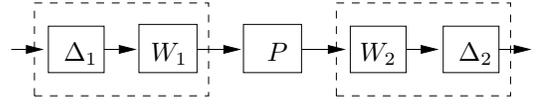


Fig. 2. Adjustment Weights

Δ_1 and Δ_2 , that would act as design variables in the optimisation problems. Adjusted weight $W_{1,adj} = W_1 \Delta_1$ (resp. $W_{2,adj} = \Delta_2 W_2$) will replace W_1 (resp. W_2).

In each optimisation phase, constraints are imposed on the singular values of the adjustment weights to nullify the effect of the optimisation in frequency regions where the particular effect is not desired. The algorithm is flexible enough to let the designer relax the performance improvement requirement in one frequency region if the original plant (or the original plant shaped by initial weights) has satisfactory behaviour in such a region. (See Section VI)

The key idea proposed in this paper can be laid out as follows.

We introduce SISO transfer functions $|w_i(j\omega)|$, $|\bar{w}_i(j\omega)|$, $|k_i(j\omega)|$, $|\underline{\delta}_i(j\omega)|$, $|\bar{\delta}_i(j\omega)|$, ($i = 1, 2$) and the constant $\underline{\epsilon} \in (0, 1)$ is specified by the designer such that:

- (i) $\underline{\epsilon}$ represents the desired minimum level of the optimal robust stability margin of the shaped plant with adjusted weights, $b_{opt}(\Delta_2 P_s \Delta_1)$,
- (ii) the frequency functions $|w_i(j\omega)|$ and $|\bar{w}_i(j\omega)|$ restrict the singular values of adjusted loop-shaping weight $W_{i,adj}(j\omega)$ for $i = 1, 2$,
- (iii) the frequency function $|k_i(j\omega)|$ bounds the condition number of loop-shaping weights $W_{i,adj}(j\omega)$ for $i = 1, 2$,
- (iv) the frequency functions $|\underline{\delta}_i(j\omega)|$ and $|\bar{\delta}_i(j\omega)|$ restrict the singular values of adjustment weight $\Delta_i(j\omega)$ for $i = 1, 2$.

Now consider the following optimisation problem which attempts to maximise the gain of $P_s \Delta_1$ at each frequency ω :

$$\max_{\Delta_1, \Delta_1^{-1} \in \mathcal{RH}_\infty} \sigma(P_s(j\omega) \Delta_1(j\omega))$$

subject to

- (a) $b_{opt}(P_s(j\omega) \Delta_1(j\omega)) > \underline{\epsilon}$,
- (b) $|\underline{\delta}_1(j\omega)| < \sigma_i(\Delta_1(j\omega)) < |\bar{\delta}_1(j\omega)| \quad \forall \omega$,
- (c) $|w_1(j\omega)| < \sigma_i(W_1(j\omega) \Delta_1(j\omega)) < |\bar{w}_1(j\omega)| \quad \forall \omega$,
- (d) $\kappa(W_1(j\omega) \Delta_1(j\omega)) < |k_1(j\omega)| \quad \forall \omega$.

Constraint (a) guarantees the minimum acceptable robust stability margin is maintained. (b) is intended to restrict the singular values of Δ_1 to a small neighbourhood of one where gain maximisation is not desirable, i.e. high frequencies. (c) and (d) provides bounds on the singular values and the condition numbers of the weights. This is important as they contribute to the upper bounds on standard closed-loop design objectives [3, Section IV B].

Also note the following points:

- to avoid Δ_1 and Δ_2 counteracting, one adjustment weight (Δ_2) is held constant at identity for all frequencies ω in this first maximisation problem described above,
- $\underline{\delta}_1(j\omega)$ and $\bar{\delta}_1(j\omega)$ are specified in such a way that they restrict the singular values of Δ_1 to a small neighbourhood of identity for $\omega > \omega_L$, thereby, gradually nullifying the maximisation of the loop gain for $\omega > \omega_L$. See the numerical example in Section VI for illustration.

This will result in an adjusted weight $W_1\Delta_1$ which maximises the gain of $P_s\Delta_1$ at low frequencies without altering its high frequency behaviour, and also maintaining a robust stability margin better than $\underline{\epsilon}$. Once Δ_1 is synthesised in the manner described above, it can be absorbed in weight W_1 . Now, to obtain adjusted weight Δ_2W_2 which achieves desired high frequency behaviour without affecting low frequency characteristics while maintaining robust stability margin above $\underline{\epsilon}$, consider a second optimisation problem:

$$\min_{\Delta_2, \Delta_2^{-1} \in \mathcal{RH}_\infty} \bar{\sigma}(\Delta_2(j\omega)P_s(j\omega))$$

subject to

- $b_{opt}(\Delta_2(j\omega)P_s(j\omega)) > \underline{\epsilon}$,
- $|\underline{\delta}_2(j\omega)| < \sigma_i(\Delta_2(j\omega)) < |\bar{\delta}_2(j\omega)| \quad \forall \omega$,
- $|\underline{w}_2(j\omega)| < \sigma_i(\Delta_2W_2(j\omega)) < |\bar{w}_2(j\omega)| \quad \forall \omega$,
- $\kappa(\Delta_2W_2(j\omega)) < |k_2(j\omega)| \quad \forall \omega$,

with analogous comments:

- Δ_1 is restricted to identity for all ω in this minimisation problem,
- $\underline{\delta}_2(j\omega)$ and $\bar{\delta}_2(j\omega)$ are specified in such a way that they restrict the singular values of Δ_2 to a small neighbourhood of identity for $\omega < \omega_H$.

Thus, the obtained adjusted weights $W_1\Delta_1$ and Δ_2W_2 will ensure that the resultant shaped plant has optimised performance for the full frequency range. Note that for the two optimisation problems to be used in conjunction in a complimentary fashion, when Δ_1 is computed in the first optimisation problem to construct $W_{1,adj} = W_1\Delta_1$, the Δ_1 is absorbed in W_1 (i.e. W_1 replaced by $W_1\Delta_1$) in the second optimisation problem.

IV. PROBLEM REFORMULATION

We follow similar algebra to that used in [1][2] to rewrite the optimisation problem in LMI format, with the modifications described in this section. In the interest of continuity we begin by restating the assumptions and definitions used in [1].

Assumption 1: Let the nominal plant $P \in \mathcal{R}^{m \times n}$ be such that $m \geq n$.

If, however, the plant has fewer outputs than inputs (i.e. $m < n$), a dual problem can be considered. Therefore, no loss of generality is incurred by this assumption.

Definition 1: Let the set of real diagonal matrices of dimension $n \times n$ be defined by:

$$\Lambda_n := \left\{ \begin{bmatrix} x_1 & & \\ & \ddots & \\ & & x_n \end{bmatrix} : x_i \in \mathbb{R} \forall i \right\}$$

Note that $b_{opt}(\Delta_2P_s\Delta_1) > \underline{\epsilon}$ if and only if there exists $C_{\infty,adj}$ such that $b(\Delta_2P_s\Delta_1, C_{\infty,adj}) > \underline{\epsilon}$. The latter inequality is true if and only if

$$\begin{aligned} & \bar{\sigma} \left(\begin{bmatrix} \Delta_2 P_s \Delta_1 \\ I \end{bmatrix} (I - C_{\infty,adj} \Delta_2 P_s \Delta_1)^{-1} \right. \\ & \quad \left. \begin{bmatrix} -C_{\infty,adj} & I \end{bmatrix} \right) < \frac{1}{\underline{\epsilon}} \quad \forall \omega \\ \Leftrightarrow & \bar{\sigma} \left(\begin{bmatrix} \Delta_2 & 0 \\ 0 & \Delta_1^{-1} \end{bmatrix} \begin{bmatrix} 0 & P_s \\ 0 & C_\infty \end{bmatrix} \right. \\ & \quad \left. \begin{bmatrix} I & P_s \\ C_\infty & I \end{bmatrix}^{-1} \begin{bmatrix} \Delta_2^{-1} & 0 \\ 0 & \Delta_1 \end{bmatrix} \right) < \frac{1}{\underline{\epsilon}} \quad \forall \omega \\ \Leftrightarrow & \underline{\epsilon}^2 \begin{bmatrix} 0 & P_s \\ 0 & I \end{bmatrix}^* \begin{bmatrix} \Delta_2^* \Delta_2 & 0 \\ 0 & \Delta_1^{-*} \Delta_1^{-1} \end{bmatrix} \begin{bmatrix} 0 & P_s \\ 0 & I \end{bmatrix} \\ & < \begin{bmatrix} I & P_s \\ C_\infty & I \end{bmatrix}^* \begin{bmatrix} \Delta_2^* \Delta_2 & 0 \\ 0 & \Delta_1^{-*} \Delta_1^{-1} \end{bmatrix} \begin{bmatrix} I & P_s \\ C_\infty & I \end{bmatrix} \quad \forall \omega \end{aligned}$$

where $C_\infty = \Delta_1 C_{\infty,adj} \Delta_2$.

For the sake of clarity, this paper restricts attention to diagonal weights. Hence Δ_1, Δ_2 are also of diagonal structure. The presented algorithm, however, can be easily modified to accommodate non-diagonal weights using the procedure in [1], readapted from [7], which uses spectral factorisation for the construction of non-diagonal weights.

In this light and remembering that Δ_2 is held at identity in the first optimisation, we can rewrite the first optimisation problem in Section III as follows.

Maximise α_ω^2 at each ω
such that \exists diagonal Δ_1 and stabilising C_∞ satisfying

- $\Delta_1, \Delta_1^{-1} \in \mathcal{RH}_\infty$,
- $\alpha_\omega^2 \Delta_1^{-*} \Delta_1^{-1} < P_s^* P_s \quad \forall \omega$,
- $\underline{\epsilon}^2 \begin{bmatrix} 0 & P_s \\ 0 & I \end{bmatrix}^* \begin{bmatrix} I & 0 \\ 0 & \Delta_1^{-*} \Delta_1^{-1} \end{bmatrix} \begin{bmatrix} 0 & P_s \\ 0 & I \end{bmatrix}$
 $< \begin{bmatrix} I & P_s \\ C_\infty & I \end{bmatrix}^* \begin{bmatrix} I & 0 \\ 0 & \Delta_1^{-*} \Delta_1^{-1} \end{bmatrix} \begin{bmatrix} I & P_s \\ C_\infty & I \end{bmatrix} \quad \forall \omega$,
- $|\bar{\delta}_1(j\omega)|^{-2} I < \Delta_1^{-*} \Delta_1^{-1} < |\underline{\delta}_1(j\omega)|^{-2} I \quad \forall \omega$,
- $\exists \underline{\xi}_{1\omega}, \bar{\xi}_{1\omega} \in \mathbb{R} :$
 $\underline{\xi}_{1\omega} (W_1^* W_1) < \Delta_1^{-*} \Delta_1^{-1} < \bar{\xi}_{1\omega} (W_1^* W_1)$,
 $|\bar{w}_1(j\omega)|^{-2} < \underline{\xi}_{1\omega}, \bar{\xi}_{1\omega} < |\underline{w}_1(j\omega)|^{-2}$,
 $\bar{\xi}_{1\omega} < |k_1(j\omega)|^2 \underline{\xi}_{1\omega} \quad \forall \omega$.

Similarly the second optimisation problem becomes,

Minimise β_ω^2 at each ω
such that \exists diagonal Δ_2 and stabilising C_∞ satisfying

- (a) $\Delta_2, \Delta_2^{-1} \in \mathcal{RH}_\infty$,
- (b) $\beta_\omega^2 I > P_s^* \Delta_2^* \Delta_2 P_s \quad \forall \omega$,
- (c) $\underline{\epsilon}^2 \begin{bmatrix} 0 & P_s \\ 0 & I \end{bmatrix}^* \begin{bmatrix} \Delta_2^* \Delta_2 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} 0 & P_s \\ 0 & I \end{bmatrix}$
 $< \begin{bmatrix} I & P_s \\ C_\infty & I \end{bmatrix}^* \begin{bmatrix} \Delta_2^* \Delta_2 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} I & P_s \\ C_\infty & I \end{bmatrix} \quad \forall \omega$,
- (d) $|\underline{\delta}_2(j\omega)|^2 I < \Delta_2^* \Delta_2 < |\bar{\delta}_2(j\omega)|^2 I \quad \forall \omega$,
- (e) $\exists \underline{\xi}_{2\omega}, \bar{\xi}_{2\omega} \in \mathbb{R}$:
 $\underline{\xi}_{2\omega} (W_2^{-*} W_2^{-1}) < \Delta_2^* \Delta_2 < \bar{\xi}_{2\omega} (W_2^{-*} W_2^{-1})$,
 $|\underline{w}_2(j\omega)|^2 < \underline{\xi}_{2\omega}, \bar{\xi}_{2\omega} < |\bar{w}_2(j\omega)|^2$,
 $\bar{\xi}_{2\omega} < |k_2(j\omega)|^2 \underline{\xi}_{2\omega} \quad \forall \omega$.

Now we parameterise the strictly positive diagonal matrix $\Delta_1^{-*} \Delta_1^{-1}$ (resp. $\Delta_2^* \Delta_2$) by $\Lambda_1 \in \Lambda_n$ (resp. $\Lambda_2 \in \Lambda_m$) to write the problems as they appear in Section V.

V. SOLUTION ALGORITHM

In this section, we iteratively solve the above two optimisation problems keeping C_∞ fixed at each optimisation and updating it after the weight is synthesised.

Inputs to the algorithm

- Scaled nominal plant P ,
- Desired minimum level of robust stability margin $\underline{\epsilon} \in (0, 1)$,
- Initial diagonal weights $W_{1,0}, W_{2,0}$ for plant P such that $b_{opt}(W_{2,0} P W_{1,0}) > \underline{\epsilon}$ (If the initial weights are of non-diagonal structure absorb them in to the plant P and use I_n (resp. I_m) as $W_{1,0}$ (resp. $W_{2,0}$)¹,
- The frequency functions $|\underline{\delta}_i(j\omega)|$ and $|\bar{\delta}_i(j\omega)|$ that restrict the singular values of adjustment weight $\Delta_i(j\omega)$ as described in Section III for $i = 1, 2$
- Frequency functions $|\underline{w}_i(j\omega)|$ and $|\bar{w}_i(j\omega)|$ that restrict the singular values of loop-shaping weight $W_i(j\omega)$ for $i = 1, 2$,
- Frequency function $|k_i(j\omega)|$ that bounds the condition number of loop-shaping weights W_i for $i = 1, 2$.

The solution algorithm

1. Synthesise $C_{\infty,0}$ such that $b(W_{2,0} P W_{1,0}, C_{\infty,0}) = b_{opt}(W_{2,0} P W_{1,0})$. Assign $P_{s,0} = W_{2,0} P W_{1,0}$ and $i = 0$.
2. Increment i by 1.
3. Solve the following quasi-convex optimisation problem at each frequency ω :

¹Also note that if $P \in \mathcal{RH}_\infty$, then there exists a sufficiently small γ such that $W_{2,0} = \gamma I$ and $W_{1,0} = I$ yield $b_{opt}(W_{2,0} P W_{1,0}) > \underline{\epsilon}$. Also, if $P \notin \mathcal{RH}_\infty$ and it is difficult to find such initial weights, then the algorithm in [1] and [2] can provide such initial weights.

Maximise α_ω^2
such that $\exists \Lambda_{1\omega} \in \Lambda_n$ satisfying

- (a) $\alpha_\omega^2 \Lambda_{1\omega} < P_{s,i-1}^* P_{s,i-1}$,
- (b) $\underline{\epsilon}^2 \begin{bmatrix} 0 & P_{s,i-1} \\ 0 & I \end{bmatrix}^* \begin{bmatrix} I & 0 \\ 0 & \Lambda_{1\omega} \end{bmatrix} \begin{bmatrix} 0 & P_{s,i-1} \\ 0 & I \end{bmatrix}$
 $< \begin{bmatrix} I & P_{s,i-1} \\ C_{\infty,i-1} & I \end{bmatrix}^* \begin{bmatrix} I & 0 \\ 0 & \Lambda_{1\omega} \end{bmatrix}$
 $\times \begin{bmatrix} I & P_{s,i-1} \\ C_{\infty,i-1} & I \end{bmatrix}$,
- (c) $|\bar{\delta}_1(j\omega)|^{-2} I < \Lambda_{1\omega} < |\underline{\delta}_1(j\omega)|^{-2} I$,
- (d) $\exists \underline{\xi}_{1\omega}, \bar{\xi}_{1\omega} \in \mathbb{R}$:
 $\underline{\xi}_{1\omega} (W_{1,i-1}^* W_{1,i-1}) < \Lambda_{1\omega} < \bar{\xi}_{1\omega} (W_{1,i-1}^* W_{1,i-1})$,
 $|\bar{w}_1(j\omega)|^{-2} < \underline{\xi}_{1\omega}, \bar{\xi}_{1\omega} < |\underline{w}_1(j\omega)|^{-2}$,
 $\bar{\xi}_{1\omega} < |k_1(j\omega)|^2 \underline{\xi}_{1\omega}$.

4. Construct a diagonal transfer function matrix $W_{1,i}(s)$ that is a unit in \mathcal{RH}_∞ by fitting a stable minimum phase transfer function on the main diagonal of $W_{1,i-1}(j\omega) \Lambda_{1\omega}^{-\frac{1}{2}}$ ²
5. Compute $b_{opt}(W_{2,i-1} P W_{1,i})$ as detailed in [4]. Synthesise a controller $C_{\infty,tmp}$ that achieves the computed robust stability margin $b(W_{2,i-1} P W_{1,i}, C_{\infty,tmp}) = b_{opt}(W_{2,i-1} P W_{1,i})$ using the state space formula given in [4]. Reassign $P_{s,i-1} = W_{2,i-1} P W_{1,i}$ and $C_{\infty,i-1} = C_{\infty,tmp}$.
6. Solve the following quasi-convex optimisation problem at each frequency ω :

Minimise β_ω^2
such that $\exists \Lambda_{2\omega} \in \Lambda_m$ satisfying

- (a) $\beta_\omega^2 I > P_{s,i-1}^* \Lambda_{2\omega} P_{s,i-1}$,
- (b) $\underline{\epsilon}^2 \begin{bmatrix} 0 & P_{s,i-1} \\ 0 & I \end{bmatrix}^* \begin{bmatrix} \Lambda_{2\omega} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} 0 & P_{s,i-1} \\ 0 & I \end{bmatrix}$
 $< \begin{bmatrix} I & P_{s,i-1} \\ C_{\infty,i-1} & I \end{bmatrix}^* \begin{bmatrix} \Lambda_{2\omega} & 0 \\ 0 & I \end{bmatrix}$
 $\times \begin{bmatrix} I & P_{s,i-1} \\ C_{\infty,i-1} & I \end{bmatrix}$,
- (c) $|\underline{\delta}_2(j\omega)|^2 I < \Lambda_{2\omega} < |\bar{\delta}_2(j\omega)|^2 I$,
- (d) $\exists \underline{\xi}_{2\omega}, \bar{\xi}_{2\omega} \in \mathbb{R}$:
 $\underline{\xi}_{2\omega} (W_{2,i-1}^{-*} W_{2,i-1}^{-1}) < \Lambda_{2\omega} < \bar{\xi}_{2\omega} (W_{2,i-1}^{-*} W_{2,i-1}^{-1})$,
 $|\underline{w}_2(j\omega)|^2 < \underline{\xi}_{2\omega}, \bar{\xi}_{2\omega} < |\bar{w}_2(j\omega)|^2$,
 $\bar{\xi}_{2\omega} < |k_2(j\omega)|^2 \underline{\xi}_{2\omega}$.

7. Construct a diagonal transfer function matrix $W_{2,i}(s)$ that is a unit in \mathcal{RH}_∞ by fitting a stable minimum phase transfer function on the main diagonal of $\Lambda_{2\omega}^{\frac{1}{2}} W_{2,i-1}(j\omega)$.

²This step along with the step 8. provide the designer with a handle on the order of the weights. The designer is, hence, able to prevent the inflation of the order with out bounds.

8. Compute $b_{opt}(W_{2,i}PW_{1,i})$ as detailed in [4] and let this value be denoted ϵ_i . Synthesise a controller $C_{\infty,i}$ that achieves the computed robust stability margin $b(W_{2,i}PW_{1,i}, C_{\infty,i}) = \epsilon_i$ using the state space formula given in [4]. Assign $P_{s,i} = W_{2,i}PW_{1,i}$.
9. Evaluate $\epsilon_i - \underline{\epsilon}$. If this difference is very small and has remained so for the last few iterations, then EXIT. Otherwise return to step 2.

Outputs from the algorithm

- Loop-shaping weights $W_{1,i}(s)$ and $W_{2,i}(s)$ that achieve optimal performance in the sense described in Section II,
- The robust stability margin $\epsilon_i (> \underline{\epsilon})$ of the plant shaped using the above weights,
- A controller $C_{\infty,i}$ that achieves this robust stability margin for the shaped plant.

Note that the algorithm is ascending with respect to the “performance” in the sense as described in Section I.

VI. NUMERICAL EXAMPLE

We use a MIMO model of a 747 jet transport aircraft from [9] to illustrate the use of the proposed algorithm. Nominal plant P has four states, two inputs and two outputs.

We will now synthesise loop shaping weights that would optimise the performance of the plant while maintaining the robust stability margin above, say for the purpose of demonstration, 0.18. Since the plant P with loop shaping weights of I has an optimal robust stability margin of 0.21, the initial weights $W_{1,0}, W_{2,0}$ can be set to I_2 . The frequency functions $|\underline{w}_i(j\omega)|, |\overline{w}_i(j\omega)|, |\overline{k}_i(j\omega)|$ that confines the singular values/condition number of adjusted pre/post compensators were set at $10^{-10}, 10^{10}, 20$ respectively for $i = 1, 2$. This effectively means that the weight bounds will play no role in the optimisation problem as the corresponding constraints never become active.

In conformity with what is described in Section III, $\overline{\delta}_i, \underline{\delta}_i$ for $i = 1, 2$ were set to following transfer functions with $\omega_L = 1 \text{ rads}^{-1}$ and $\omega_H = 4 \text{ rads}^{-1}$. (See Figure 4.)

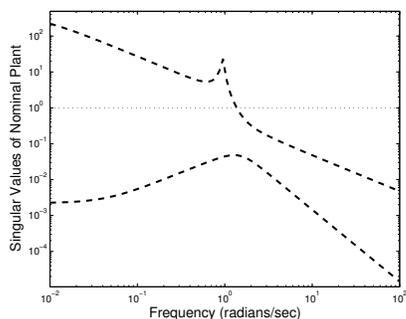


Fig. 3. Singular values of nominal plant P

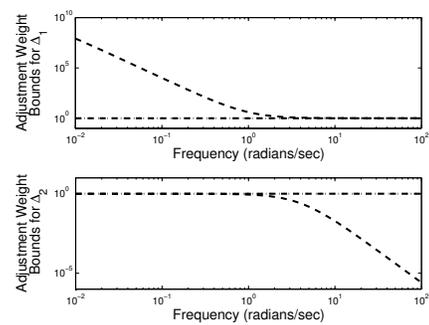


Fig. 4. Adjustment weight bounds

$$\underline{\delta}_1(s) = 1, \quad \overline{\delta}_1(s) = \left(\frac{1.01 \frac{1}{10} s + \omega_L}{s + (10^{-10}) \frac{1}{10} \times \omega_L} \right)^{10}$$

$$\underline{\delta}_2(s) = \left(\frac{s + (10^{-10}) \frac{1}{10} \times \omega_H}{1.01 \frac{1}{10} s + \omega_H} \right)^{10}, \quad \overline{\delta}_2(s) = 1$$

We have chosen $\omega_L < \omega_H$ to avoid changing the behaviour of the loop gain near crossover. If an increase in bandwidth is to be desired ω_L and ω_H can both be set to a higher value than the bandwidth of the nominal plant, as necessary. Also note that the algorithm optimises the performance by maximising the minimum singular value, $\underline{\sigma}(P_s)$, in low frequencies and minimising the maximum singular value, $\overline{\sigma}(P_s)$, in high frequencies. To avoid any other singular value being adversely affected, the bounds $\underline{\delta}_1$ and $\overline{\delta}_2$ were set to 1 in this example so that only maximisation in low frequencies and minimisation in high frequencies is allowed. If this is not required these bound can be relaxed so that they do not present active constraints.

After 4 iterations an improved loop shape, as depicted in Figure 5, was achieved. The results are summarised in Table I and the singular values of the adjusted weights are shown in Figure 6.

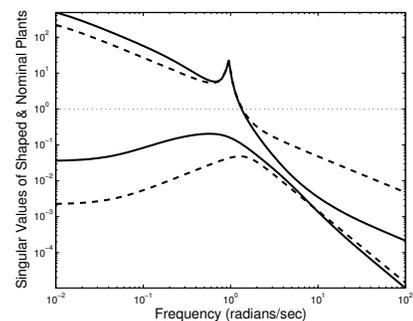


Fig. 5. New loop shape with adjusted weights (solid), loop shape of the nominal plant P (dashed)

No. of iterations	4
Order of weight W_1 (inc. adjuttment)	2 states
Condition number of weight W_1	$< 13.29 \quad \forall \omega$
Order of weight W_2 (inc. adjuttment)	1 state
Condition number of weight W_2	$< 20 \quad \forall \omega$
Order of controller C_∞	6 states
Order of controller C	9 states
Robust stability margin	0.2056

TABLE I
SUMMARY OF RESULTS

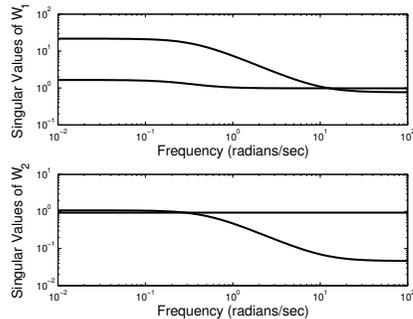


Fig. 6. Singular values of the adjusted weights

VII. CONCLUSION

We conclude by pointing out that the presented procedure for loop shaping weight synthesis carries through the virtues of the one presented in [1], with an alternate objective. The approach taken here is to optimise the performance while maintaining a minimum acceptable level of robustness. This is in contrast to [1], where the emphasis is on achieving the maximum possible robustness while maintaining a minimum acceptable level of performance³. The former approach is deemed to be more practical as it aligns with a designer's typical goal of achieving the best possible performance with the required level of robustness. This procedure can also be used to optimise pre-designed weights with respect to the performance. Thereby it provides one with a tool to fine tune one's designs.

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³Here, we further add that the robust performance goal is also met as it is inherently guaranteed by loop shaping.

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