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A Frequency Domain Optimisation Algorithm for Simultaneous Design of Performance Weights and Controllers in μ-Synthesis

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Abstract

A novel iterative procedure is proposed in this paper as a valuable alternative/complement to D-K iterations frequently used in μ-synthesis. This new procedure gives both a robustly stabilising controller and performance weights which together achieve a closed-loop μ-value which is slightly less than unity, thereby immediately guaranteeing robust performance. The performance weights given by this algorithm maximise the achieved performance for the particular plant set considered according to some sensible cost function. The designer is only required to specify the desired directionality of the optimisation according to his engineering insight. Of course, a sensible plant set and a sensible directionality for the optimisation are still required for a sensible control problem. The algorithm proposed here thus eliminates the trial and error process usually adopted by designers in selecting such performance weights.

Keywords: performance weights selection, maximise achieved performance, D-K iterations, $\mathscr{H}_\infty$-control, μ-synthesis.

1 Introduction

It is well known that the choice of performance weights in $\mathscr{H}_\infty$ and μ-synthesis frameworks is a non-trivial task. The resulting controller and closed-loop performance are very much dependent on the particular performance weights chosen and usually, the designer is required to find such suitable performance weights by a long and tedious trial and error process using his engineering experience and intuition.

The D-K iteration procedure [5] is probably the most popular method used in μ-synthesis to design robustly stabilising controllers. Other methods with different computational benefits have later been proposed, such as μ-K iterations in [9], E-K iterations in [4] and L-R iterations in [10]. However, all these methods assume that the performance weights have already been chosen. Some authors have suggested methods for choosing such performance weights for specific design problems [8, 11]. However, all this work heavily relies on the designer's experience and the final performance weights used always come after a long trial and error process.

In 1992, Fan and Tits [6] introduced a new mathematical quantity, closely related to μ, which solves the problem: "Determine the smallest α such that for any uncertainty bounded by unity, an $\mathscr{H}_\infty$ performance level of α is guaranteed". Although this was an initial step towards maximising the achieved performance (i.e. the determination of the smallest α) for a given uncertainty set, the value α is a constant bound over all frequency and spatial direction. This paper solves the more general problem: "Determine the largest performance weights (in some sense, over frequency and spatial direction) such that for any uncertainty bounded by unity, an $\mathscr{H}_\infty$ performance level of unity is guaranteed".

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2 Preliminaries

Let $\mathbb{R}$ and $\mathbb{C}$ be the fields of real and complex numbers respectively. Also let $\mathbb{R}_+$ ($\overline{\mathbb{R}}_+$) be the field of strictly-positive (non-negative) real numbers and $\overline{\mathbb{C}}_+$ be the closed right-half plane. Furthermore, define $\mathbb{R}^n_+$ as the field of n-dimensional vectors with entries in $\mathbb{R}_+$ and $\mathbb{R}^{p\times q}$ ($\mathbb{C}^{p\times q}$) as the field of real (complex) matrices of dimension $p \times q$. Let A^* denote the complex conjugate transpose of matrix A and $A > 0$ denote a positive-definite matrix A. Moreover, $\overline{\sigma}(A)$ is used to denote the largest singular value of matrix A and $\|A\|_F$ is used to denote the Frobenius norm of matrix A. Furthermore, let $\mathscr{RH}_\infty$ be the real-rational subspace of $\mathscr{H}_\infty$ and define $\mathcal{F}_l(\cdot,\cdot)$ as the lower Linear Fractional Transformation (LFT). Denote by $\text{diag}(A_1, A_2, \ldots, A_n)$ the block-diagonal matrix with matrices A_i on its main diagonal and let $\begin{pmatrix} Q & S \\ * & R \end{pmatrix}$ be shorthand notation for $\begin{pmatrix} Q & S \\ S^* & R \end{pmatrix}$. Finally, define $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ as the state-space realisation $C(sI - A)^{-1}B + D$. All the rest is standard.

3 Problem Formulation

Consider the Linear Time-Invariant (LTI) system depicted in Figure 1. This LFT framework is very general as any linear

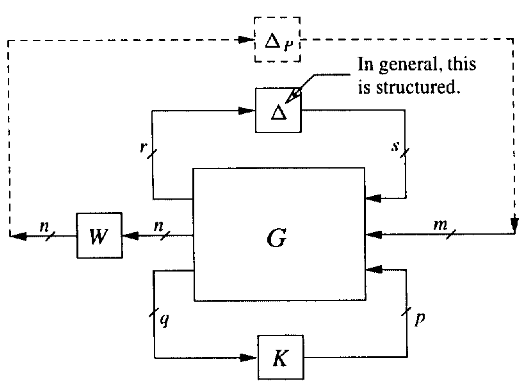

Figure 1: General $\mathscr{H}_\infty$ or μ-synthesis LFT framework

interconnection of systems, inputs, outputs and model uncertainties can be recast into this framework [12]. First, define the following sets:

$$\begin{aligned}
\boldsymbol{\Delta} &:= \left\{\operatorname{diag}\left(\delta_1 I_{\alpha_1}, \ldots, \delta_R I_{\alpha_R}, \Delta_1, \ldots, \Delta_F\right) : \right. \\
&\qquad \left. \delta_i \in \mathbb{C}, \Delta_j \in \mathbb{C}^{\beta_j \times \upsilon_j}\right\} \\
\boldsymbol{\Delta}_P &:= \left\{\Delta_P \in \mathbb{C}^{m \times n}\right\} \\
\boldsymbol{\Delta}_{TOT} &:= \left\{\operatorname{diag}(\Delta, \Delta_P) : \Delta \in \boldsymbol{\Delta}, \Delta_P \in \boldsymbol{\Delta}_P\right\} \\
\mathbf{B}\boldsymbol{\Delta}^{TF} &:= \left\{\Delta(s) \in \mathscr{RH}_\infty : \Delta(s_o) \in \boldsymbol{\Delta}\ \forall s_o \in \overline{\mathbb{C}}_+, \right. \\
&\qquad \left. \|\Delta\|_\infty \leq 1\right\} \\
\mathbf{B}\boldsymbol{\Delta}_P^{TF} &:= \left\{\Delta_P(s) \in \mathscr{RH}_\infty : \Delta_P(s_o) \in \boldsymbol{\Delta}_P\ \forall s_o \in \overline{\mathbb{C}}_+, \right. \\
&\qquad \left. \|\Delta_P\|_\infty \leq 1\right\}.
\end{aligned}$$

For consistency in dimensions, $\sum_{i=1}^{R} \alpha_i + \sum_{j=1}^{F} \beta_j = s$ and $\sum_{i=1}^{R} \alpha_i + \sum_{j=1}^{F} \upsilon_j = r$. The results presented in this paper can be easily extended to the case where the set $\boldsymbol{\Delta}_P$ has a block-diagonal structure and/or to the case where the uncertainty sets defined above have mixed real/complex uncertainty. This, however, will not be done here for the sake of simplicity. Now, let $\mathcal{K}$ denote the set of internally stabilising controllers and define the set $\mathcal{W}$ as:

$$\begin{aligned}
\mathcal{W} := \{&\operatorname{diag}[p_1(s), p_2(s), \ldots, p_n(s)] : \\
&p_i(s), p_i(s)^{-1} \in \mathscr{RH}_\infty,\ p_i(s_o) \in \mathbb{C}\ \forall s_o \in \overline{\mathbb{C}}_+\}.
\end{aligned}$$

Then in Figure 1,

- G denotes the LTI generalised plant. Before forming this generalised plant, the designer should have already decided what "plant set" to design for. This "plant set" is determined from the nominal plant model, the structure, type and size of the uncertainty and any 'a priori' knowledge about the frequency content of the exogenous signals. Here, it is assumed that all this has already been done and $G(s)$ is already given. See [2] for a detailed explanation on how to construct such a generalised plant.

- $K \in \mathcal{K}$ denotes an LTI controller which is to be designed. Besides achieving the required closed-loop μ-value, $K(s)$ must also be *internally stabilising*.

- $W \in \mathcal{W}$ is a stable minimum-phase invertible (and hence bi-proper) diagonal square transfer function matrix denoting the performance weight. It is required to find the "biggest" $W(s)$, in some sense, such that some constraints are not violated.

- $\Delta \in \mathbf{B}\boldsymbol{\Delta}^{TF}$ is the uncertainty in the system. Thus, Δ is an unknown stable LTI system which has a block-diagonal structure and satisfies $\|\Delta\|_\infty \leq 1$.

- $\Delta_P \in \mathbf{B}\boldsymbol{\Delta}_P^{TF}$ is the "performance uncertainty". This uncertainty is fictitious and is only used to transform the robust performance problem into an equivalent robust stability problem. Thus, Δ_P is also an unknown stable LTI system which satisfies $\|\Delta_P\|_\infty \leq 1$.

Furthermore, the Robust Performance Theorem (e.g. [1]) states that robust performance is achieved for the setup shown in Figure 1 for all $\Delta \in \mathbf{B}\boldsymbol{\Delta}^{TF}$ and $\Delta_P \in \mathbf{B}\boldsymbol{\Delta}_P^{TF}$ if and only if

$$\sup_\omega \mu_{\boldsymbol{\Delta}_{TOT}} \left[\begin{pmatrix} I_r & 0 \\ 0 & W(j\omega) \end{pmatrix} \mathcal{F}_l\left(G(j\omega), K(j\omega)\right)\right] < 1.$$

Now, define the function $\Gamma_{\omega_L}^{\omega_H} : \mathscr{RH}_\infty \to \overline{\mathbb{R}}_+$ with $0 < \omega_L < \omega_H < \infty$ as:

$$\Gamma_{\omega_L}^{\omega_H}(P) := \sqrt{\int_{\log_{10}\omega_L}^{\log_{10}\omega_H} \|P(j\omega)\|_F^2 \, d(\log_{10}\omega)}.$$

$\Gamma_{\omega_L}^{\omega_H}$ is a semi-norm which in some sense measures the size of $P(j\omega)$ in the frequency range $[\omega_L, \omega_H]$. Logarithmic frequency is chosen as the variable of integration so that $\Gamma_{\omega_L}^{\omega_H}(P)$ can have easy interpretation when the singular values of P are plotted on a Bode diagram. This follows from the fact that the square of the Frobenius norm of a matrix is equal to the sum of the squares of all singular values of that matrix.

Now consider the following optimisation problem:

$$\begin{aligned}
&\max_{W \in \mathcal{W}} \frac{1}{\Gamma_{\omega_L}^{\omega_H}\left(\tilde{C} W^{-1}\right)^2} \\
&\text{such that} \qquad\qquad\qquad\qquad\qquad\qquad (1) \\
&\min_{K \in \mathcal{K}} \sup_\omega \mu_{\boldsymbol{\Delta}_{TOT}} \left[\begin{pmatrix} I_r & 0 \\ 0 & W(j\omega) \end{pmatrix} \mathcal{F}_l\left(G(j\omega), K(j\omega)\right)\right] < 1.
\end{aligned}$$

Here, $[\omega_L, \omega_H]$ is the frequency range where maximisation of $W(s)$ is required. This frequency range should be chosen sensibly and a good rule-of-thumb is to consider two or three decades below and above the required closed-loop bandwidth.

Also, the constraint of optimisation (1) states that maximisation of the performance weight $W(s)$ is limited by the fact that there must exist some internally stabilising controller $K(s)$ which guarantees robust performance.

Furthermore, $\tilde{C}(s)$ is defined to be a diagonal stable transfer function matrix chosen by the designer so as to direction the maximisation as desired. Since maximisation will only take place in the frequency range $[\omega_L, \omega_H]$, then $\tilde{C}(j\omega)$ is only relevant in this frequency range. Each diagonal element of $\tilde{C}(j\omega)$ influences the amount of maximisation required in the corresponding element of the performance weight $W(j\omega)$. In fact, the i-th diagonal element of $\tilde{C}(j\omega)$ will be chosen large (resp. small) where the corresponding diagonal element of $W(j\omega)$ is required to be large (resp. small). This does *not* make $\tilde{C}(j\omega)$ a substitute for the performance weights in normal μ-synthesis, as here $\tilde{C}(j\omega)$ only states the desired directionality for the optimisation. The absolute size of each diagonal element in $\tilde{C}(j\omega)$ is completely irrelevant, as this will only affect the value of the cost associated with optimisation (1). Only the shape across frequency and the relative sizes amongst the different diagonal entries of $\tilde{C}(j\omega)$ are important. Moreover, optimisation (1) will always give a closed-loop μ-value which is slightly less than unity, as guaranteed by the constraint on the above maximisation. Sensible choice of $\tilde{C}(j\omega)$ is of course still necessary (this is however much easier than choosing the actual performance weights) so as to obtain a controller which performs sensibly and satisfies reasonable stability/performance requirements (e.g. small sensitivity at low frequency and small complementary sensitivity at high frequency).

If $\tilde{C}$ and W were frequency independent, then justification for the choice of cost function in optimisation (1) would be simple. This is because $1/\Gamma_{\omega_L}^{\omega_H}(\tilde{C}W^{-1})^2 \propto 1/\|\tilde{C}W^{-1}\|_F^2$, and hence the steepest ascent is achieved by always maximising the smallest $|w_i/\tilde{c}_i|^2$, where w_i (resp. $\tilde{c}_i$) is the i-th diagonal element of W (resp. $\tilde{C}$). This reasoning can be extended to the case where $\tilde{C}$ and W are transfer function matrices and hence functions of frequency, but more elaborate arguments are required to justify the choice of cost function there.

4 Deriving a Sub-Optimal Problem

Optimisation (1) in the previous section is however non-convex (due to the μ constraint) and hence not easily computable. In this section, a convex sub-optimal problem with tighter constraints is derived. The derivation is split up into several sub-sections for clarity.

4.1 Transforming the Optimisation Problem

Here, optimisation problem (1) is transformed to an equivalent problem which is easier to work on. First of all, since interest is in the arguments of the optimisation and not in the value of the maximum cost, then optimisation (1) can be rewritten as:

$$\min_{W \in \mathcal{W}} \Gamma_{\omega_L}^{\omega_H}(\tilde{C}W^{-1})^2$$
$$\text{such that } \exists K \in \mathcal{K} \text{ satisfying} \tag{2}$$
$$\mu_{\Delta_{TOT}}\left[\begin{pmatrix} I_r & 0 \\ 0 & W(j\omega)\end{pmatrix}\mathcal{F}_l\left(G(j\omega), K(j\omega)\right)\right] < 1\ \forall\omega.$$

Now, partition the state-space representation of $G(s)$ consistently with Figure 1 as follows:

$$G = \left[\begin{array}{c|cc|c} A & B_1 & B_2 & B_3 \\ \hline C_1 & D_{11} & D_{12} & D_{13} \\ C_2 & D_{21} & D_{22} & D_{23} \\ \hline C_3 & D_{31} & D_{32} & D_{33} \end{array}\right].$$

Standard Assumption *Suppose (A, B_3) is stabilisable and (C_3, A) is detectable.*

This assumption is necessary and sufficient for the existence of an internally stabilising controller [7].

Fact 4.1 *Let F and L be such that $A + B_3F$ and $A + LC_3$ are stable, and define $T(s)$ as:*

$$T := \left[\begin{array}{cc|cc|c} A + B_3F & -B_3F & B_1 & B_2 & B_3 \\ 0 & A + LC_3 & B_1 + LD_{31} & B_2 + LD_{32} & 0 \\ \hline C_1 + D_{13}F & -D_{13}F & D_{11} & D_{12} & D_{13} \\ C_2 + D_{23}F & -D_{23}F & D_{21} & D_{22} & D_{23} \\ \hline 0 & C_3 & D_{31} & D_{32} & 0 \end{array}\right].$$

Let $G(s)$, $K(s)$ and $W(s)$ be the generalised plant, controller and performance weight respectively as defined in Section 3. Then the following statements are equivalent:

(a) $\exists K \in \mathcal{K}$ satisfying
$$\mu_{\Delta_{TOT}}\left[\begin{pmatrix} I_r & 0 \\ 0 & W(j\omega)\end{pmatrix}\mathcal{F}_l\left(G(j\omega), K(j\omega)\right)\right] < 1\ \forall\omega.$$

(b) $\exists Q \in \mathcal{RH}_\infty : I + D_{33}Q(\infty)$ is invertible and
$$\mu_{\Delta_{TOT}}\left[\begin{pmatrix} I_r & 0 \\ 0 & W(j\omega)\end{pmatrix}\mathcal{F}_l\left(T(j\omega), Q(j\omega)\right)\right] < 1\ \forall\omega.$$

Proof This immediately follows from the parametrisation of all stabilising controllers and closed-loops (see Theorems 12.8 and 12.16 in [13]). □

Note that the transfer function matrix $T(s)$ defined above is stable and $\mathcal{F}_l\left(T(s), Q(s)\right)$ is affine in $Q(s)$. Moreover, $T(s)$ can be computed before the optimisation. Using Fact 4.1, optimisation (2) may be rewritten as:

$$\min_{W \in \mathcal{W}} \int_{\log_{10}\omega_L}^{\log_{10}\omega_H} \|\tilde{C}(j\omega)W(j\omega)^{-1}\|_F^2\, d(\log_{10}\omega)$$
$$\text{such that } \exists Q \in \mathcal{RH}_\infty \text{ satisfying} \tag{3}$$
$$I + D_{33}Q(\infty) \text{ invertible},$$
$$\mu_{\Delta_{TOT}}\left[\begin{pmatrix} I_r & 0 \\ 0 & W(j\omega)\end{pmatrix}\mathcal{F}_l\left(T(j\omega), Q(j\omega)\right)\right] < 1\ \forall\omega.$$

In the following sub-sections, each different part of optimisation (3) will be investigated separately.

4.2 A Sufficient Condition so that $\exists Q \in \mathcal{RH}_\infty$

A sufficient condition which ensures that $\exists Q(s) \in \mathcal{RH}_\infty$ is obtained by parametrising a subspace of $\mathcal{RH}_\infty$ as:

$$Q(s) = \bar{Q}.B(s) \tag{4}$$

in which

$$\bar{Q} := \begin{bmatrix} Q_0 & Q_1 & Q_2 & \cdots & Q_N \end{bmatrix} \in \mathbb{R}^{p\times(N+1)q}$$
$$B(s) := \begin{bmatrix} I_q & \left(\frac{\frac{2}{\tau}-s}{\frac{2}{\tau}+s}\right)I_q & \left(\frac{\frac{2}{\tau}-s}{\frac{2}{\tau}+s}\right)^2 I_q & \cdots & \left(\frac{\frac{2}{\tau}-s}{\frac{2}{\tau}+s}\right)^N I_q \end{bmatrix}^T.$$

Here τ is chosen *sufficiently small* so that all dynamics can be accurately captured and N is chosen *sufficiently large* so that there are enough parameters Q_i to be able to closely model most transfer functions in $\mathcal{RH}_\infty$.

4.3 A Sufficient Condition so that $I + D_{33}Q(\infty)$ is invertible

A sufficient condition which ensures that $I + D_{33}Q(\infty)$ is invertible is given by the following set of steps, where $\rho \in \overline{\mathbb{R}}_+$.

$$\begin{aligned} & I + D_{33}Q(\infty) \text{ is invertible} \\ \Leftrightarrow\quad & (\rho+1)I + (Q(\infty)D_{33} - \rho I) \text{ is invertible}, \\ \Leftarrow\quad & \overline{\sigma}(Q(\infty)D_{33} - \rho I) < (\rho+1), \\ \Leftrightarrow\quad & (Q(\infty)D_{33} - \rho I)^*(Q(\infty)D_{33} - \rho I) < (\rho+1)^2 I, \\ \Leftrightarrow\quad & \begin{bmatrix} (\rho+1)I & (Q(\infty)D_{33} - \rho I) \\ * & (\rho+1)I \end{bmatrix} > 0. \end{aligned}$$

The above derivation makes use of Corollary 2.2.3 in [7] and Schur's Inequality found in [3]. Note also that conservativeness of the above sufficient condition reduces as ρ increases. Furthermore, if $Q(s)$ is restricted to be of the form suggested in Section 4.2, then $Q(\infty) = \bar{Q}B(\infty)$.

4.4 A Sufficient Condition so that $\mu(\cdot) < 1$

A sufficient condition which ensures that:

$$\mu_{\mathbf{\Delta}_{TOT}}\left[\begin{pmatrix} I_r & 0 \\ 0 & W(j\omega)\end{pmatrix} \mathcal{F}_l\left(T(j\omega), Q(j\omega)\right)\right] < 1 \;\forall\omega$$

is presented in this sub-section. The condition given here is also necessary whenever the uncertainty set $\mathbf{\Delta}$ is such that $2R + F \leq 2$. This can be seen by a straightforward application of Theorem 11.5 in [13]. Now, define the set:

$$\begin{aligned}\mathcal{D} := \{&\mathrm{diag}\left(D^l, D^r, d\right) : \\ &D^l = \mathrm{diag}(\tilde{D}_1, \ldots, \tilde{D}_R, \tilde{d}_1 I_{\upsilon_1}, \ldots, \tilde{d}_F I_{\upsilon_F}), \\ &D^r = \mathrm{diag}(\tilde{D}_1, \ldots, \tilde{D}_R, \tilde{d}_1 I_{\beta_1}, \ldots, \tilde{d}_F I_{\beta_F}), \\ &\tilde{D}_i \in \mathbb{C}^{\alpha_i \times \alpha_i}, \tilde{D}_i = \tilde{D}_i^* > 0 \text{ and } d, \tilde{d}_j \in \mathbb{R}_+\}.\end{aligned}$$

Then, at each *fixed* frequency ω, choose a particular element $D_\omega := \mathrm{diag}(D^l_\omega, D^r_\omega, d_\omega) \in \mathcal{D}$ with D^l_ω, D^r_ω and d_ω having dimensions $r \times r$, $s \times s$ and 1×1 respectively. Subsequently, define the left and right D-scales at each *fixed* frequency ω for the robust performance problem depicted in Figure 1 as:

$$\text{Left D-scale} := \begin{pmatrix} D^l_\omega & 0 \\ 0 & d_\omega I_n \end{pmatrix},$$

$$\text{Right D-scale} := \begin{pmatrix} D^r_\omega & 0 \\ 0 & d_\omega I_m \end{pmatrix}.$$

Now note that, at each *fixed* frequency ω, an upper bound for μ is given by (see for example [13]):

$$\begin{aligned}&\mu_{\mathbf{\Delta}_{TOT}}\left[\begin{pmatrix} I_r & 0 \\ 0 & W(j\omega)\end{pmatrix} \mathcal{F}_l\left(T(j\omega), Q(j\omega)\right)\right] \\ &\leq \inf_{D_\omega \in \mathcal{D}} \bar{\sigma}\left[\begin{pmatrix} D^l_\omega & 0 \\ 0 & d_\omega W(j\omega)\end{pmatrix} \mathcal{F}_l\left(T(j\omega), Q(j\omega)\right) \begin{pmatrix} D^r_\omega & 0 \\ 0 & d_\omega I_m\end{pmatrix}^{-1}\right].\end{aligned}$$

Since μ is not easily computable, this upper bound is used instead. Now define the set:

$$\mathcal{X} := \left\{\mathrm{diag}\left[f_1, f_2, \ldots, f_n\right] \text{ with } f_i : \mathbb{R} \to \mathbb{R}_+\right\},$$

and let

$$\begin{aligned}X(\omega) &:= \left[W(j\omega)^* W(j\omega)\right]^{-1} \\ &= \mathrm{diag}\left(|w_1(j\omega)|^{-2}, \ldots, |w_n(j\omega)|^{-2}\right) \in \mathcal{X}.\end{aligned} \tag{5}$$

Then, $\mu_{\mathbf{\Delta}_{TOT}}\left[\begin{pmatrix} I_r & 0 \\ 0 & W(j\omega)\end{pmatrix} \mathcal{F}_l\left(T(j\omega), Q(j\omega)\right)\right] < 1 \;\forall\omega$ if

$$\inf_{D_\omega \in \mathcal{D}} \bar{\sigma}\left[\begin{pmatrix} D^l_\omega & 0 \\ 0 & d_\omega W(j\omega)\end{pmatrix} \mathcal{F}_l\left(T(j\omega), Q(j\omega)\right) \begin{pmatrix} D^r_\omega & 0 \\ 0 & d_\omega I_m\end{pmatrix}^{-1}\right] < 1 \;\;\forall\omega,$$

which is true if and only if $\exists D_\omega \in \mathcal{D}$ such that

$$\begin{bmatrix} \begin{pmatrix} (D^{l*}_\omega D^l_\omega)^{-1} & 0 \\ 0 & d_\omega^{-2} X(\omega)\end{pmatrix} & \mathcal{F}_l\left(T(j\omega), Q(j\omega)\right) \\ * & \begin{pmatrix} (D^{r*}_\omega D^r_\omega) & 0 \\ 0 & d_\omega^2 I_m\end{pmatrix}\end{bmatrix} > 0 \;\;\forall\omega.$$

As before, if $Q(s)$ is restricted to be of the form suggested in Section 4.2, then $Q(j\omega) = \bar{Q}B(j\omega)$.

Now, in order to reduce conservativeness in the above sufficient condition, $D_\omega \in \mathcal{D}$ (one D_ω for each *fixed* frequency ω) is chosen as the argument of the following minimisation:

$$\min_{D_\omega \in \mathcal{D}} \bar{\sigma}\left[\begin{pmatrix} D^l_\omega & 0 \\ 0 & d_\omega W(j\omega)\end{pmatrix} \mathcal{F}_l\left(T(j\omega), \bar{Q}B(j\omega)\right) \begin{pmatrix} D^r_\omega & 0 \\ 0 & d_\omega I_m\end{pmatrix}^{-1}\right].$$

This can be rewritten as:

$$\begin{gathered}\text{Minimise } \gamma_\omega \\ \text{for each fixed } \omega \in \mathbb{R} \\ \text{such that} \\ \bar{\sigma}\left[\begin{pmatrix} D^l_\omega & 0 \\ 0 & d_\omega W(j\omega)\end{pmatrix} \mathcal{F}_l\left(T(j\omega), \bar{Q}B(j\omega)\right) \begin{pmatrix} D^r_\omega & 0 \\ 0 & d_\omega I_m\end{pmatrix}^{-1}\right] < \gamma_\omega,\end{gathered}$$

which after some algebra yields the equivalent minimisation:

$$\begin{gathered}\text{Minimise } \gamma_\omega^2 \\ \text{for each fixed } \omega \in \mathbb{R} \\ \text{such that} \\ \mathcal{F}_l\left(T(j\omega), \bar{Q}B(j\omega)\right)^* \begin{pmatrix} D^{l*}_\omega D^l_\omega & 0 \\ 0 & d_\omega^2 X(\omega)^{-1}\end{pmatrix} \mathcal{F}_l\left(T(j\omega), \bar{Q}B(j\omega)\right) \\ < \gamma_\omega^2 \begin{pmatrix} D^{r*}_\omega D^r_\omega & 0 \\ 0 & d_\omega^2 I_m\end{pmatrix}.\end{gathered} \tag{6}$$

4.5 An Equivalent Objective Function

This sub-section presents an equivalent objective function to:

$$\min_{W \in \mathcal{W}} \int_{\log_{10}\omega_L}^{\log_{10}\omega_H} \left\|\tilde{C}(j\omega) W(j\omega)^{-1}\right\|_F^2 \, d(\log_{10}\omega)$$

given in optimisation (3). To this end, first define

$$\begin{aligned}C(\omega) &:= \left[\tilde{C}(j\omega)^* \tilde{C}(j\omega)\right] \\ &= \mathrm{diag}\left(|\tilde{c}_1(j\omega)|^2, |\tilde{c}_2(j\omega)|^2, \ldots, |\tilde{c}_n(j\omega)|^2\right).\end{aligned}$$

Then define the set $\mathbf{\Upsilon} := \left\{f : \mathbb{R} \to \mathbb{R}^n_+\right\}$ and let

$$x(\omega) := X(\omega)\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1\end{bmatrix} \in \mathbf{\Upsilon} \text{ and } c(\omega) := C(\omega)\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1\end{bmatrix}.$$

Thus $x(\omega)$ and $c(\omega)$ are n-dimensional vector functions containing the n diagonal elements of the diagonal matrix functions $X(\omega)$ and $C(\omega)$ respectively. Also, since $\tilde{C}(j\omega)$ is chosen by the designer prior to the optimisation, then so is $c(\omega)$. Fact 4.2 gives such an equivalent objective function.

Fact 4.2 *Let $x(\omega)$ and $c(\omega)$ be as defined above. Then,*

$$\begin{aligned}&\min_{W \in \mathcal{W}} \int_{\log_{10}\omega_L}^{\log_{10}\omega_H} \left\|\tilde{C}(j\omega) W(j\omega)^{-1}\right\|_F^2 \, d(\log_{10}\omega) \\ &\qquad = \min_{x \in \mathbf{\Upsilon}} \int_{\log_{10}\omega_L}^{\log_{10}\omega_H} c(\omega)^T x(\omega) \, d(\log_{10}\omega).\end{aligned}$$

Proof Simple algebraic manipulations immediately show that the integrands are equivalent and the rest trivially follows from the definitions of the sets $\mathcal{W}$ and $\mathbf{\Upsilon}$. □

4.6 A Sub-Optimal Problem with Tighter Constraints

Using the results obtained in Sections 4.2 to 4.5, it can be seen that the following problem:

$$\min_{x \in \Upsilon} \int_{\log_{10}\omega_L}^{\log_{10}\omega_H} c(\omega)^T x(\omega)\, d(\log_{10}\omega)$$

such that

$$\exists \bar{Q} \in \mathbb{R}^{p \times (N+1)q},\ \exists D_\omega \in \mathcal{D} \text{ satisfying} \tag{7}$$

$$\begin{bmatrix} (\rho+1)I & \bar{Q}B(\infty)D_{33} - \rho I \\ * & (\rho+1)I \end{bmatrix} > 0$$

$$\begin{bmatrix} \begin{pmatrix} (D_\omega^{l*}D_\omega^l)^{-1} & 0 \\ 0 & d_\omega^{-2}X(\omega) \end{pmatrix} & \mathcal{F}_l\left(T(j\omega), \bar{Q}B(j\omega)\right) \\ * & \begin{pmatrix} (D_\omega^{r*}D_\omega^r) & 0 \\ 0 & d_\omega^2 I_m \end{pmatrix} \end{bmatrix} > 0 \ \forall\omega,$$

is sub-optimal to optimisation (3) presented in Section 4.1 as it has tighter constraints. Furthermore, restraining $x(\omega)$ to be in set Υ in the above minimisation is superfluous as this is implicitly ensured by the last constraint. Recall also that $\mathcal{F}_l\left(T(j\omega), \bar{Q}B(j\omega)\right)$ appearing above is affine in $\bar{Q}$, as seen from the definition of $T(s)$ in Fact 4.1.

5 The Solution Algorithm

Since optimisations (6) and (7) are interdependent, then the solution algorithm described below alternates between these two minimisations resulting into an iterative scheme. This iterative scheme is not guaranteed to converge to the global minimum (similarly to D-K iterations) but will at least be monotonically non-increasing as explained below.

Furthermore, optimisations (6) and (7) involve a search over a functional set with constraints holding for all $\omega \in \mathbb{R}$. Thus, in order to reduce these minimisation problems to finite-dimensional problems, frequency gridding is required. Of course, this gridding must be chosen *dense* enough so as not to miss rapid changes in the transfer function matrices. Also, the frequency gridding of the constraints can be chosen to be denser, if so desired, than that of the objective function in (7) and over a larger frequency range (say, from a decade below ω_L to a decade above ω_H), as this gives some better confidence that the actual functional constraints are not violated. Further details will be omitted from this paper for the sake of brevity. Hereafter, let the grid-point frequencies be denoted by ω_i.

Then, the proposed iterative procedure is summarised by the following set of steps:

1. Select a feasible initial starting point for the algorithm. That is, select some $D_{\omega_i} = \mathrm{diag}(D_{\omega_i}^l, D_{\omega_i}^r, d_{\omega_i}) \in \mathcal{D}\ \forall\omega_i$ such that the constraints of the optimisation in Step 2 below are not violated. A systematic procedure for selecting such a feasible initial starting point is available but will not be discussed here due to space restrictions. Setting $D_{\omega_i}^l = I_r$, $D_{\omega_i}^r = I_s$ and d_{ω_i} as a sufficiently large number for all ω_i is usually good enough.

2. Solve the finite-dimensional LMI minimisation problem obtained by frequency gridding optimisation (7) and holding $D_{\omega_i}^l$, $D_{\omega_i}^r$ and d_{ω_i} fixed at the values obtained in the previous step. This optimisation outputs the value of the minimum cost, and the LMI variables $\bar{Q}$ and $X(\omega_i)$ for all ω_i. If the minimum cost has not decreased in the last few iterations, then STOP here.

3. Solve the finite-dimensional LMI minimisation problem obtained by frequency gridding optimisation (6) and holding $\bar{Q}$ and $X(\omega_i)$ fixed at the values obtained in the previous step. This optimisation outputs the value of the minimum γ_{ω_i} at each frequency ω_i, and the LMI variables $D_{\omega_i}^l$, $D_{\omega_i}^r$ and d_{ω_i} $\forall\omega_i$.

4. Go back to Step 2.

Note that Step 2 ensures that $\max_{\omega_i}\gamma_{\omega_i} \leq 1$ and Step 3 minimises γ_{ω_i} further. This immediately guarantees Robust Performance. Moreover, as the iterations proceed, the minimum cost obtained at Step 2 is monotonically non-increasing. This is because the values of $\bar{Q}$ and $X(\omega_i)$ obtained at Step 2 in the current iteration always satisfy the constraints of the optimisation in the same Step 2 during the next iteration. Finally, the controller $K(s) \in \mathcal{K}$ which achieves this Robust Performance can be computed when the iterations are over using $Q(s) = \bar{Q}B(s)$ and the parametrisation of all stabilising controllers (see for example, Theorem 12.8 in [13]).

6 Numerical Example

The iterative procedure proposed above will now be illustrated by a numerical example. Consider the block diagram shown in Figure 2. The actual plant is uncertain but is known to be-

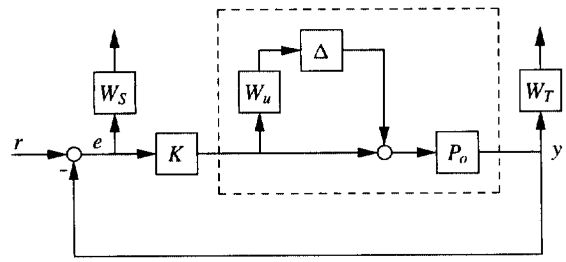

Figure 2: Block diagram for a typical S/T problem

long to the plant set $\{P_o(1 + \Delta W_u) : \Delta \in \mathcal{RH}_\infty, \|\Delta\|_\infty \leq 1\}$ parametrised by Δ. Here, the nominal plant P_o was chosen as $\frac{10}{s(s+10)}$ and the uncertainty weight W_u as $\frac{10(s+5)}{(s+100)}$. The chosen W_u allows the magnitude of the actual plant to differ from that of the nominal plant by as much as 50% in the low-frequency region and by as much as 1000% in the high-frequency region.

It is required to find some internally stabilising controller $K(s)$ such that the performance weights W_S and W_T are maximised according to a pre-stated desired directionality. For a sensible control problem, this directionality should be chosen so that W_S is maximised in the low-frequency region and W_T is maximised in the high-frequency region. The directionality used in this example is shown in Figure 3.

The frequency range $[\omega_L, \omega_H]$ selected for maximising the performance weights was $[10^{-2}, 10^2]$, whereas that selected for gridding the constraints was $[10^{-3}, 10^3]$. Furthermore, $T = 0.2$ and $N = 1$ were found to be sufficiently good, as smaller values of T and larger values of N did not give any

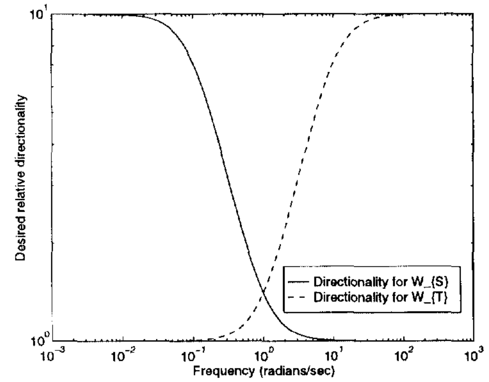

Figure 3: Desired directionality for the optimisation

improvement. The value of ρ was completely irrelevant here, as P_o is strictly proper. Five iterations were found to be sufficient and the total iteration time taken was 7 minutes.

Figure 4 shows plots for the final $|W_S^{-1}|$ and $|W_T^{-1}|$, obtained after completing the iterations. The Sensitivity function for the actual plant will always be below the $|W_S^{-1}|$ plot, whereas that for Complementary Sensitivity will always be below the $|W_T^{-1}|$ plot.

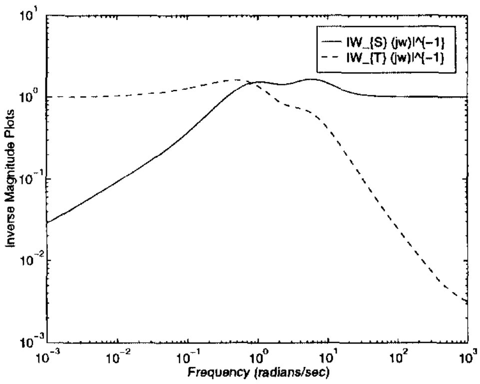

Figure 4: Inverse of magnitude of performance weights

The controller which achieves this maximised performance was computed after all five iterations have been completed and is given by the following state-space representation:

$$K(s) = \left[\begin{array}{cccc|c} -5.80 & 2.09 & -51.63 & 26.83 & -7.02 \\ -3.06 & -10.07 & 2.21 & -0.87 & 0.28 \\ 0 & 1.00 & -1.74 & 0 & -0.17 \\ 0 & 0 & 20.00 & -10.00 & 2.00 \\ \hline -3.06 & -0.07 & 1.68 & -0.87 & 0.23 \end{array}\right].$$

This controller, together with the above performance weights, gave a flat curve across frequency for the computed upper bound of μ. This confirms that performance has been maximised for the particular plant set considered while robust stability was maintained.

7 Conclusions

The problem of maximising performance weights in the frequency range of interest while maintaining robust stability was posed as an optimisation problem, to which a sub-optimal solution was given. A major advantage of this procedure over existing methods (such as D-K iterations) is that the performance weights which maximise some sensible cost function and a robustly stabilising controller are synthesised simultaneously by one algorithm. The resulting closed-loop μ-curve is usually flat across frequency and very close to, but slightly less than, unity. This confirms that robust performance is maximised without the need for a long and tedious trial and error process by the designer to select "good" performance weights.

The proposed scheme does, however, have some important disadvantages. The Laguerre-like parametrisation $Q(s) = \bar{Q}B(s)$ usually causes high-order controllers to be synthesised. Furthermore, frequency gridding causes loss of information between grid-points and hence a dense grid can only give confidence that $\mu < 1$ rather than absolute certainty. These issues are currently under investigation by the authors and state-space techniques are being developed to address these problems.

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