Authors' Camera Ready Manuscript. To appear in the
Proceedings of the 59th IEEE Conference on Decision and Control, Jeju Island, Republic of Korea, Dec 2020,
Please cite using bibliographic data of the associated published version.

Time-domain output negative imaginary systems and its connection to dynamic dissipativity

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***Abstract*— This paper introduces the notion of time-domain output (not necessarily strictly) negative imaginary systems that encompasses all the existing (not only stable) subsets of the negative imaginary systems class. A time-domain *dynamic dissipative* framework is developed in this paper to describe the full class of output negative imaginary systems (i.e. including systems having poles on the imaginary axis). This dynamic dissipative framework also leads to an LMI-based state-space characterization which can be conveniently used to test the output negative imaginary property of a given LTI system. It is further established that for time-domain output negative imaginary systems, the proposed dynamic dissipative framework is equivalent to the classical dissipative property of this class of systems with respect to the supply rate $w(u, \dot{\bar{y}})$. Interestingly, the class of time-domain output *strictly* negative imaginary systems now captures systems that are not necessarily stable – it is found that the majority of the SISO, LTI transfer functions with a simple pole at the origin satisfy this property. Numerical examples are provided throughout the paper for better illustration.**

I. INTRODUCTION

Negative Imaginary (NI) systems theory was introduced in [1] and was primarily inspired by the positive position feedback control of highly resonant mechanical structures with colocated position sensors and force actuators [2]. NI theory has formalised and unified many existing vibration control techniques (e.g. graphical techniques, integral resonant control schemes) developed for lightly damped flexible structures using positive position feedback [1]. Negative imaginary systems property is closely related to counter-clockwise input-output dynamics in a nonlinear setting [3], to input-output Hamiltonian systems in both linear and nonlinear setting [4], [5] and to a class of dissipative systems involving the input to the system and the time-derivative of the system's output [6]. NI systems theory has earned its popularity mainly due its simple internal stability condition that depends only on the DC loop gain information and hence the theory can be easily applied to practical systems for which the exact mathematical model is unavailable. NI theory finds potential applications in vibration control of lightly damped flexible structures [1], cantilever beams [7], large space structures [8] and robotic manipulators [8], in control of nano-positioning systems [9], in control of large vehicle platoons [10], etc.

Negative imaginary control techniques can also be considered to fall in the arena of energy-based control similar to passivity-based control and hence it has strong resemblance with the dissipativity theory [11]. In [6], the connections between the Output Strictly Negative Imaginary (OSNI) systems property and classical dissipativity with respect to a particular supply rate $w(u, \dot{\bar{y}})$ with an index $\delta > 0$ has been investigated. In addition to OSNI systems, all stable NI systems are shown to be dissipative in [6] with respect to the same supply rate except for $\delta = 0$. [6] has also corrected the earlier supply rates available in the literature [12], [13] to define OSNI systems. However, the analysis of [6] was confined to only stable OSNI systems. In this paper, the notion of time-domain Output Negative Imaginary (ONI) systems is introduced that includes an wide variety of LTI systems with possible poles on the imaginary axis compared to [6], [12]–[14]. The time-domain definition has the advantage that it does not impose any additional constraints on the underlying system, such as, asymptotic stability, minimality, full normal rank, etc., and owing to this, the proposed ONI class accommodates the entire class of NI systems within it. Note however that with the constraints back in place, the proposed ONI definition will specialise to the existing frequency-domain conditions. Surprisingly, the strict subset of the proposed time-domain ONI class, called as time-domain OSNI (with $\delta > 0$) systems, captures also the systems with a simple pole at the origin [e.g. $\frac{1}{s}$, $\frac{1}{s(s+1)}$].

A dynamic dissipative framework (see Fig. 3), termed as (Σ, $\bar{Q}$)-dissipativity where Σ is a specific LTI operator and $\bar{Q} = \bar{Q}^T$ is a real matrix of appropriate dimension, is developed in this paper to describe the full class of time-domain ONI systems (i.e., including the systems with poles on the imaginary axis). The dynamic dissipative framework also leads to derive an LMI-based state-space characterisation to test the time-domain ONI property of a given LTI system. Finally, it is established that for time-domain ONI systems, dynamic dissipativity with respect to (Σ, $\bar{Q}$), classical dissipativity (in the sense of Willems [11]) with respect to the supply rate $w(u, \dot{\bar{y}})$ and the proposed state-space characterisation all are equivalent. Note that when restricted to only stable ONI systems, the present theoretical analysis captures the findings of [6].

Notation: The notations are standard throughout. $\mathbb{R}_{\geq 0}$ and $\mathbb{R}_{>0}$ denote respectively the sets of all non-negative and all positive real numbers. The space of all real-valued, absolutely square integrable, time-domain functions is de-

This work was supported by the Engineering and Physical Sciences Research Council (EPSRC) [grant number EP/R008876/1]. All research data supporting this publication are directly available within this publication.

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fined by $\mathbb{L}_2^m = \{f : \mathbb{R} \to \mathbb{R}^m : f(t) = 0 \text{ when } t < 0, \int_0^\infty f(t)^T f(t)\,\mathrm{d}t < \infty\}$, while the space of all real-valued, locally square integrable, time-domain functions is defined as $\mathbb{L}_{2e}^m = \{f : \mathbb{R} \to \mathbb{R}^m : f(t) = 0 \text{ when } t < 0, \int_0^T f(t)^T f(t)\,\mathrm{d}t < \infty\ \forall T \in [0,\infty)\}$. An energy supply rate function $w(u, y)$ is an abstraction of the energy inflow into a physical system which is expressed by the mapping $w : \mathbb{U} \times \mathbb{Y} \to \mathbb{R}$, where the input space $\mathbb{U} \in \mathbb{L}_{2e}^m$ and the output space $\mathbb{Y} \in \mathbb{L}_{2e}^p$, and satisfies the property $\int_0^T w(u, y)\,\mathrm{d}t < \infty$ for all admissible $(u, y) \in \mathbb{U} \times \mathbb{Y}$ and $\forall T \in [0,\infty)$. In particular, $\int_0^\infty w(u, y)\,\mathrm{d}t < \infty$ when $(u, y) \in \mathbb{L}_2^m \times \mathbb{L}_2^p$. A function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is termed as C^1 function if it is differentiable in its argument over the entire domain.

II. Technical Preliminaries

In this section, we present essential technical preliminaries, definitions and lemmas which underpin the proofs of the main results of the paper.

A. Definitions for negative imaginary systems theory

We will now recall the definitions of NI and SNI systems.

Definition 1: (**NI System**) [8], [15] Let $M(s)$ be the real, rational and proper transfer function matrix of a finite-dimensional, square and causal system without any poles in the open right-half plane. $M(s)$ is said to be NI if

- $j[M(j\omega) - M(j\omega)^*] \geq 0$ for all $\omega \in (0,\infty)$ except the values of ω where $s = j\omega$ is a pole of $M(s)$;
- If $s = j\omega_0$ with $\omega_0 \in (0,\infty)$ is a pole of $M(s)$, then it is at most a simple pole and the residue matrix $\lim_{s \to j\omega_0} (s - j\omega_0) j M(s)$ is Hermitian and positive semidefinite;
- If $s = 0$ is a pole of $M(s)$, then $\lim_{s\to 0} s^k M(s) = 0$ for all $k \geq 3$ and $\lim_{s\to 0} s^2 M(s)$ is Hermitian and positive semidefinite.

Definition 2: (**SNI System**) [1] Let $M(s)$ be the real, rational and proper transfer function matrix of a finite-dimensional, square and causal system. $M(s)$ is said to be Strictly Negative Imaginary (SNI) if $M(s)$ has no poles in $\{s \in \mathbb{C} : \Re[s] \geq 0\}$ and $j[M(j\omega) - M(j\omega)^*] > 0$ for all $\omega \in (0,\infty)$.

B. Dissipative systems notations and definition

The finite-dimensional, causal, LTI systems studied in this paper are described by the equations

$$M : \begin{cases} \dot{x} = Ax + Bu, \quad x(0) = x_0; \\ y = Cx + Du. \end{cases} \tag{1}$$

The admissible inputs are considered to be in the space $\mathbb{L}_{2e}^m$ such that the unique solution of the state trajectory $x(t)$ exists forward in time $t \geq 0$ and $x \in \mathbb{L}_{2e}^n$. Therefore, the output $y(t)$ also exists and $y \in \mathbb{L}_{2e}^p$. Let us introduce the state transition function Φ, associated with M, being a mapping from $\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \times \mathbb{R}^n \times \mathbb{R}^m$ into $\mathbb{R}^n$. Here, $\Phi(t_1, t_0, x(t_0), u(t))$ denotes the state $x(t_1)$ at time t_1 when the system M starts from an initial state $x(t_0) \in \mathbb{R}^n$ at time $t = t_0$ and an admissible input $u(t)$ is applied on M for the time interval $t \in [t_0, t_1]$.

Let us recall the notion of dissipativity of finite-dimensional, causal, LTI systems introduced in [11].

Definition 3: (**Dissipative systems**) [11] A dynamical system M, given in (1), is said to be dissipative with respect to an energy supply rate $w(u, y)$ if there exists a function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, called the storage function, such that

$$V(x(0)) + \int_0^T w(u, y)\,\mathrm{d}t \geq V(x(T)) \tag{2}$$

for any $T \in [0,\infty)$, any initial condition $x(0) \in \mathbb{R}^n$ and any input $u \in \mathbb{L}_{2e}^m$ where $x(T) = \Phi(T, 0, x(0), u(t))$ and $w(u, y)$ has been evaluated along any trajectory of (1).

Inequality (2) is known as the 'dissipation inequality' in the sense of Willems. If $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is a differentiable storage function, then the dissipation inequality (2) can be expressed in the differential form as

$$w(u, y) \geq \dot{V}(x). \tag{3}$$

If the supply rate function is specialised to $w(u, y) = y^T Q y + 2y^T S u + u^T R u$ following Hill-Moylan's framework [16] where $Q = Q^T \in \mathbb{R}^{p\times p}$, $S \in \mathbb{R}^{p\times m}$ and $R = R^T \in \mathbb{R}^{m\times m}$, then (3) takes the form

$$y^T Q y + 2y^T S u + u^T R u \geq \dot{V}(x). \tag{4}$$

Note that for finite-dimensional LTI systems with a minimal state-space realisation, the storage function $V(x)$ can be characterized with a quadratic form $x^T P x$, without loss of generality, where $P = P^T \geq 0$ [11], [17]. Also, in the LTI setting, the storage function $V(x)$ can always be assumed to be a differentiable function of $x \in \mathbb{R}^n$ [16].

For a dissipative system with a completely controllable state-space, the 'required supply' is defined as [11], [18]

$$V_r(x_1) = \inf_{\substack{x^* \to x_1 \\ u(\cdot),\, T \leq 0}} \int_T^0 w(u, y)\,\mathrm{d}t \tag{5}$$

where $x^* \in \mathbb{R}^n$ represents the point of minimum storage. In general, the origin of a state-space is the point of minimum storage where $V(x^*) = V(0) = 0$. The 'required supply' is the least amount of energy required to excite a system to a desired state from the state of minimum energy level [18]. $V_r(x)$ is a possible storage function for any dissipative system with a completely controllable state-space.

C. Dynamic dissipativity theory

We will now set the notations and definition for dynamic dissipativity. A system M is said to be *dynamically dissipa-*

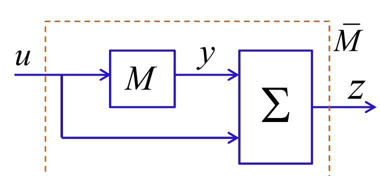

Fig. 1: Cascaded interconnection of M and Σ.

tive if the system M cascaded with another given dynamic system Σ is dissipative [19]. The concept is depicted through Fig. 1.

Definition 4: A finite-dimensional, causal, square and LTI system M is said to be dynamically dissipative, termed as $(\Sigma, \bar{Q})$-dissipative where $\bar{Q} = \bar{Q}^T \in \mathbb{R}^{\hat{l}\times\hat{l}}$ and the LTI operator Σ has a real, rational transfer function representation $\Sigma(s)$, if the cascaded interconnection $(\bar{M})$ of M and Σ shown in Fig. 1 is dissipative with respect to the supply rate $z^T\bar{Q}z$ where z is evaluated along the trajectories of the combined system $\bar{M}$ (i.e. $z = \Sigma\begin{pmatrix} y \\ u \end{pmatrix}$).

Let $x(t) \in \mathbb{R}^n$ and $\hat{x}(t) \in \mathbb{R}^{\hat{n}}$ denote respectively the states of M and Σ. Definition 4 implies that for a $(\Sigma, \bar{Q})$-dissipative system M with respect to particular $\bar{Q} = \bar{Q}^T \in \mathbb{R}^{\hat{l}\times\hat{l}}$ and $\Sigma(s)$ there will always exist a storage function $V : \mathbb{R}^{n+\hat{n}} \to \mathbb{R}_{\geq 0}$ such that

$$V(\bar{x}(0)) + \int_0^T z^T\bar{Q}z\,\mathrm{d}t \geq V(\bar{x}(T)) \tag{6}$$

for all $T \in [0,\infty)$, all $u \in \mathbb{L}_{2e}^m$ and where $\bar{x} = [x\ \hat{x}]^T$. In general, for LTI cases, the storage function $V(\bar{x})$ can be assumed to be a differentiable function of $\bar{x} \in \mathbb{R}^{n+\hat{n}}$ and owing to this (6) becomes equivalent to $z^T\bar{Q}z \geq \dot{V}(\bar{x})$. Note that in particular applications, Σ can also be a linear dynamical operator, such as, $\dfrac{d}{\mathrm{d}t}(\cdot)$ or $\int(\cdot)\,\mathrm{d}t$.

III. Time-domain output negative imaginary systems

At its inception, OSNI/ONI systems property was defined only for stable LTI systems [6], [14]. In this paper, ONI systems has been defined in time-domain and in doing so, it captures a bigger class of LTI systems (i.e., with possible poles on the $j\omega$ axis) compared to its earlier definitions [6], [14]. The proposed ONI definition covers the full set of NI systems and does not impose any a priori assumptions on the system. Note that in this section, the admissible inputs u are considered to be in the space $\mathbb{L}_{2e}^m$ along with sufficient smoothness properties such that unique solution of the state trajectory $x(t)$ exists forward in time $t \geq 0$ and also $x \in \mathbb{L}_{2e}^n$. Hence $\dot{\bar{y}}(t) = C\dot{x}(t) = CAx(t) + CBu(t)$ also exists forward in time $\forall t \geq 0$ and $\dot{\bar{y}} \in \mathbb{L}_{2e}^m$.

Definition 5: (**Time-domain ONI systems**) Let M be a finite-dimensional, square and causal system given by the state-space equations $\dot{x} = Ax + Bu$, $x(0) = 0$ and $y = Cx + Du$. Define $\bar{y} = y - Du$. Then M is said to be a time-domain Output Negative Imaginary (ONI) system with a level of output strictness $\delta \geq 0$ if

$$\int_0^T 2\dot{\bar{y}}^T u - \delta\,\dot{\bar{y}}^T\dot{\bar{y}}^T\,\mathrm{d}t \geq 0 \tag{7}$$

for all admissible $u \in \mathbb{L}_{2e}^m$ and all $T \in [0,\infty)$. M is said to be a time-domain Output Strictly Negative Imaginary (OSNI) system when $\delta > 0$.

Example 1: Consider the transfer function $\dfrac{1}{s+1}$ with a minimal state-space description $\dot{x} = -x + u,\ x(0) = 0;$ and $y = x$. Now we compute $2\dot{y}u - \delta\,\dot{y}^2 = 2\dot{x}u - \delta\,\dot{x}^2 = 2x\dot{x} + 2\dot{x}^2 - \delta\dot{x}^2$ where $\delta \geq 0$ is the output negative-imaginary index. Integrating the above terms with respect to t from 0 to $T \in [0,\infty)$, we get $\int_0^T(2\dot{y}u - \delta\,\dot{y}^2)\mathrm{d}t = \int_{x^2(0)}^{x^2(T)}\mathrm{d}(x^2) + (2-\delta)\int_0^T \dot{x}^2\,\mathrm{d}t = [x^2(T) - x^2(0)] + (2-\delta)\int_0^T \dot{x}^2\,\mathrm{d}t \geq 0$ for all $T \in [0,\infty)$, all $u \in \mathbb{L}_{2e}$ and for any $\delta \in (0,2]$. Hence, $\dfrac{1}{s+1}$ is an OSNI system with $\delta \in (0,2]$ via Definition 5. Note that the above inequality also holds for $\delta = 0$ which signifies that the system is also a stable ONI (or simply a stable NI) system.

Example 2: Consider a single integrator plant $\dfrac{1}{s}$ having a state-variable representation $\dot{x} = u,\ x(0) = 0;$ and $y = x$. Similar to Example 1 it can be shown that $\dfrac{1}{s}$ satisfies OSNI property with $\delta \in (0,2]$.

Example 3: Consider a double integrator system $\dfrac{1}{s^2}$ with a minimal state-space description $\dot{x_1} = x_2,\ \dot{x_2} = u,\ x_1(0) = x_2(0) = 0$ and $y = x_1$. Following the similar procedure as before, we find $2\dot{y}u - \delta\dot{y}^2 = 2\dot{x}_1u - \delta\,\dot{x}_1^2 = 2x_2u - \delta\,x_2^2 = 2x_2\dot{x}_2 - \delta\,x_2^2$. Integrating the above expression with respect to t from 0 to $T \in [0,\infty)$ we have, $\int_0^T(2\dot{y}u - \delta\dot{y}^2)\mathrm{d}t = \int_{x_2^2(0)}^{x_2^2(T)}\mathrm{d}(x_2^2) - \delta\int_0^T x_2^2\,\mathrm{d}t = [x_2^2(T) - x_2^2(0)] - \delta\int_0^T x_2^2\,\mathrm{d}t \ngeq 0$ for any $\delta > 0$. Hence $\dfrac{1}{s^2}$ is not an OSNI system. However, $\int_0^T 2\dot{y}u\,\mathrm{d}t = x_2^2(T) \geq 0$ for all $T \in [0,\infty)$ and all $u \in \mathbb{L}_{2e}$ since $x_2(0) = 0$. Hence, $\dfrac{1}{s^2}$ is an ONI system with $\delta = 0$ via Definition 5.

A. Time-domain ONI systems in Willems's dissipative framework

In this subsection, we establish that any time-domain ONI system M with a completely controllable state-space realization satisfies the dissipativity property (in the sense of Willems [11]) with respect to the supply rate $w(u,\dot{\bar{y}}) = 2\dot{\bar{y}}^Tu - \delta\dot{\bar{y}}^T\dot{\bar{y}}$ for some $\delta \geq 0$ where $\bar{y} = y - M(\infty)u$ is an auxiliary output of the system M.

Theorem 1: Let M be a finite-dimensional, causal and square system given by the state-space equations $\dot{x} = Ax + Bu$ and $y = Cx + Du$ with zero initial condition and the state-space being completely controllable. Also let $\delta \geq 0$ and define $\bar{y} = y - Du$. Then M is dissipative with respect to the supply rate $w(u,\dot{\bar{y}}) = 2\dot{\bar{y}}^Tu - \delta\dot{\bar{y}}^T\dot{\bar{y}}$ and $D = D^T$ if M is a time-domain ONI system with a level of output strictness δ.

Proof. To show that an ONI system M with $\delta \geq 0$ is dissipative with respect to the supply rate $w(u,\dot{\bar{y}}) = 2\dot{\bar{y}}^Tu - \delta\dot{\bar{y}}^T\dot{\bar{y}}$, we have to establish that there exists a storage function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ with $V(0) = 0$ such that M satisfies the dissipation inequality (2). Since the state-space is assumed to be completely controllable, there exists an admissible input $u(t)$ defined as

$$u(t) = \begin{cases} 0 & \text{when } t < t_{-1}; \\ \tilde{u}(t) & \text{when } t_{-1} \leq t \leq 0; \\ 0 & \text{when } t > 0, \end{cases}$$

which steers the system from $x(t_{-1}) = 0$ to any $x(0) \in \mathbb{R}^n$. In this proof, let $y(t)$ be the corresponding output and define $\bar{y} = y - Du$. Now,

$$\int_{t_{-1}}^{0} w(u,\dot{\bar{y}})\,\mathrm{d}t = \int_{t_{-1}}^{0} (2\dot{\bar{y}}^T u - \delta\dot{\bar{y}}^T\dot{\bar{y}})\,\mathrm{d}t$$
$$= \int_{t_{-1}}^{T} (2\dot{\bar{y}}^T u - \delta\dot{\bar{y}}^T\dot{\bar{y}})\,\mathrm{d}t + \delta\int_{0}^{T} \dot{\bar{y}}^T\dot{\bar{y}}\,\mathrm{d}t \qquad \forall T \in [0,\infty)$$
[since M is causal and time-invariant]
$$\geq \int_{t_{-1}}^{T} (2\dot{\bar{y}}^T u - \delta\dot{\bar{y}}^T\dot{\bar{y}})\,\mathrm{d}t \quad \forall T \in [0,\infty) \quad [\text{since } \delta \geq 0]$$
$$= \int_{0}^{\bar{T}} (2\dot{\bar{y}}^T u - \delta\dot{\bar{y}}^T\dot{\bar{y}})\,\mathrm{d}\tau \geq 0 \qquad \forall \bar{T} \in [0,\infty)$$

via Definition 5 applying a change of time variable $\tau = t - t_{-1}$ where $-\infty < t_{-1} \leq 0$ and denoting $\bar{T} = T - t_{-1}$. Hence, for arbitrary $t_{-1} \leq 0$ and $x(t_{-1}) = 0$ we have, $\int_{t_{-1}}^{0} w(u,\dot{\bar{y}})\,\mathrm{d}t \geq 0$. We now construct the required supply function as $V_r(x) = \inf\limits_{\substack{x^*=0\to x\\ u(\cdot),\, t_{-1}\leq 0}} \int\limits_{t_{-1}}^{0} w(u,\dot{\bar{y}})\,\mathrm{d}t \geq 0$ where origin is the point of minimum storage (i.e., $x^* = 0$). Thus, $V_r(x)$ can be considered as a storage function candidate associated with the ONI system M.

It remains to be shown that $V_r(x)$ satisfies the dissipation inequality (2). Towards this end, note that in taking the system from $x = 0$ at $t = 0$ to $x_1 \in \mathbb{R}^n$ at $t = t_1$, we could first take it to $x_0 \in \mathbb{R}^n$ at time t_0 while minimizing the energy, and then take it to x_1 at time t_1 along the path for which the dissipation inequality is to be evaluated. This is possible due to M being a causal and time-invariant system. Since $V_r(x_1)$ gives the infimum of the amount of energy required to reach x_1 at $t = t_1$ from $x = 0$ at $t = 0$, hence the energy required to reach the same destination x_1 from the same starting point $x = 0$ via any other path will be greater than or equal to $V_r(x_1)$. Therefore, $V_r(x_0) + \int_{t_0}^{t_1} w(u,\dot{\bar{y}})\,\mathrm{d}t \geq V_r(x_1)$ follows. It can hence be concluded that the ONI system M is dissipative with respect to the supply rate $w(u,\dot{\bar{y}}) = 2\dot{\bar{y}}^T u - \delta\dot{\bar{y}}^T\dot{\bar{y}}$ for the same δ. ■

B. Equivalence between dissipativity of the time-domain ONI systems and its state-space characterization

This subsection proves that the dissipative property of the time-domain ONI systems with respect to the supply rate $w(u,\dot{\bar{y}}) = 2\dot{\bar{y}}^T u - \delta\,\dot{\bar{y}}^T\dot{\bar{y}}$ is equivalent to satisfying the LMI condition (8) [referred to as ONI Lemma]. This LMI condition can be viewed as a state-space characterisation of such systems and can be conveniently used to test the ONI property of a given LTI system.

Theorem 2: Let M be a finite-dimensional, causal, square system described by the state-space equations $\dot{x} = Ax + Bu$ and $y = Cx + Du$ where $x(0) = 0$, $D = D^T$, (A,B,C,D) is minimal and A has no eigenvalues in $\{s \in \mathbb{C} : \Re[s] > 0\}$. Define $\bar{y} = y - Du$ and let a scalar $\delta \geq 0$. Then the following statements are equivalent:

i) M is a time-domain ONI system;
ii) there exists a real matrix $P = P^T \geq 0$ such that

$$\begin{bmatrix} \begin{pmatrix} -PA - A^TP \\ -\delta A^TC^TCA \end{pmatrix} & \begin{pmatrix} -PB + A^TC^T \\ -\delta A^TC^TCB \end{pmatrix} \\ \begin{pmatrix} -PB + A^TC^T \\ -\delta A^TC^TCB \end{pmatrix}^T & \begin{pmatrix} CB + B^TC^T \\ -\delta B^TC^TCB \end{pmatrix} \end{bmatrix} \geq 0. \tag{8}$$

iii) M is dissipative with respect to the supply rate $w(u,\dot{\bar{y}}) = 2\dot{\bar{y}}^T u - \delta\,\dot{\bar{y}}^T\dot{\bar{y}}$.

The storage function in part iii) is chosen as $V(x) = x^TPx$ with $P = P^T \geq 0$ from part ii).

Proof: ii) $\Rightarrow$ iii) For the sake of convenience, we denote the following shorthand

$$\Pi = \begin{bmatrix} \begin{pmatrix} -PA - A^TP \\ -\delta A^TC^TCA \end{pmatrix} & \begin{pmatrix} -PB + A^TC^T \\ -\delta A^TC^TCB \end{pmatrix} \\ \begin{pmatrix} -PB + A^TC^T \\ -\delta A^TC^TCB \end{pmatrix}^T & \begin{pmatrix} CB + B^TC^T \\ -\delta B^TC^TCB \end{pmatrix} \end{bmatrix}. \tag{9}$$

There exists $P = P^T \geq 0$ such that $\Pi \geq 0$

$\Leftrightarrow$ there exists $P = P^T \geq 0$ such that
$\begin{bmatrix} x^T & u^T \end{bmatrix} \Pi \begin{bmatrix} x \\ u \end{bmatrix} \geq 0$ for all $\begin{bmatrix} x \\ u \end{bmatrix} \in \mathbb{R}^{n+m}$

$\Leftrightarrow$ there exists $P = P^T \geq 0$ such that $2(CAx + CBu)^Tu$ $- \delta(CAx + CBu)^T(CAx + CBu) \geq x^T(PA + A^TP)x$ $+ 2x^TPBu$ for all $x \in \mathbb{R}^n$ and all $u \in \mathbb{R}^m$

$\Rightarrow$ there exists $P = P^T \geq 0$ such that the C^1 storage function $V(x) = x^TPx$ satisfies $2\dot{\bar{y}}^Tu - \delta\dot{\bar{y}}^T\dot{\bar{y}} \geq \dot{V}(x)$ evaluated along any trajectory of M subjected to any admissible input $u \in \mathbb{L}_{2e}^m$

$\Leftrightarrow$ there exists a C^1 storage function $V(x) = x^TPx$ with $P = P^T \geq 0$ such that $\int_0^T 2\dot{\bar{y}}^Tu - \delta\dot{\bar{y}}^T\dot{\bar{y}}\,\mathrm{d}t \geq V(x(T))$ $- V(x(0))$ for all $T \in [0,\infty)$ and evaluated along any trajectory of M subjected to any admissible $u \in \mathbb{L}_{2e}^m$

$\Leftrightarrow$ M is dissipative with respect to the supply rate $w(u,\dot{\bar{y}})$ $= 2\dot{\bar{y}}^Tu - \delta\dot{\bar{y}}^T\dot{\bar{y}}$ and a specific C^1 storage function $V(x) = x^TPx$ with $P = P^T \geq 0$.

iii) $\Rightarrow$ ii) This part follows directly from [6, Theorem 2] except the difference that P is now positive semidefinite.

i) $\Rightarrow$ iii) This implication is already established in Theorem 1.

iii) $\Rightarrow$ i) Since M is dissipative with respect to the supply rate $w(u,\dot{\bar{y}}) = \dot{\bar{y}}^Tu - \delta\,\dot{\bar{y}}^T\dot{\bar{y}}$ and has a minimal state-space realization, it ensures that there always exists a storage function $V(x) = x^TPx$ with $P = P^T \geq 0$ such that

$$V(x(0)) + \int_0^T (2\dot{\bar{y}}^Tu - \delta\dot{\bar{y}}^T\dot{\bar{y}})\mathrm{d}t \geq V(x(T)) \tag{10}$$

for all $T \in [0,\infty)$, all $u \in \mathbb{L}_{2e}^m$ and for some $\delta \geq 0$. The above expression implies

$$\int_0^T (2\dot{\bar{y}}^T u - \delta \dot{\bar{y}}^T \dot{\bar{y}}) \mathrm{d}t \geq 0 \tag{11}$$

for all $T \in [0,\infty)$ and all $u \in \mathbb{L}_{2e}^m$ since $x(0) = 0$ and $V(x(T)) \geq 0$. Hence, M is a time-domain ONI system with a level of output strictness $\delta \geq 0$ according to Definition 5.

Combining all the aforementioned parts, it can now be confirmed that i) $\Leftrightarrow$ ii) $\Leftrightarrow$ iii). ∎

C. Classification of time-domain ONI systems

We will now classify the strict ($\delta > 0$) and non-strict ($\delta = 0$) subsets of the time-domain ONI systems on the basis of the parameter $\delta \geq 0$ by applying the LMI condition (7). Fig. 2 illustrates the classification through a comprehensive Venn diagram.

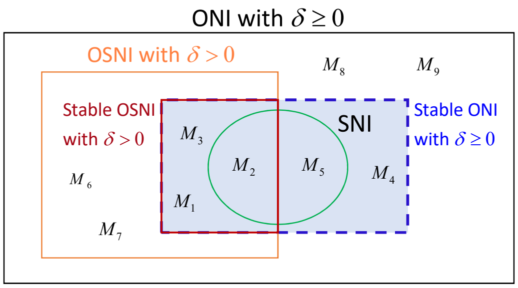

Fig. 2: Relationship among the subsets of the class of time-domain ONI systems. $M_1, M_2, \cdots, M_9$ denote the examples of the corresponding subsets.

1) Stable OSNI [6] if A is Hurwitz and there exist $P = P^T > 0$, $\delta > 0$ such that (8) holds. For example, $M_1(s) = \frac{s^2+12.5}{s^4+s^3+42.5s^2+12.5s+150}$, $M_2(s) = \frac{1}{s+1}$, $M_3(s) = \frac{1}{s+1}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$;
2) Stable ONI if A is Hurwitz and there exist $P = P^T > 0$, $\delta \geq 0$ such that (8) holds. The stable ONI class contains i) the stable NI systems (with $\delta = 0$) [1] which are neither SNI nor OSNI, e.g., $M_4(s) = \frac{2s^2+s+1}{(s+1)(2s+1)(s^2+2s+5)}$; ii) the SNI systems which are not OSNI [6], e.g., $M_5(s) = \frac{s+4}{s^2+8s+32}$; and iii) all stable OSNI with $\delta > 0$ systems [6].
3) OSNI with a simple pole at origin if $\det[A] = 0$ and there exist $P = P^T \geq 0$, $\delta > 0$ such that (8) holds. For example, $M_6(s) = \frac{1}{s}$, $M_7(s) = \frac{s+4}{s(s+1)}$.
4) ONI with imaginary axis poles if i) $\det[A] \neq 0$ and there exist $P = P^T > 0$, $\delta = 0$ such that (8) holds, e.g., $M_8(s) = \frac{4}{s^2+4}$; ii) $\det[A] = 0$ and there exist $P = P^T \geq 0$, $\delta = 0$ such that (8) holds, e.g., $M_9(s) = \frac{1}{s^2}$; iii) all OSNI systems with a simple pole at $s = 0$.

IV. Time-domain ONI systems in a dynamic dissipative framework

This section underpins the key contribution of this paper. Motivated by [19], a dynamic dissipative framework is developed in Fig. 3 to characterise the time-domain ONI systems. The linear operator Σ has been designed to have a particular configuration represented by the transfer function $\Sigma(s) = \begin{bmatrix} sI_m & 0 \\ 0 & I_m \end{bmatrix}$ such that the output of the cascaded system $\bar{M}$ becomes $z = \begin{bmatrix} \dot{\bar{y}}^T & u^T \end{bmatrix}^T$. $\bar{M}$ has the following state-space realization:

$$\bar{M} : \begin{cases} \dot{x} = Ax + Bu, \quad x(0) = x_0; \\ z = \bar{C}x + \bar{D}u, \end{cases} \tag{12}$$

where $\bar{C} = \begin{pmatrix} CA \\ 0 \end{pmatrix}$ and $\bar{D} = \begin{pmatrix} CB \\ I \end{pmatrix}$.

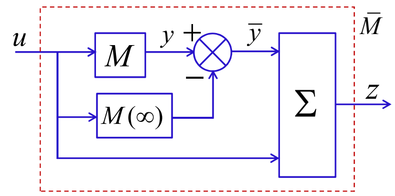

Fig. 3: A dynamic dissipative framework for characterizing time-domain ONI systems.

Theorem 3: Let M be a finite-dimensional, causal, square system described by the state-space equations $\dot{x} = Ax + Bu$ and $y = Cx + Du$ where $x(0) = 0$, $D = D^T$, (A, B, C, D) is minimal and A has no eigenvalues in $\{s \in \mathbb{C} : \Re[s] > 0\}$. Define $\bar{y} = y - Du$ and let a scalar $\delta \geq 0$. Then the following statements are equivalent:

I. M is $(\Sigma, \bar{Q})$-dissipative where Σ has the real, rational transfer function representation $\Sigma(s) = \begin{bmatrix} sI_m & 0 \\ 0 & I_m \end{bmatrix}$ and $\bar{Q} = \begin{bmatrix} -\delta I_m & I_m \\ I_m & 0 \end{bmatrix}$;
II. M is dissipative with respect to the supply rate $w(u, \dot{\bar{y}}) = 2\dot{\bar{y}}^T u - \delta \dot{\bar{y}}^T \dot{\bar{y}}$;
III. there exists $P = P^T \geq 0$ such that $\Pi \geq 0$ where Π is defined in (9);
IV. M is a time-domain ONI system.

The storage function in parts I and II can be chosen as $V(x) = x^T P x$ with $P = P^T \geq 0$ from part III.

Proof. <u>I $\Leftrightarrow$ II</u> We have the following set of equivalent statements:

M is $(\Sigma, \bar{Q})$-dissipative with respect to

$\Sigma(s) = \begin{bmatrix} sI_m & 0 \\ 0 & I_m \end{bmatrix}$ and $\bar{Q} = \begin{bmatrix} -\delta I_m & I_m \\ I_m & 0 \end{bmatrix}$

$\Leftrightarrow$ there exists a C^1 storage function $V(x) = x^T P x$ with $P = P^T \geq 0$ such that $z^T \bar{Q} z \geq \dot{V}(x)$ [using Definition 4]

$$\Leftrightarrow \begin{bmatrix} \dot{\bar{y}}^T & u^T \end{bmatrix}^T \begin{bmatrix} -\delta I_m & I_m \\ I_m & 0 \end{bmatrix} \begin{bmatrix} \dot{\bar{y}} \\ u \end{bmatrix} \geq \dot{x}^T P x + x^T P \dot{x} \tag{13}$$

$\Leftrightarrow M$ is dissipative with respect to the supply rate $w(u,\dot{\bar{y}}) = 2\dot{\bar{y}}^T u - \delta\, \dot{\bar{y}}^T \dot{\bar{y}}$ and a specific C^1 storage function $V(x) = x^T P x$ with $P = P^T \geq 0$.

<u>I $\Leftrightarrow$ III</u> From the first part of the proof, it follows that

$$(13) \Leftrightarrow \exists P = P^T \geq 0 \text{ such that } \Pi \geq 0$$

$[(\Leftarrow)$ part is trivial; $(\Rightarrow)$ part holds via Theorem 2$]$.

<u>II $\Leftrightarrow$ IV</u> Follows from Definition 4 via Theorem 2.

Therefore, it is now established that i) $\Leftrightarrow$ ii) $\Leftrightarrow$ iii) $\Leftrightarrow$ iv). This completes the proof. ■

The following corollaries are immediate consequences of Theorem 3 and underpin significant specialisations of the results derived in Theorem 3 towards stable OSNI [6] systems and NI systems without pole(s) at the origin [15].

Corollary 1: Let M be a finite-dimensional, causal, square system described by the state-space equations $\dot{x} = Ax+Bu$ and $y = Cx+Du$ where $x(0) = 0$ and (A,B,C,D) is minimal. Define $\bar{y} = y - Du$ and let a scalar $\delta > 0$. Then M is stable OSNI with a level of output strictness δ if and only if A is Hurwitz, $D = D^T$ and M is $(\Sigma, \bar{Q})$-dissipative with $\bar{Q} = \begin{bmatrix} -\delta I_m & I_m \\ I_m & 0_{m\times m} \end{bmatrix}$ and $\Sigma(s) = \begin{bmatrix} sI_m & 0 \\ 0 & I_m \end{bmatrix}$.

Proof. This corollary follows from Theorem 3 under the additional assumptions: A is Hurwitz and $\delta > 0$. ■

Corollary 2: Let M be a finite-dimensional, causal, square system described by the state-space equations $\dot{x} = Ax+Bu$ and $y = Cx+Du$ where $x(0) = 0$ and (A,B,C,D) is minimal. Define $\bar{y} = y - Du$. Then M is NI without poles at the origin if and only if A has no eigenvalue in the entire open right-half plane or at the origin, $D = D^T$ and M is $(\Sigma, \bar{Q})$-dissipative with $\bar{Q} = \begin{bmatrix} 0_{m\times m} & I_m \\ I_m & 0_{m\times m} \end{bmatrix}$ and $\Sigma(s) = \begin{bmatrix} sI_m & 0 \\ 0 & I_m \end{bmatrix}$.

Proof. The proof follows directly from Theorem 3 on setting $\delta = 0$ and assuming A has no eigenvalue in the entire open right-half plane or at the origin. ■

V. CONCLUSIONS

This paper defines the class of time-domain Output Negative Imaginary (ONI) systems. This set contains within it the classes of NI systems defined in the previous literature [6], [14] and includes additional systems having poles on the imaginary axis. In doing so, it unifies all the existing subclasses of the NI systems class. This paper also formalises the time-domain definition of ONI (and hence NI) systems and resembles the existing frequency-domain definitions for different subclasses within the NI class which impose additional constraints on the underlying system, such as, stability, minimality, full normal rank, etc. Interestingly, the class of time-domain OSNI systems (with $\delta > 0$) now captures transfer functions with a simple pole at the origin [e.g., $\frac{1}{s}$, $\frac{1}{s(s+1)}$] in addition to stable transfer functions. This finding is in contrast with existing notions which state that any strict NI property can only be defined for stable NI systems. This paper also demonstrates a complete dissipativity analysis of time-domain ONI systems invoking the classical dissipative approach (i.e. following the framework of Willems [11]) and also a new dynamic dissipative approach built on [19]. Finally, the connections among the ONI state-space characterization, classical dissipativity and dynamic dissipativity have been established.

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