Authors' Camera Ready Manuscript. To appear in the
Proceedings of the 57th IEEE Conference on Decision and Control, Miami, FL, USA, Dec 2018,
Please cite using bibliographic data of the associated published version.

Robust output consensus for networks of homogeneous negative imaginary systems

Ola Skeik and Alexander Lanzon

***Abstract*— This paper is concerned with robust output consensus for networks of homogeneous negative imaginary (NI) systems under $\mathfrak{L}_2$ external disturbances and model uncertainty in a generalised framework. By removing certain assumptions which had been imposed in earlier studies, we derive generalised conditions that guarantee robust output consensus of the networked systems by means of recently published generalised internal stability results for NI systems. The proposed conditions are shown to reduce to earlier conditions in literature by imposing the same assumptions. A convergence analysis is also provided which is in agreement with the conclusions of previous literature. An example that demonstrates the effectiveness of the results is also provided.**

I. INTRODUCTION

A dc loop gain condition has been the (one) condition used to test the internal stability of two systems connected in a positive feedback interconnection where one system is negative imaginary (NI) and the other system is strictly negative imaginary (SNI) provided two assumptions at infinite frequency hold. This result, which can be found in [1] and [2], is applicable for the case when the NI system may include poles on the imaginary axis but not at the origin. For the case where the NI system may also include poles at the origin, conditions for internal stability have been proposed in [3] but with the disadvantage of being proposed under restrictive assumptions. Recently, new results for the internal stability of the same feedback interconnection but with the aforementioned assumptions lifted have been published in [4]. These new internal stability results involve conditions that depend on both zero and infinite frequency and generalise the existing results in literature. It is possible therefore, with these new results, to extend the work in [5] on robust output consensus for networked NI systems which was mainly based on the previously existing internal stability results, and this motivates our work in this paper.

Negative imaginary systems are systems with negative imaginary frequency response. The NI systems theory was first introduced in [1] in the interest to tackle stability issues related to flexible structures with co-located force actuators and position sensors in a systematic framework. This area of research has been dominating the field of robust control over the past decade and rapid developments have been witnessed in both theoretical and application sides such as [4], [6], [7], [5], [8], to name a few.

Collective control of multi-agent systems is another area of interest to the control systems community. In particular, consensus control of networked systems has been the centre of much attention over the past two decades. Consensus control is concerned with the design of distributed control laws (also known as protocols) such that all agents in the network reach an agreement on a certain quantity of interest. In relation to consensus/synchronization problems for homogeneous multi-agent systems using relative output measurements we mention [9], [10], [11], [12], [13]. The problems in those papers were addressed from a state-space perspective, observer-based consensus protocols were considered and protocol design involved the solution of Riccati equations or/and linear matrix inequalities. More detailed and comprehensive surveys relating to the topic in general can be found in [14], [15], [16], [17].

In applications where an individual NI system is unable to achieve a desired goal on its own, cooperation among several agents is convenient. The work in [5], [8] explored this issue where the robust output consensus problem was addressed for networks of homogeneous and heterogeneous NI systems, respectively. Specifically, the issue was addressed by reformulating the consensus problem into an internal stability problem, owing to properties of Laplacian matrix of the network graph, and thus providing a solution by means of NI systems robust stability results. However, as mentioned earlier, the results are applicable when certain assumptions hold.

In this paper we address the robust output consensus problem for networks of NI systems by lifting the assumptions imposed in [5]. In particular, we derive generalised results, by use of the generalised internal stability results developed in [4], such that robust output consensus is guaranteed for a network of homogeneous NI systems under $\mathfrak{L}_2$ external disturbances and model uncertainty.

Notation: Let $\mathbb{R}^{m\times n}$ denote the set of $m \times n$ real matrices. Given a matrix A, A^T and A^* are the transpose and the complex conjugate transpose of A respectively. $\bar{\lambda}(A)$ denotes the largest eigenvalue (when the matrix has only real eigenvalues) of A. $\Re[\cdot]$ is the real part of a complex number. I_N is the identity matrix of dimension $N \times N$ and 1_N is an $N \times 1$ vector with entries 1. $A \otimes B$ denotes the Kronecker product of matrices A and B. diag(A_i) represents a block-diagonal matrix with matrices A_i for all $i \in \{1, \dots, N\}$ on the main diagonal. $[P, K]$ represents a positive feedback interconnection between systems P and K. Finally, CLHP

This work was supported by the Engineering and Physical Sciences Research Council (EPSRC) [grant number EP/R008876/1] and a University of Manchester President's Doctoral Scholar Award. All research data supporting this publication are directly available within this publication.

The authors are with the School of Electrical and Electronic Engineering, University of Manchester, Manchester M13 9PL, UK. ola.skeik@manchester.ac.uk, Alexander.Lanzon@manchester.ac.uk

and OLHP stand for closed left half plane and open left half plane, respectively.

II. Preliminaries

A. Negative Imaginary Systems

Negative imaginary systems are defined as follows.

Definition 1 ([3]): A square real, rational, proper transfer function matrix $P(s)$ is termed negative imaginary if

1) $P(s)$ has no poles in $\Re[s] > 0$;
2) $j[P(j\omega) - P^*(j\omega)] \geq 0$ for all $\omega \in (0, \infty)$ except values of ω where $j\omega$ is a pole of $P(s)$;
3) if $j\omega_0$ with $\omega_0 \in (0, \infty)$ is a pole of $P(s)$, then it is a simple pole and the residue matrix $K_0 \triangleq \lim_{s \to j\omega_0}(s - j\omega_0)jP(s)$ is Hermitian and positive semidefinite;
4) if $s = 0$ is a pole of $P(s)$, then $\lim_{s \to 0} s^k P(s) = 0$ $\forall k \geq 3$ and $\lim_{s \to 0} s^2 P(s)$ is Hermitian and positive semidefinite.

Strictly negative imaginary systems are defined as follows.

Definition 2 ([1]): A square real, rational, proper transfer function matrix $K(s)$ is termed strictly negative imaginary if

1) $K(s)$ has no poles in $\Re[s] \geq 0$;
2) $j[K(j\omega) - K^*(j\omega)] > 0$ for all $\omega \in (0, \infty)$.

B. Graph theory

In this paper, we are concerned with undirected graphs. An undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ consists of a nonempty finite vertex set $\mathcal{V} = \{v_1, v_2, \ldots, v_N\}$ and an edge set $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ of unordered pairs of vertices, called edges. An edge in $\mathcal{G}$ is denoted by (v_i, v_j). If $(v_i, v_j) \in \mathcal{E}$, then vertices (i.e., agents) v_i and v_j are adjacent (or neighbours) and can obtain information from each other. The set of neighbours of vertex v_i is defined as $\mathcal{N}_i = \{v_j \in \mathcal{V} : (v_j, v_i) \in \mathcal{E}\}$. Self edges are not allowed, that is, $(v_i, v_i) \notin \mathcal{E}$. A path in a graph from v_i to v_j is a sequence of edges of the form $(v_i, v_{i+1}), (v_{i+1}, v_{i+2}), \ldots, (v_{j-1}, v_j)$. An undirected graph is connected if there is an undirected path between every pair of distinct vertices. The adjacency matrix $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N \times N}$ of $\mathcal{G}$ is defined as $a_{ij} = a_{ji} = 1$ if $(v_i, v_j) \in \mathcal{E}$, 0 otherwise. The Laplacian matrix $\mathcal{L} = [l_{ij}] \in \mathbb{R}^{N \times N}$ of $\mathcal{G}$ is defined as $l_{ij} = -a_{ij}$, for $i \neq j$ and $l_{ii} = \sum_{j=1}^{N} a_{ij}$ for all $i \in \{1, \ldots, N\}$. It is well known that $\mathcal{L}$ is symmetric and has nonnegative eigenvalues when the graph is undirected, i.e., $\mathcal{L}$ is positive semidefinite. Furthermore, for undirected graphs, zero is a simple eigenvalue of $\mathcal{L}$ and the associated eigenvector is 1_N if and only if the undirected graph is connected [18], [17].

Let μ_i be the ith eigenvalue of an $\mathcal{L}$ associated with an undirected and connected graph. Then the eigenvalues of $\mathcal{L}$ can be arranged as

$$0 = \mu_1 < \mu_2 \leq \mu_3 \leq \cdots \leq \mu_N = \bar{\lambda}(\mathcal{L}). \tag{1}$$

III. Problem Description

Consider a network of N negative imaginary systems with external disturbances acting on each system. The dynamics of the ith NI system are described as

$$y_i = d_{o_i} + P(s)\left(d_{in_i} + u_i\right) \quad \forall i \in \{1, \ldots, N\} \tag{2}$$

where $P(s)$ is an $n \times n$ transfer function matrix of the ith NI system, u_i, y_i, d_{in_i} and d_{o_i} are all vector signals with "n" elements and d_{in_i} and d_{o_i} are also energy-bounded in an H_2 (or in the time domain $\mathfrak{L}_2[0, \infty)$) sense and are control input, output of the ith NI system, input and output disturbances respectively.

It is assumed that relative output measurements with respect to neighbouring agents are available to each system. The network graph which models the information exchange among the systems is assumed fixed and satisfies the following assumption.

Assumption 1: The network graph $\mathcal{G}$ is undirected and connected.

Following [5], the distributed control protocol for the ith NI system is given by

$$\begin{aligned} u_i &= K(s) z_i, \\ z_i &= \sum\nolimits_{j=1}^{N} a_{ij}(y_i - y_j), \end{aligned} \quad \forall i \in \{1, \ldots, N\} \tag{3}$$

where $K(s)$ is the transfer function matrix of an SNI feedback controller, z_i represents the signal of relative measurements of neighbouring agents with respect to system i and a_{ij} are the elements of the adjacency matrix associated with the network graph $\mathcal{G}$.

The collective network dynamics can thus be written as

$$y = d_o + \left(I_N \otimes P(s)\right)\left(d_{in} + u\right), \tag{4}$$

and

$$\begin{aligned} u &= (I_N \otimes K(s)) z, \\ z &= (\mathcal{L} \otimes I_n) y, \end{aligned} \tag{5}$$

where $z = \left[z_1^T, \ldots, z_N^T\right]^T$, $y = \left[y_1^T, \ldots, y_N^T\right]^T$, $u = \left[u_1^T, \ldots, u_N^T\right]^T$, $d_{in} = \left[d_{in_1}^T, \ldots, d_{in_N}^T\right]^T$ and $d_o = \left[d_{o_1}^T, \ldots, d_{o_N}^T\right]^T$ are all vector signals with "nN" elements and d_{in} and d_o are also energy-bounded in an H_2 (or in the time domain $\mathfrak{L}_2[0, \infty)$) sense. $\mathcal{L} \in \mathbb{R}^{N \times N}$ is the Laplacian matrix associated with the network graph $\mathcal{G}$. A block diagram of the closed loop networked multi-agent system is depicted in Fig. 1. Let $\bar{P}(s) = (\mathcal{L} \otimes I_n)(I_N \otimes P(s))$ denote the transfer function matrix from u to z and let $\bar{K}(s) = I_N \otimes K(s)$ denote the transfer function matrix from z to u. According to [5, Lemma 3], $\bar{P}(s)$ is NI if and only if $P(s)$ is NI with $\mathcal{G}$ satisfying Assumption 1. Similarly, $\bar{K}(s)$ is SNI since $K(s)$ is SNI. We consider the robust output consensus problem for networks of NI systems. Before stating a formal definition of the problem, it is important to state how model uncertainties are captured in this framework.

Remark 1: Model uncertainties are captured in this framework by noting that any additive NI perturbations to a nominal NI system results in an NI perturbed systems.

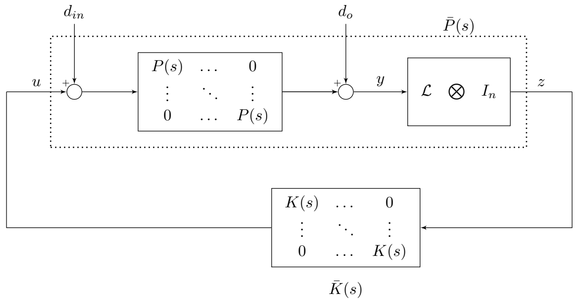


Fig. 1. Closed loop networked multi-agent system.

Other forms of feedback uncertainties are also possible that preserve NI properties (see e.g. [6], [19]). Hence, $P(s)$ is regarded interchangeably as a nominal or perturbed plant as long as it fulfils the robust output consensus conditions which will be shown to depend on the dc and infinite frequency gains of the systems as well as the network graph but not on the precise dynamics of the systems.

Definition 3 ([5], [8]): For a family of NI plant dynamics and for all $\mathfrak{L}_2[0,\infty)$ disturbances acting on the plant input and/or output, robust output consensus is said to be achieved with distributed control protocol (3) for a network of NI systems when $y_i \to y_{ss}\ \forall i \in \{1,\dots,N\}$ with no external disturbances and when $y_i \to y_{ss} + \delta\ \forall i \in \{1,\dots,N\}$ with $\delta \in \mathfrak{L}_2[0,\infty)$, where y_{ss} is the final convergence trajectory. By properties of $\mathcal{L}$, the output consensus problem can be addressed as an internal stability problem for the interconnection $[\bar{P}(s),\bar{K}(s)]$. Our objective is therefore to derive generalised conditions, by means of generalised internal stability results in [4], for which robust output consensus is guaranteed for a network of homogeneous NI systems under $\mathfrak{L}_2$ external disturbances and model uncertainty.

IV. Robust Output Consensus

In this section, we address the robust output consensus problem for networked NI systems with no poles at the origin.

Theorem 1: Consider a network of homogeneous NI systems $P(s)$ without poles at the origin, a network graph $\mathcal{G}$ that satisfies Assumption 1 and an SNI feedback controller $K(s)$ for each NI agent. Let μ_i for all $i \in \{1,\dots,N\}$ be the eigenvalues of the Laplacian matrix $\mathcal{L}$ associated with $\mathcal{G}$ ordered as in (1). Then, robust output consensus is achieved via control protocol (5) for networked system (4) as shown in Fig. 1 (or in a distributed manner (3) for each system (2)) under any external disturbances $d_{in}, d_o \in \mathfrak{L}_2^{nN}[0,\infty)$ and model uncertainty that retains the NI property of the perturbed system $P(s)$ if and only if for all $i \in \{2,\dots,N\}$

$$\begin{aligned}&I_n - \mu_i P(\infty)K(\infty) \text{ is nonsingular,}\\ &\bar{\lambda}[[I_n - \mu_i P(\infty)K(\infty)]^{-1}P(\infty)[K(0)-K(\infty)]] < \frac{1}{\mu_i}, \text{and}\\ &\bar{\lambda}[[I_n - \mu_i K(0)P(\infty)]^{-1}K(0)[P(0)-P(\infty)]] < \frac{1}{\mu_i}.\end{aligned}$$

Proof: Let $\bar{P}(s) = \mathcal{L}\otimes P(s)$ and $\bar{K}(s) = I_N \otimes K(s)$. Now $\bar{P}(s)$ is NI by [5, Lemma 3] and has no poles at the origin since $P(s)$ has no poles at origin. Also, $\bar{K}(s)$ is SNI since $K(s)$ is SNI. As in the proof of [5, Th. 1], the internal stability of $[\bar{P}(s),\bar{K}(s)]$ implies output consensus when $d_{in} = d_o = 0$, by noting that $z \to 0 \Leftrightarrow y \to 1_N \otimes y_{ss}$ since Assumption 1 holds. According to [4, Th. 9], $[\bar{P}(s),\bar{K}(s)]$ is internally stable if and only if

$$\begin{aligned}&I_{Nn} - \bar{P}(\infty)\bar{K}(\infty) \text{ is nonsingular,}\\ &\bar{\lambda}[[I_{Nn} - \bar{P}(\infty)\bar{K}(\infty)]^{-1}[\bar{P}(\infty)\bar{K}(0) - I_{Nn}]] < 0, \text{ and}\\ &\bar{\lambda}[[I_{Nn} - \bar{K}(0)\bar{P}(\infty)]^{-1}[\bar{K}(0)\bar{P}(0) - I_{Nn}]] < 0.\end{aligned}$$

Now $\mathcal{L}$ is a real symmetric matrix due to Assumption 1. Thus, $\mathcal{L}$ can be written as $\mathcal{L} = U\Lambda U^T$ where U is an orthogonal matrix and Λ is a diagonal matrix with eigenvalues of $\mathcal{L}$ on the diagonal. Then,

$$\begin{aligned}&I_{Nn} - \bar{P}(\infty)\bar{K}(\infty)\\ &= I_{Nn} - (\mathcal{L}\otimes P(\infty))(I_N \otimes K(\infty))\\ &= I_{Nn} - (\mathcal{L}\otimes P(\infty)K(\infty))\\ &= I_{Nn} - (U\Lambda U^T \otimes P(\infty)K(\infty))\\ &= (U\otimes I_n)[I_{Nn} - (\Lambda\otimes P(\infty)K(\infty))](U^T\otimes I_n)\\ &= (U\otimes I_n)\mathrm{diag}(I_n - \mu_i P(\infty)K(\infty))(U^T \otimes I_n)\\ &\qquad\qquad \forall i \in \{1,2,\dots,N\}.\end{aligned}$$

So,

$$\begin{aligned}&I_{Nn} - \bar{P}(\infty)\bar{K}(\infty) \text{ is nonsingular}\\ &\Leftrightarrow I_n - \mu_i P(\infty)K(\infty)\ \forall i \in \{2,\dots,N\} \text{ is nonsingular}\\ &\quad \text{(due to the fact that } U \text{ and } U^T \text{ are nonsingular matrices}\\ &\quad \text{and for } \mu_1 = 0,\ I_n \text{ is nonsingular),}\end{aligned}$$

$$\begin{aligned}&\bar{\lambda}[[I_{Nn} - \bar{P}(\infty)\bar{K}(\infty)]^{-1}[\bar{P}(\infty)\bar{K}(0) - I_{Nn}]] < 0\\ &\Leftrightarrow \bar{\lambda}[[I_{Nn} - (\mathcal{L}\otimes P(\infty)K(\infty))]^{-1}\\ &\qquad \times [(\mathcal{L}\otimes P(\infty)K(0)) - I_{Nn}]] < 0\\ &\Leftrightarrow \bar{\lambda}[[I_{Nn} - (U\Lambda U^T\otimes P(\infty)K(\infty))]^{-1}\\ &\qquad \times [(U\Lambda U^T \otimes P(\infty)K(0)) - I_{Nn}]] < 0\\ &\Leftrightarrow \bar{\lambda}[(U\otimes I_n)[I_{Nn} - (\Lambda\otimes P(\infty)K(\infty))]^{-1}(U^T\otimes I_n)\\ &\qquad \times (U\otimes I_n)[(\Lambda\otimes P(\infty)K(0)) - I_{Nn}](U^T\otimes I_n)] < 0\\ &\Leftrightarrow \bar{\lambda}[[I_{Nn} - (\Lambda\otimes P(\infty)K(\infty))]^{-1}\\ &\qquad \times [(\Lambda\otimes P(\infty)K(0)) - I_{Nn}]] < 0\\ &\Leftrightarrow \max_{i=1,\dots,N} \bar{\lambda}[[I_n - \mu_i P(\infty)K(\infty)]^{-1}\\ &\qquad \times [\mu_i P(\infty)K(0) - I_n]] < 0\\ &\quad \text{(since the matrix in the previous step is block diagonal)}\\ &\Leftrightarrow \bar{\lambda}[[I_n - \mu_i P(\infty)K(\infty)]^{-1}[\mu_i P(\infty)K(0) - I_n]] < 0\\ &\qquad \forall i \in \{2,\dots,N\}\\ &\quad \text{(since for } \mu_1 = 0, \text{the condition is trivally fulfilled)}\\ &\Leftrightarrow \bar{\lambda}[\mu_i[I_n - \mu_i P(\infty)K(\infty)]^{-1}P(\infty)K(0)\\ &\qquad - [I_n - \mu_i P(\infty)K(\infty)]^{-1}] < 0 \quad \forall i \in \{2,\dots,N\}\end{aligned}$$

$$\Leftrightarrow \bar{\lambda}[\mu_i[I_n - \mu_i P(\infty)K(\infty)]^{-1}P(\infty)K(0) - \mu_i[I_n - \mu_i P(\infty)K(\infty)]^{-1}P(\infty)K(\infty) - I_n] < 0 \quad \forall i \in \{2,\dots,N\}$$

$$\Leftrightarrow \bar{\lambda}[\mu_i[I_n - \mu_i P(\infty)K(\infty)]^{-1}P(\infty)[K(0) - K(\infty)]] < 1 \quad \forall i \in \{2,\dots,N\}$$

$$\Leftrightarrow \bar{\lambda}[[I_n - \mu_i P(\infty)K(\infty)]^{-1}P(\infty)[K(0) - K(\infty)]] < 1/\mu_i \quad \forall i \in \{2,\dots,N\}$$

and

$$\bar{\lambda}[[I_{Nn} - \bar{K}(0)\bar{P}(\infty)]^{-1}[\bar{K}(0)\bar{P}(0) - I_{Nn}]] < 0$$

$$\Leftrightarrow \bar{\lambda}[[I_{Nn} - \mathcal{L}\otimes K(0)P(\infty)]^{-1} \times [\mathcal{L}\otimes K(0)P(0) - I_{Nn}]] < 0$$

$$\Leftrightarrow \bar{\lambda}[[I_{Nn} - (U\Lambda U^T \otimes K(0)P(\infty))]^{-1} \times [(U\Lambda U^T \otimes K(0)P(0)) - I_{Nn}]] < 0$$

$$\Leftrightarrow \bar{\lambda}[(U\otimes I_n)[I_{Nn} - (\Lambda\otimes K(0)P(\infty))]^{-1}(U^T\otimes I_n) \times (U\otimes I_n)[(\Lambda\otimes K(0)P(0)) - I_{Nn}](U^T\otimes I_n)] < 0$$

$$\Leftrightarrow \bar{\lambda}[[I_{Nn} - (\Lambda\otimes K(0)P(\infty))]^{-1} \times [(\Lambda\otimes K(0)P(0)) - I_{Nn}]] < 0$$

$$\Leftrightarrow \max_{i=1,\dots,N} \bar{\lambda}[[I_n - \mu_i K(0)P(\infty)]^{-1} \times [\mu_i K(0)P(0) - I_n]] < 0$$

(since the matrix in the previous step is block diagonal)

$$\Leftrightarrow \bar{\lambda}[[I_n - \mu_i K(0)P(\infty)]^{-1}[\mu_i K(0)P(0) - I_n]] < 0 \quad \forall i \in \{2,\dots,N\}$$

(since for $\mu_1 = 0$, the condition is trivally fulfilled)

$$\Leftrightarrow \bar{\lambda}[\mu_i[I_n - \mu_i K(0)P(\infty)]^{-1}K(0)P(0) - \mu_i[I_n - \mu_i K(0)P(\infty)]^{-1}K(0)P(\infty) - I_n] < 0 \quad \forall i \in \{2,\dots,N\}$$

$$\Leftrightarrow \bar{\lambda}[\mu_i[I_n - \mu_i K(0)P(\infty)]^{-1}K(0)[P(0) - P(\infty)]] < 1 \quad \forall i \in \{2,\dots,N\}$$

$$\Leftrightarrow \bar{\lambda}[[I_n - \mu_i K(0)P(\infty)]^{-1}K(0)[P(0) - P(\infty)]] < 1/\mu_i \quad \forall i \in \{2,\dots,N\}$$

The proof for robust output consensus under external disturbances and model uncertainties then follows similar to that in the proof of [5, Th.1]. ■

Remark 2: We shall show that $I_n - \mu_i K(0)P(\infty)$ in the third condition of Theorem 1 is nonsingular $\forall i \in \{2,\dots,N\}$ as for $\mu_i = 0$ it is clear. From the proof above for the second condition, we have $\bar{\lambda}[[I_n - \mu_i P(\infty)K(\infty)]^{-1}P(\infty)[K(0)-K(\infty)]] < 1/\mu_i$ if and only if $\bar{\lambda}[[I_n - \mu_i P(\infty)K(\infty)]^{-1}[\mu_i P(\infty)K(0) - I_n]] < 0$ $\forall i \in \{2,\dots,N\}$. From this condition and by the first condition of Theorem 1, the matrix $I_n - \mu_i K(0)P(\infty)$ $\forall i \in \{2,\dots,N\}$ is guaranteed nonsingular.

Remark 3: It turns out that removing the assumptions in [5] makes the conditions more involved as they now depend on all nonzero eigenvalues and not only on the largest eigenvalue of $\mathcal{L}$. However, this is not an issue since our work is limited to fixed graphs and finite number of agents. Moreover, since all nonzero eigenvalues are positive, then if the left hand side of the inequalities of the second and third conditions are negative $\forall i \in \{2,\dots,N\}$, the inequalities need not be checked as they are clearly satisfied.

Remark 4: For special classes of undirected graphs, the conditions simplify significantly. This includes complete and star graphs (see e.g. [20]). For complete graphs, the nonzero eigenvalues are all equal to N. Hence the conditions of Theorem 1 reduce to $I_n - NP(\infty)K(\infty)$ is nonsingular, $\bar{\lambda}[[I_n - NP(\infty)K(\infty)]^{-1}P(\infty)[K(0)-K(\infty)]] < \frac{1}{N}$, and $\bar{\lambda}[[I_n - NK(0)P(\infty)]^{-1}K(0)[P(0)-P(\infty)]] < \frac{1}{N}$. For a star graph, $\mu_i = 1$ $\forall i \in \{2,\dots,\mu_{N-1}\}$ and $\mu_N = N$ and the conditions simplify accordingly.

The robust consensus conditions in Theorem 1 reduce to the one in [5] by imposing the same two assumptions at infinite frequency as follows.

Corollary 1: Let the hypotheses of Theorem 1 hold and furthermore let $P(\infty)K(\infty) = 0$ and $K(\infty) \geq 0$. Then, robust output consensus is achieved via control protocol (5) for networked system (4) as shown in Fig. 1 (or in a distributed manner (3) for each system (2)) under any external disturbances $d_{in}, d_o \in \mathfrak{L}_2^{nN}[0,\infty)$ and model uncertainty that retains the NI property of the perturbed system $P(s)$ if and only if

$$\bar{\lambda}[P(0)K(0)] < \frac{1}{\bar{\lambda}(\mathcal{L})}.$$

Proof: The proof will be published elsewhere. ■

A different set of conditions for robust output consensus can be obtained by applying [4, Th. 14] rather than [4, Th. 9] as follows.

Theorem 2: Consider a network of homogeneous NI systems $P(s)$ without poles at the origin, a network graph $\mathcal{G}$ that satisfies Assumption 1, and an SNI feedback controller $K(s)$ for each NI agent. Let μ_i for all $i \in \{1,\dots,N\}$ be the eigenvalues of the Laplacian matrix $\mathcal{L}$ associated with $\mathcal{G}$ ordered as in (1). Then, robust output consensus is achieved via control protocol (5) for networked system (4) as shown in Fig. 1 (or in a distributed manner (3) for each system (2)) under any external disturbances $d_{in}, d_o \in \mathfrak{L}_2^{nN}[0,\infty)$ and model uncertainty that retains the NI property of the perturbed system $P(s)$ if and only if $\forall i \in \{2,\dots,N\}$

$I_n - \mu_i P(\infty)K(\infty)$ is nonsingular,

$\bar{\lambda}[[P(0) - P(\infty)]K(\infty)[I_n - \mu_i P(\infty)K(\infty)]^{-1}] < \dfrac{1}{\mu_i}$, and

$\bar{\lambda}[[K(0) - K(\infty)]P(0)[I_n - \mu_i K(\infty)P(0)]^{-1}] < \dfrac{1}{\mu_i}$.

Proof: The proof is similar to the proof in Theorem 1 but we here apply [4, Th. 14] instead of [4, Th. 9]. ■

V. CONVERGENCE ANALYSIS

In this section, we study convergence of the networked systems under distributed control protocol (3). We show that the same conclusion as in [5] for the final convergence can be drawn here which states that the steady state behaviour of the closed loop networked system is determined by the eigenvalues of the closed loop networked system on the imaginary axis. In doing so, the external disturbances and model uncertainty will not be considered in this section.

Let a minimal realisation for the ith NI system $P(s)$ be

$$\begin{aligned}\dot{x}_i &= Ax_i + Bu_i \\ y_i &= Cx_i + Du_i\end{aligned}, \qquad i \in \{1, \dots, N\} \tag{6}$$

and a minimal realisation for the ith SNI controller $K(s)$ be

$$\begin{aligned}\dot{\bar{x}}_i &= \bar{A}\bar{x}_i + \bar{B}\bar{u}_i \\ \bar{y}_i &= \bar{C}\bar{x}_i + \bar{D}\bar{u}_i\end{aligned}, \qquad i \in \{1, \dots, N\} \tag{7}$$

where $A \in \mathbb{R}^{p\times p}$, $B \in \mathbb{R}^{p\times n}$, $C \in \mathbb{R}^{n\times p}$, $D \in \mathbb{R}^{n\times n}$, $\bar{A} \in \mathbb{R}^{q\times q}$, $\bar{B} \in \mathbb{R}^{p\times n}$, $\bar{C} \in \mathbb{R}^{n\times q}$ and $\bar{D} \in \mathbb{R}^{n\times n}$. Define $R = (I_{Nn} - \mathcal{L} \otimes D\bar{D})$. Unlike [5], the closed loop system, with the assumption $D\bar{D} = 0$ removed, is now given by

$$\begin{bmatrix} \dot{x} \\ \dot{\bar{x}} \end{bmatrix} = \Psi_{cl} \begin{bmatrix} x \\ \bar{x} \end{bmatrix} \tag{8}$$

where

$$\Psi_{cl} = \begin{bmatrix} \Psi_{cl_{11}} & \Psi_{cl_{12}} \\ \Psi_{cl_{21}} & \Psi_{cl_{22}} \end{bmatrix},$$

$\Psi_{cl_{11}} = (I_N \otimes A) + (\mathcal{L} \otimes B\bar{D})R^{-1}(I_N \otimes C)$, $\Psi_{cl_{12}} = (I_N \otimes B\bar{C}) + (\mathcal{L} \otimes B\bar{D})R^{-1}(I_N \otimes D\bar{C})$,$\Psi_{cl_{21}} = (\mathcal{L} \otimes \bar{B})R^{-1}(I_N \otimes C)$, and $\Psi_{cl_{22}} = (I_N \otimes \bar{A}) + (\mathcal{L} \otimes \bar{B})R^{-1}(I_N \otimes D\bar{C})$. Define $\tilde{R}_i = (I_n - \mu_i D\bar{D})$ and $\tilde{S}_i = (I_n - \mu_i \bar{D}D)$. The following Lemma yields information about the spectrum of Ψ_{cl}, which can be considered a generalisation of [5, Lemma 5].

Lemma 1: Let μ_i be the ith eigenvalue of $\mathcal{L}$ associated with eigenvector $v_{\mathcal{L}}^i$. The spectrum of Ψ_{cl} is given by the union of the spectra of the following matrices:

$$\psi_i = \begin{bmatrix} A + \mu_i B\bar{D}\tilde{R}_i^{-1}C & B\bar{C} + \mu_i B\bar{D}\tilde{R}_i^{-1}D\bar{C} \\ \mu_i \bar{B}\tilde{R}_i^{-1}C & \bar{A} + \mu_i \bar{B}\tilde{R}_i^{-1}D\bar{C} \end{bmatrix} \quad \forall i \in \{1, \dots, N\}.$$

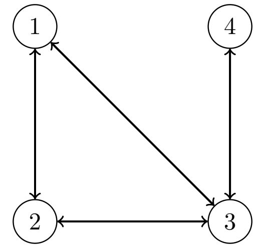
Furthermore, let $\begin{bmatrix} v_1^{i\,*} & v_2^{i\,*} \end{bmatrix}^*$ be an eigenvector of ψ_i. Then, the corresponding eigenvector of Ψ_{cl} is $\begin{bmatrix} v_{\mathcal{L}}^i \otimes v_1^i \\ v_{\mathcal{L}}^i \otimes v_2^i \end{bmatrix}$.

Proof: The proof will be published elsewhere. ■

The importance of Lemma 1 is that it characterises the spectrum of Ψ_{cl} which plays an essential role in determining the final convergence of system (8). In what follows we show that the steady-state behaviour of the closed loop system (8) is in particular determined by the eigenvalues of A on the imaginary axis. For $\mu_1 = 0$, $\psi_1 = \begin{bmatrix} A & B\bar{C} \\ 0 & \bar{A} \end{bmatrix}$. The eigenvalues of ψ_1 are the union of the eigenvalues of A and $\bar{A}$ which are in the CLHP excluding origin and OLHP, respectively. For $\mu_i > 0 \; \forall i \in \{2, \dots, N\}$, using [2, Lemma 7] and [2, Lemma 8], ψ_i can be written as

$$\psi_i = \begin{bmatrix} A & B\bar{C} \\ 0 & \bar{A} \end{bmatrix} + \mu_i \begin{bmatrix} B\bar{D} \\ \bar{B} \end{bmatrix} \tilde{R}_i^{-1} \begin{bmatrix} C & D\bar{C} \end{bmatrix} = \Phi T_i$$

where $T_i = \begin{bmatrix} Y^{-1} - \mu_i C^* \bar{D}\tilde{R}_i^{-1}C & -C^*\tilde{S}_i^{-1}\bar{C} \\ -\mu_i \bar{C}^*\tilde{R}_i^{-1}C & \bar{Y}^{-1} - \mu_i \bar{C}^*\tilde{R}_i^{-1}D\bar{C} \end{bmatrix}$, $\Phi = \begin{bmatrix} AY & 0 \\ 0 & \bar{A}\bar{Y} \end{bmatrix}$, and in a similar manner as [4, Th. 9] ψ_i is Hurwitz if and only if the three conditions in Theorem 1 are satisfied. Thus the eigenvalues of Ψ_{cl} on the imaginary axis are the eigenvalues of A on the imaginary axis and all the remaining eigenvalues of Ψ_{cl} are in the OLHP. Let n_0 be the number of eigenvalues of Ψ_{cl} on the imaginary axis denoted by λ_A and let v_A^r and v_A^l be the right and left eigenvectors of A associated with λ_A. The steady state expression of system (8) is given in the following Lemma.



$$\mathcal{L} = \begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

Fig. 2. Network Graph and associated Laplacian matrix.

Lemma 2: The steady state trajectory of system (8) is given by

$$\begin{bmatrix} x(t) \\ \bar{x}(t) \end{bmatrix} \xrightarrow{t\to\infty} \begin{bmatrix} w_1 & \dots & w_{n_0} \end{bmatrix} e^{J't} \begin{bmatrix} v_1^* \\ \vdots \\ v_{n_0}^* \end{bmatrix} \begin{bmatrix} x(0) \\ \bar{x}(0) \end{bmatrix} \tag{9}$$

where J' is the Jordan block associated with λ_A, and $\forall j \in \{1, \dots, n_0\}$

$$w_j = \begin{bmatrix} 1_N \otimes v_A^r \\ 0_{N_q \times 1} \end{bmatrix}, v_j = \begin{bmatrix} 1_N \otimes \frac{1}{N} v_A^l \\ 1_N \otimes \frac{1}{N}(\lambda_A I_q - \bar{A})^{-*}\bar{C}^* B^* v_A^l \end{bmatrix}$$

are the right and left eigenvectors of Ψ_{cl} associated with λ_A.

Proof: Similar to proof of [5, Th. 2]. ■

Since we are concerned with output consensus, internal stability guarantees that $y \to 1_N \otimes y_{ss}$. Thus, the final output convergence is given by

$$\begin{aligned} y(t) &= R^{-1}(I_N \otimes C)x(t) + R^{-1}(I_N \otimes D\bar{C})\bar{x}(t) \\ Ry(t) &= (I_N \otimes C)x(t) + (I_N \otimes D\bar{C})\bar{x}(t) \\ 1_N \otimes y_{ss} &= \begin{bmatrix} (I_N \otimes C) & (I_N \otimes D\bar{C}) \end{bmatrix} \begin{bmatrix} x(t) \\ \bar{x}(t) \end{bmatrix}. \end{aligned}$$

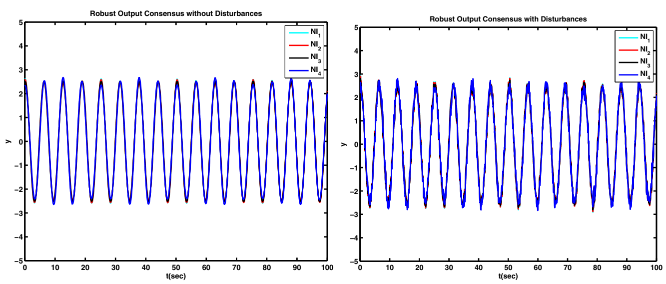
VI. Illustrative Example

Consider a group of $N = 4$ homogeneous NI systems connected over the network topology shown in Fig. 2. The Laplacian matrix $\mathcal{L}$ associated with the network graph is also given in Fig. 2. The nonzero eigenvalues of $\mathcal{L}$ are $\{1, 3, 4\}$. Data is borrowed from [4] where each NI system is undamped and has a transfer function given by $P(s) = \frac{1}{s^2+1} + 2$. The SNI feedback controller to each plant is chosen to be $K(s) = \frac{1}{s+1} + 5$. Since $P(\infty)K(\infty) = 10 \neq 0$, the results in [5, Th. 1] can not be used to

TABLE I

CONDITIONS OF THEOREM 1 CHECKED FOR ALL NONZERO EIGENVALUES OF $\mathcal{L}$

i	μ_i	condition 1 $\neq 0$	condition 2 $< 1/\mu_i$	condtion 3 $< 1/\mu_i$
2	1	-9	$-2/9$	$-6/11$
3	3	-29	$-2/29$	$-6/35$
4	4	-39	$-2/39$	$-6/47$



(a) (b)

Fig. 3. Robust output consensus. (a) Without disturbances and (b) With external output disturbances.

determine whether robust output consensus of the networked NI systems can be achieved or not. On the other hand, robust output consensus of the networked NI systems can be easily concluded via Theorem 1 since the conditions are satisfied for all nonzero eigenvalues as shown in Table I. Simulation results are shown in Fig. 3. The initial conditions are chosen as $x(0) = [1, 2, 3, 4]^T$ and $\dot{x}(0) = [0.1, 0.2, 0.3, 0.4]^T$ for the NI systems and $\bar{x}(0) = [0, 0, 0, 0]^T$ for the SNI controllers. It can be seen from Fig. 3a and Fig. 3b that robust output consensus is achieved when no external disturbances are present and when external output disturbances are present, respectively.

VII. CONCLUSION

In this paper we derived generalised conditions for which robust output consensus is guaranteed for a network of homogeneous NI systems with no poles at the origin under $\mathfrak{L}_2$ external disturbances and model uncertainty. We showed that these results reduce to earlier results proposed in literature when imposing the same assumptions. A convergence analysis for the NI systems' outputs was also given. It was shown that the final convergence depends on the eigenvalues of the closed loop networked system on the imaginary axis which is in agreement with the conclusion provided in earlier literature. An illustrative example was given to demonstrate the capability of the generalised results over earlier results when earlier assumptions fail to hold. Finally, it is worth mentioning that robust output consensus of Networked NI systems with poles at the origin is currently being studied.

REFERENCES

[1] A. Lanzon and I. R. Petersen, "Stability robustness of a feedback interconnection of systems with negative imaginary frequency response," *IEEE Transactions on Automatic Control*, vol. 53, no. 4, pp. 1042–1046, May 2008.

[2] J. Xiong, I. R. Petersen, and A. Lanzon, "A negative imaginary lemma and the stability of interconnections of linear negative imaginary systems," *IEEE Transactions on Automatic Control*, vol. 55, no. 10, pp. 2342–2347, Oct. 2010.

[3] M. A. Mabrok, A. G. Kallapur, I. R. Petersen, and A. Lanzon, "Generalizing negative imaginary systems theory to include free body dynamics: Control of highly resonant structures with free body motion," *IEEE Transactions on Automatic Control*, vol. 59, no. 10, pp. 2692–2707, Oct. 2014.

[4] A. Lanzon and H. J. Chen, "Feedback stability of negative imaginary systems," *IEEE Transactions on Automatic Control*, vol. 62, no. 11, pp. 5620–5633, Nov. 2017.

[5] J. Wang, A. Lanzon, and I. R. Petersen, "Robust output feedback consensus for networked negative-imaginary systems," *IEEE Transactions on Automatic Control*, vol. 60, no. 9, pp. 2547–2552, Sep. 2015.

[6] A. Ferrante, A. Lanzon, and L. Ntogramatzidis, "Foundations of not necessarily rational negative imaginary systems theory: Relations between classes of negative imaginary and positive real systems," *IEEE Transactions on Automatic Control*, vol. 61, no. 10, pp. 3052–3057, Oct. 2016.

[7] I. R. Petersen, "Negative imaginary systems theory and applications," *Annual Reviews in Control*, vol. 42, pp. 309–318, 2016.

[8] J. Wang, A. Lanzon, and I. R. Petersen, "Robust cooperative control of multiple heterogeneous negative-imaginary systems," *Automatica*, vol. 61, pp. 64–72, Nov. 2015.

[9] X. Li, Y. C. Soh, and L. Xie, "Output-feedback protocols without controller interaction for consensus of homogeneous multi-agent systems: A unified robust control view," *Automatica*, vol. 81, pp. 37–45, Jul. 2017.

[10] H. Jongsma, H. L. Trentelman, and M. Camlibel, "Robust synchronization of coprime factor perturbed networks," *Systems and Control Letters*, vol. 95, pp. 62–69, Sep. 2016.

[11] H. L. Trentelman, K. Takaba, and N. Monshizadeh, "Robust synchronization of uncertain linear multi-agent systems," *IEEE Transactions on Automatic Control*, vol. 58, no. 6, pp. 1511–1523, Jun. 2013.

[12] Z. Li, X. Liu, P. Lin, and W. Ren, "Consensus of linear multi-agent systems with reduced-order observer-based protocols," *Systems and Control Letters*, vol. 60, no. 7, pp. 510–516, 2011.

[13] Z. Li, Z. Duan, G. Chen, and L. Huang, "Consensus of Multiagent Systems and Synchronization of Complex Networks: A Unified Viewpoint," *IEEE Transactions on Circuits and Systems I: Regular Papers*, vol. 57, no. 1, pp. 213–224, 2010.

[14] S. Knorn, Z. Chen, and R. H. Middleton, "Overview: Collective Control of Multiagent Systems," *IEEE Transactions on Control of Network Systems*, vol. 3, no. 4, pp. 334–347, 2016.

[15] Y. Chen, J. Lu, X. Yu, and D. J. Hill, "Multi-agent systems with dynamical topologies: Consensus and applications," *IEEE Circuits and Systems Magazine*, vol. 13, no. 3, pp. 21–34, Jan. 2013.

[16] Y. Cao, W. Yu, W. Ren, and G. Chen, "An overview of recent progress in the study of distributed multi-agent coordination," *IEEE Transactions on Industrial Informatics*, vol. 9, no. 1, pp. 427–438, Feb. 2013.

[17] R. Olfati-Saber, J. A. Fax, and R. M. Murray, "Consensus and cooperation in networked multi-agent systems," *Proceedings of the IEEE*, vol. 95, no. 1, pp. 215–233, 2007.

[18] W. Ren, R. Beard, and E. Atkins, "Information consensus in multivehicle cooperative control," *IEEE Control Systems Magazine*, vol. 27, no. 2, 2007.

[19] H. J. Chen, A. Lanzon, and I. R. Petersen, "Conditions for preserving negative imaginary properties in feedback interconnections and an application to multi-agent systems," in *Proceedings of the 25th Mediterranean Conference on Control and Automation*, Valletta, Malta, Jul. 2017, pp. 1143–1147.

[20] M. Mesbahi and M. Egerstedt, *Graph Theoretic Methods in Multiagent Networks*. Princeton University Press, 2010.