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Robust Output Feedback Consensus for Multiple Heterogeneous Negative-Imaginary Systems

Jianan Wang, Alexander Lanzon, and Ian R. Petersen

***Abstract*— This paper investigates a robust output feedback consensus for a wide class of linear time-invariant (LTI) systems, namely Negative-Imaginary (NI) systems. A positive-feedback interconnection with Strictly Negative-Imaginary (S-NI) controllers is applied through the network topology to achieve robust output feedback consensus of heterogeneous multi-input-multi-output (MIMO) NI systems. Robustness to external disturbances and model uncertainty is guaranteed via NI system theory. Numerical examples for various scenarios are given to demonstrate the effectiveness of proposed consensus algorithm.**

I. Introduction

NI systems theory has drawn much attention ([12], [1], [5], [7], [17], [18]) since it was introduced in [9], [19], [11] and [13]. It is vital because there are a wide class of LTI systems with negative imaginary frequency response, for which applications can be easily found in a variety of fields including aerospace, large space structures, multi-link robotic arms usually with co-located position sensors and force actuators and nano-positioning [10], etc. Cooperative control of multiple NI systems arise with the development of NI systems' applications where one single NI system is incapable of achieving the mission goals, for example, the load is too heavy to be carried by one multi-link robotic arm.

There are a number of existing works on robust cooperative control of LTI systems, most of which appeared recently. In terms of heterogeneous network of systems: [3] studies a cooperative control problem for a string of coupled heterogeneous NI subsystems. Such systems can arise in vehicle platoons. However, the systems considered are constrained to SISO systems (due to the mathematics of continued fractions used) and do not allow poles on the imaginary axis, and also the graph is only restricted to string connections. [16] solves a cooperative robust output regulation problem for a class of LTI systems with minimum phase dynamics. A combination of simultaneous high-gain state feedback control and a distributed high-gain observer is adopted to achieve cooperative output regulation under particular parameter uncertainty as well as particular external disturbances. [21] discusses a full-state feedback robust consensus protocol for heterogeneous second-order multi-agent systems.

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It can be seen that the existing published literature on robust cooperative control of heterogeneous multi-agent systems is restricted to either only SISO plants, minimum phase LTI plants or full-state feedback second order plants. This paper solves a more general problem, which is the robust output feedback cooperative control of heterogeneous MIMO NI systems (possibly with poles on the imaginary axis) under $\mathfrak{L}_2$ external disturbances and SNI model uncertainty. We impose no minimum phase assumption. The communication graph can be any general undirected and connected graph rather than any specific graph, like the string connection in [3]. Towards this end, NI system theory is adopted to derive conditions for robust output feedback consensus for a wide class of LTI systems, namely NI systems.

II. Preliminaries

A. Graph Theory

A graph can be mathematically expressed by $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ where $\mathcal{V} = \{v_1, v_2, \ldots, v_n\}$ is a nonempty finite set of n nodes and an edge set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is used to model the l communications links among nodes. A sequence of successive edges of $\mathcal{E}$ in the form of $\{(v_i, v_k), (v_k, v_l), \ldots, (v_m, v_j)\}$ is defined as a directed path from node i to node j. An undirected path in an undirected graph is defined analogously but bidirectionally. An undirected graph is said to be connected if there is a path from node i to node j for all the distinct nodes $v_i, v_j \in \mathcal{V}$.

The incidence matrix $\mathcal{Q}$ of $\mathcal{G}$ is a $|\mathcal{V}| \times |\mathcal{E}|$ $(n \times l)$ matrix, which can be attained by first letting each edge in the graph have an arbitrary but fixed orientation and then

$$\mathcal{Q} := \begin{cases} q_{ve} = 1, & \text{if } v \text{ is the initial vertex of edge } e \\ q_{ve} = -1, & \text{if } v \text{ is the terminal vertex of edge } e \\ q_{ve} = 0, & \text{if } v \text{ is not connected to } e. \end{cases}$$

For an undirected graph $\mathcal{G}$, $\mathcal{Q}$ is not unique but the corresponding Laplacian matrix is unique and given by: $\mathcal{L}_n = \mathcal{Q}\mathcal{Q}^T$, and the edge-weighted Laplacian is also unique given by: $\mathcal{L}_e = \mathcal{Q}\mathcal{K}\mathcal{Q}^T$, where $\mathcal{K} \geq 0$ is the diagonal edge weighting matrix. It is also shown in [2] that $\mathrm{rank}(\mathcal{Q}) = n - 1 = \mathrm{rank}(\mathcal{L}_n)$ when $\mathcal{G}$ is connected and $\mathrm{rank}(\mathcal{Q}) = n - 1 = \mathrm{rank}(\mathcal{L}_e)$ when $\mathcal{G}$ is connected and $\det(\mathcal{K}) \neq 0$. It is well-known [4] that $\mathcal{L}_n$ and $\mathcal{L}_e$ will both have one unique zero eigenvalue associated with the eigenvector $\mathbf{1}_n$ and all the other eigenvalues are positive and real, when $\det(\mathcal{K}) \neq 0$, $\mathcal{G}$ is undirected and connected. In this case, $\mathcal{L}_n \geq 0$, $\mathcal{L}_e \geq 0$, and

$$\mathrm{Ker}(\mathcal{L}_n) = \mathrm{Ker}(\mathcal{L}_e) = \mathrm{Ker}(\mathcal{Q}^T) = \mathrm{span}\{\mathbf{1}_n\}. \quad (1)$$

The following lemmas are needed for the main result of this paper:

Lemma 1: Given an undirected and connected graph $\mathcal{G}$, any principle submatrix of the Laplacian matrix $\mathcal{L}_n$ or $\mathcal{L}_e$ is positive definite and thus full rank.

Proof: It can be directly seen from positive semi-definite matrices with a Kernel dimension of 1. ■

As a consequence, the following lemma is given as:

Lemma 2: Given an undirected and connected graph $\mathcal{G}$, any row removal of $\mathcal{Q}$ or column removal of $\mathcal{Q}^T$ yields a full row rank $\mathcal{Q}$ or a full column rank $\mathcal{Q}^T$ respectively.

Proof: It is straightforward to see from Lemma 1 and the relation of $\mathcal{L}_n$ or $\mathcal{L}_e$ with $\mathcal{Q}$ as shown above in II-A. ■

B. Negative-Imaginary Systems

Before proceeding to the main result, let us first recall the definitions of NI and SNI systems:

Definition 1: ([11]) A square, real, rational, proper transfer function matrix $P(s)$ is NI if the following conditions are satisfied: 1) $P(s)$ has no pole in $\mathrm{Re}[s] > 0$; 2) $\forall \omega > 0$ such that $j\omega$ is not a pole of $P(s)$, $j(P(j\omega) - P(j\omega)^*) \geq 0$; 3) If $s = j\omega_0$ where $\omega_0 > 0$ is a pole of $P(s)$, then it is a simple pole and the residue matrix $K = \lim_{s \to j\omega_0} (s - j\omega_0) jP(s)$ is Hermitian and positive semi-definite; 4) If $s = 0$ is a pole of $P(s)$, then $\lim_{s\to 0} s^k P(s) = 0 \; \forall k \geq 3$ and $P_2 = \lim_{s \to 0} s^2 P(s)$ is Hermitian and positive semi-definite.

It can be observed that Definition 1 for NI systems captures earlier definitions in [9] and [19]. This definition also includes free body dynamics which often refers to dynamical models with poles at the origin, such as $\frac{s^2+1}{s^2(s^2+2)}$. Examples of NI systems can be found in [11], and these include a single-integrator system, a double-integrator system, second-order systems such as those that arise in undamped and damped flexible structures or inertial systems, to name a few typically considered in the consensus literature.

Definition 2: A square, real, rational, proper transfer function matrix $P_s(s)$ is SNI if the following conditions are satisfied: 1) $P_s(s)$ has no pole in $\mathrm{Re}[s] \geq 0$; 2) $\forall \omega > 0, j(P_s(j\omega) - P_s(j\omega)^*) > 0$.

Examples of SNI systems include $\frac{1}{s+a}$ where $a > 0$, $\frac{a}{s^2+bs+c}$ where $a, b, c > 0$ or non-minimum phase systems such as $\frac{1-s}{2+s}$.

III. Main Results

In this section, we will consider robust output feedback consensus for multiple heterogeneous NI systems under $\mathfrak{L}_2$ external disturbance and additive SNI model uncertainty (as would arise in spill-over dynamics for truncated order flexible structures). First of all, let us begin with the problem formulation:

For multiple heterogeneous NI systems (in general MIMO) with $n > 1$ agents, the transfer function of agent i is given as $\hat{\boldsymbol{y}}_i = \hat{P}_i(s)\hat{\boldsymbol{u}}_i, \;\; i = 1, \cdots, n$, where $\hat{\boldsymbol{y}}_i \in \mathbb{R}^{m_i \times 1}$ and $\hat{\boldsymbol{u}}_i \in \mathbb{R}^{m_i \times 1}$ are the output and input of agent i respectively. In order to deal with the consensus of different dimensional inputs/outputs, $\hat{P}_i(s)$ can be padded with zeros up to $m = \max_{i=1}^{n}\{m_i\}$ and the locations of padding zeros depend on which output needs to be coordinated, for instance, $P_i(s) = \begin{bmatrix} \hat{P}_i(s) & 0 \\ 0 & 0 \end{bmatrix}$ has m dimension such that the first m_i outputs are to be coordinated, or $P_i(s) = \begin{bmatrix} 0 & 0 \\ 0 & \hat{P}_i(s) \end{bmatrix}$ has m dimension such that the last m_i outputs are to be coordinated instead. Accordingly, the input $\hat{\boldsymbol{u}}_i$ and output $\hat{\boldsymbol{y}}_i$ are extended to be $\boldsymbol{u}_i = \begin{bmatrix} \hat{\boldsymbol{u}}_i^T & 0 \end{bmatrix}^T$ or $\begin{bmatrix} 0 & \hat{\boldsymbol{u}}_i^T \end{bmatrix} \in \mathbb{R}^{m \times 1}$ and $\boldsymbol{y}_i = \begin{bmatrix} \hat{\boldsymbol{y}}_i^T & 0 \end{bmatrix}^T$ or $\begin{bmatrix} 0 & \hat{\boldsymbol{y}}_i^T \end{bmatrix}^T \in \mathbb{R}^{m \times 1}$, respectively. It can be easily seen that the above manipulation would preserve the NI property by checking Definition 1. Therefore, the overall plant can be described as Fig. 1:

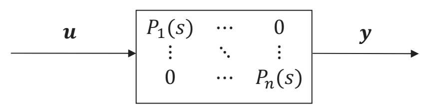

Fig. 1. Multiple Heterogeneous NI Plants

where $\boldsymbol{y} = \begin{bmatrix} \boldsymbol{y}_1^T & , \cdots, & \boldsymbol{y}_n^T \end{bmatrix}^T \in \mathbb{R}^{nm \times 1}$ and $\boldsymbol{u} = \begin{bmatrix} \boldsymbol{u}_1^T & , \cdots, & \boldsymbol{u}_n^T \end{bmatrix}^T \in \mathbb{R}^{nm\times 1}$. In general, robust output feedback consensus is defined as follows:

Definition 3: A distributed output feedback control law achieves robust output feedback consensus for a network of systems if for a family of plant dynamics and for all $\mathfrak{L}_2[0, \infty)$ disturbances on the plant input and/or plant output, $\boldsymbol{y}_i - \boldsymbol{y}_{ss} \in \mathfrak{L}_2[0, \infty) \; \forall i \in \{1, \cdots, n\}$. Here $\boldsymbol{y}_{ss}$ is the final convergence trajectory, which can be a function of time depending on the plant and controller dynamics. If there are no disturbances, then $\boldsymbol{y}_i - \boldsymbol{y}_{ss} \to \mathbf{0} \; \forall i \in \{1, \cdots, n\}$ retrieves the typical consensus meaning in the literature.

Observe that if one were to construct the overall networked plant dynamics involving the heterogeneous multiple agents $P_i(s)$ and the communications graph represented by a Laplacian matrix $\mathcal{L}_n$ as $(\mathcal{L}_n \otimes I_m) \cdot \operatorname{diag}_{i=1}^{n}\{P_i(s)\}$, then the overall networked plant is not NI any more due to the asymmetry despite each heterogeneous agent being individually NI. This would then make NI systems theory inapplicable. Given the preliminaries in Section II, we can utilize the incidence matrix $\mathcal{Q}$ instead of $\mathcal{L}_n$ to reformulate the overall networked plant as shown in Fig. 2:

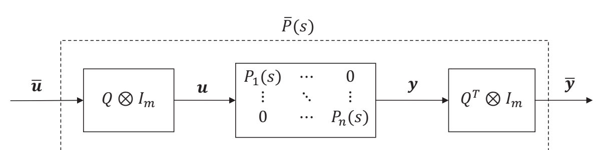

Fig. 2. Overall Network Plant

The augmented networked system can be derived as

$$\bar{\boldsymbol{y}} = \bar{P}(s)\bar{\boldsymbol{u}} = (\mathcal{Q}^T \otimes I_m) \operatorname{diag}_{i=1}^{n}\{P_i(s)\}(\mathcal{Q} \otimes I_m)\bar{\boldsymbol{u}} \qquad (2)$$

where $\bar{\boldsymbol{y}} = \left[\bar{\boldsymbol{y}}_1^T \quad, \cdots, \quad \bar{\boldsymbol{y}}_l^T\right]^T \in \mathbb{R}^{lm\times 1}$ and $\bar{\boldsymbol{u}} = \left[\bar{\boldsymbol{u}}_1^T \quad, \cdots, \quad \bar{\boldsymbol{u}}_l^T\right]^T \in \mathbb{R}^{lm\times 1}$ are the output and input for the overall system. It can be concluded that the overall system $\bar{P}(s)$ is still NI due to the following lemmas:

Lemma 3: $\operatorname*{diag}_{i=1}^{n}\{P_i(s)\}$ is NI if and only if $P_i(s)$ are all NI $\forall i = 1, \cdots, n$.

Proof: It is straightforward to see by definition. ∎

The same argument applies for SNI functions. The following lemma is needed as well:

Lemma 4: Given any MIMO $P(s)$ being NI, then $\bar{P}(s) = FP(s)F^*$ is still NI for any constant matrix F.

Proof: It is straightforward to see by definition. ∎

It can be seen that the output $\boldsymbol{y} \in \mathbb{R}^{nm\times 1}$ reaches consensus when $\bar{\boldsymbol{y}} \to \mathbf{0} \in \mathbb{R}^{lm\times 1}$ by noticing the properties of the incidence matrix $\mathcal{Q}$ given in (1). This formulation actually converts the output consensus problem to an internal stability problem which is usually easier to tackle and investigate the robustness property via standard control theoretic methods. We now impose the following assumptions throughout the rest of this paper:

Assumption 1: $\mathcal{G}$ is undirected and connected.

Another assumption is also needed throughout this paper:

Assumption 2: Let $\Delta_i(s)$ $\forall i = 1, \cdots, n$ be arbitrary SNI systems satisfying $\bar{\lambda}(\Delta_i(0)) < \mu, \Delta_i(\infty) = 0$ $\forall i = 1, \cdots, n$, where μ is a constant value.

In the sequel, robust output feedback consensus will be discussed along two directions: NI plants without or with free body dynamics to cover all the heterogeneous types of NI systems.

A. NI plants without free body dynamics

In this subsection, the NI plants without free body dynamics will be firstly considered, which also means $\hat{P}_i(s)$ has no poles at the origin. The following lemmas are needed:

Lemma 5: ([8]) Given $M \in \mathbb{R}^{n\times m}$, $\bar{\lambda}(MM^T) = \bar{\lambda}(M^TM)$.

Lemma 6: Assume M is Hermitian with $\bar{\lambda}(M) \geq 0$ and $N \geq 0$, we have $\bar{\lambda}(MN) \leq \bar{\lambda}(M)\bar{\lambda}(N)$.

Proof: Since $M \leq \bar{\lambda}(M)I$, we obtain $N^{\frac{1}{2}}MN^{\frac{1}{2}} \leq \bar{\lambda}(M)N$. With the conditon of $\bar{\lambda}(M) \geq 0$, $N^{\frac{1}{2}}MN^{\frac{1}{2}} \leq \bar{\lambda}(M)N \leq \bar{\lambda}(M)\bar{\lambda}(N)I$. Thus, $\bar{\lambda}(MN) = \bar{\lambda}(N^{\frac{1}{2}}MN^{\frac{1}{2}}) \leq \bar{\lambda}(M)\bar{\lambda}(N)$. ∎

Lemma 7: ([9], [19]) Given an NI transfer function $P(s)$ and an SNI function $P_s(s)$ with $P(s)$ having no pole(s) at the origin, $P(\infty)P_s(\infty) = 0$ and $P_s(\infty) \geq 0$. $[P(s), P_s(s)]$ is internally stable if and only if $\bar{\lambda}(P(0)P_s(0)) < 1$.

Next we present the first main result of this paper:

Theorem 1: Given a graph $\mathcal{G}$ with incidence matrix $\mathcal{Q}$, satisfying Assumption 1 and modelling the communication links among multiple NI agents $\hat{P}_i(s)$ with no pole(s) at the origin which are appropriately padded with rows and columns of zeros to give $P_i(s)$ in Fig. 3. Define $\bar{P}_s(s) = \operatorname*{diag}_{j=1}^{l}\{P_{s,j}(s)\}$ where $P_{s,j}(s)$ are arbitrary SNI compensators. Robust output feedback consensus is achieved via the output

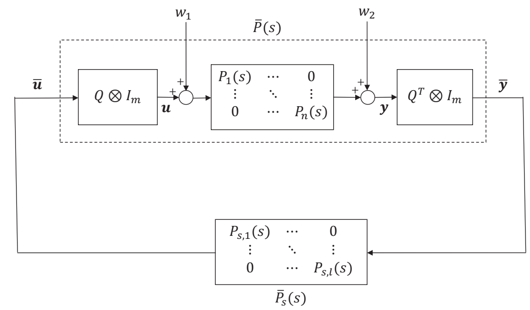

Fig. 3. Positive-feedback interconnection with SNI compensators through the network topology

feedback control law

$$\boldsymbol{u} = (\mathcal{Q} \otimes I_m)\bar{P}_s(s)(\mathcal{Q}^T \otimes I_m)\boldsymbol{y} \tag{3}$$

(or in a distributed manner for agent i via

$$\boldsymbol{u}_i = \sum_{k=1}^{n} a_{ik}P_{s,j}(s)(\boldsymbol{y}_i - \boldsymbol{y}_k), \tag{4}$$

where a_{ik} are the elements of the adjacency matrix[1] and j is the edge connecting vertex i to vertex k) under any external disturbances $w_1 \in \operatorname{Im}_{\mathfrak{L}_2}(\mathcal{Q} \otimes I_m)$ and $w_2 \in \mathfrak{L}_2$ if $\exists i \in \{1, \cdots, n\} : \ \bar{\lambda}(P_i(0)) \geq 0$ and $\forall i \in \{1, \cdots, n\}, j \in \{1, \cdots, l\}$ all the following conditions hold:

$$\bar{\lambda}(P_i(0))\bar{\lambda}(P_{s,j}(0)) < \frac{1}{\bar{\lambda}(\mathcal{L}_n)}, \tag{5}$$

$P_i(\infty)P_{s,j}(\infty) = 0$ (where i is the vertex of edge j) and $P_{s,j}(\infty) \geq 0$. The output feedback consensus control law (3) will be robust to all model uncertainty $\Delta_i(s), i = 1, \cdots, n$ satisfying Assumption 2 if the D.C. gain of the SNI compensator $\bar{P}_s(s)$ is tuned more stringently such that $\forall i \in \{1, \cdots, n\}, j \in \{1, \cdots, l\}$

$$\bar{\lambda}(P_i(0)) + \mu < \frac{1}{\bar{\lambda}(\mathcal{L}_n)\bar{\lambda}(P_{s,j}(0))}. \tag{6}$$

Proof: From Fig. 3, Lemmas 3 and 4, it can be seen that $\bar{P}(s)$ is NI without pole(s) at the origin and $\bar{P}_s(s)$ is SNI. Applying Lemma 6, we obtain

$$\begin{aligned}
&\bar{\lambda}(\bar{P}(0)\bar{P}_s(0)) \\
=&\bar{\lambda}((\mathcal{Q}^T \otimes I_m)\operatorname*{diag}_{i=1}^{n}\{P_i(0)\}(\mathcal{Q} \otimes I_m)\operatorname*{diag}_{j=1}^{l}\{P_{s,j}(0)\}) \\
\leq&\bar{\lambda}((\mathcal{Q}^T \otimes I_m)\operatorname*{diag}_{i=1}^{n}\{P_i(0)\}(\mathcal{Q} \otimes I_m)) \max_{j=1}^{l}\{\bar{\lambda}(P_{s,j}(0))\} \\
\leq&\max_{i=1}^{n}\{\bar{\lambda}(P_i(0))\}\bar{\lambda}(\mathcal{Q}^T\mathcal{Q}) \max_{j=1}^{l}\{\bar{\lambda}(P_{s,j}(0))\} \quad \text{(requires } \exists i : \bar{\lambda}(P_i(0)) \geq 0\text{)} \\
=&\max_{i=1}^{n}\{\bar{\lambda}(P_i(0))\} \max_{j=1}^{l}\{\bar{\lambda}(P_{s,j}(0))\}\bar{\lambda}(\mathcal{L}_n) \qquad \text{(by Lemma 5)}
\end{aligned}$$

since $\bar{\lambda}(\bar{P}(0)) \geq 0$ (because $\exists i : \bar{\lambda}(P_i(0)) \geq 0$) and $P_{s,j}(0) > P_{s,j}(\infty) \geq 0$ $\forall j \in \{1, \cdots, l\}$ (due to Lemma 2 in [9] with the assumption of $P_{s,j}(\infty) \geq 0$). Thus, since $\exists i \in \{1, \cdots, n\} : \bar{\lambda}(P_i(0)) \geq 0$ and $\forall i = 1, \cdots, n$ and $j = 1, \cdots, l$, all of the following hold: $\bar{\lambda}(P_i(0))\bar{\lambda}(P_{s,j}(0)) < \frac{1}{\bar{\lambda}(\mathcal{L}_n)}$, $P_i(\infty)P_{s,j}(\infty) = 0$ (where i is the vertex of edge

[1] See [14] for definition

j) and $P_{s,j}(\infty) \geq 0$, $[\bar{P}(s), \bar{P}_s(s)]$ is internally stable via NI systems theory in Lemma 7. This then implies nominal output consensus when the disturbances $\boldsymbol{w}_1$ and $\boldsymbol{w}_2$ are set to zero by noting that $\bar{\boldsymbol{y}} \to \mathbf{0} \Leftrightarrow \boldsymbol{y} \to \mathbf{1}_n \otimes \boldsymbol{y}_{ss}$, i.e., $\boldsymbol{y}_i \to \boldsymbol{y}_{ss}$ since the graph $\mathcal{G}$ is undirected and connected.

In addition, internal stability of $[\bar{P}(s), \bar{P}_s(s)]$ and superposition principle of linear systems ([20]) guarantee that $\boldsymbol{y}_i \to \boldsymbol{y}_{ss} + \delta$ with $\delta \in \mathfrak{L}_2$ for all $\mathfrak{L}_2$ exogenous signal injections perturbing signals $\bar{\boldsymbol{u}}$ and $\bar{\boldsymbol{y}}$, which in turn means that any $w_1 \in \mathrm{Im}_{\mathfrak{L}_2}(\mathcal{Q} \otimes I_m)$ and any $w_2 \in \mathfrak{L}_2$ can be injected in Fig. 3. Hence, the control protocol (3) will achieve a perturbed $\mathfrak{L}_2$ consensus signal on output $\boldsymbol{y}$ (due to superposition principle of linear systems) for all disturbances $w_1 \in \mathrm{Im}_{\mathfrak{L}_2}(\mathcal{Q} \otimes I_m)$ and $w_2 \in \mathfrak{L}_2$.

Additive model uncertainties $\Delta_i(s)$ $\forall i \in \{1, \cdots, n\}$ satisfying Assumption 2 can be dealt with as in [15], which is shown in Fig. 4.

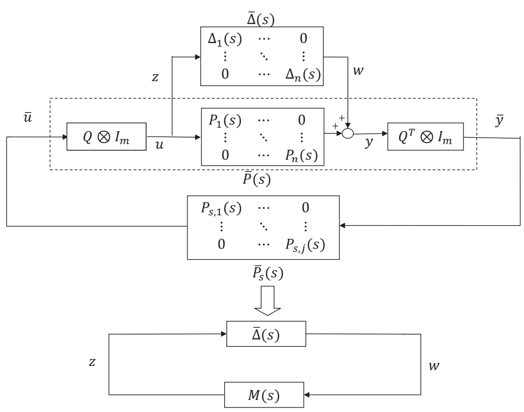

Fig. 4. Robustness to model uncertainty via NI system theory

Fig. 4 (top) can be manipulated to Fig. 4 (bottom) with $M(s) = (\mathcal{Q} \otimes I_m)\bar{P}_s(s)(I - \bar{P}(s)\bar{P}_s(s))^{-1}(\mathcal{Q}^T \otimes I_m)$. Internal stability already yields $M(s) \in \mathcal{RH}_\infty$ and $M(s)$ is NI via Lemma 7 and Lemma 19 in [6] in general, or via Lemma 7 and Theorem 2 in [13] when $P_i(s)$ have no poles on the imaginary axis. This NI system $M(s)$ is connected with $\bar{\Delta}(s)$ which fulfills Assumption 2. Now

$$
\begin{aligned}
&\bar{\lambda}(\Delta(0)M(0)) \\
\leq &\bar{\lambda}(\Delta(0))\bar{\lambda}[(\mathcal{Q} \otimes I_m)\bar{P}_s(0)(I - \bar{P}(0)\bar{P}_s(0))^{-1}(\mathcal{Q}^T \otimes I_m)] \\
\leq &\mu\bar{\lambda}(\mathcal{L}_n)\bar{\lambda}[\bar{P}_s(0)(I - \bar{P}(0)\bar{P}_s(0))^{-1}] \\
\leq &\frac{\mu\bar{\lambda}(\mathcal{L}_n)\bar{\lambda}(\bar{P}_s(0))}{1 - \bar{\lambda}(\bar{P}(0)\bar{P}_s(0))} \\
\leq &\frac{\mu\bar{\lambda}(\mathcal{L}_n)\max\limits_{j=1}^{l}\{\bar{\lambda}(P_{s,j}(0))\}}{1 - \max\limits_{i=1}^{n}\{\bar{\lambda}(P_i(0))\}\bar{\lambda}(\mathcal{L}_n)\max\limits_{j=1}^{l}\{\bar{\lambda}(P_{s,j}(0))\}}
\end{aligned}
$$

It is then clear that inequality (6) guarantees $\bar{\lambda}(\Delta(0)M(0)) < 1$ which in turn implies robust stability for all uncertainties that satisfy Assumption 2. ■

Remark 1: There is clearly a huge class of permissible dynamic perturbations to the nominal dynamics as Assumption 2 only imposes a restriction on $\Delta_i(s)$ only at the frequency $\omega = 0$ and $\omega = \infty$ and the SNI class has no gain (as long as it is finite gain) or order restriction [9]. The result in Theorem 1 is for additive perturbations, but similar analysis can be performed for other types of perturbations that preserve the NI class. A few examples of permissible perturbations that preserve the NI class include additive perturbations where the uncertainty is also NI [9], feedback perturbations where both systems in the feedback interconnection are NI [13] and more general perturbations based Redheffer Star-products and Linear Fractional Transformations [13]. For example, $\frac{1}{s+5}$ and $\frac{(2s^2+s+1)}{(s^2+2s+5)(s+1)(2s+1)}$ are both SNI with the same D.C. gain.

B. NI plants with free body dynamics

In this subsection, we will consider more general NI plants by including free body dynamics (i.e. poles at the origin) under the assumption of strict properness, i.e. $P_i(\infty) = 0$. The NI class restricts the number of such poles at the origin to be at most 2. The following residue matrices carrying information about the properties of the free body motion for the NI system $\boldsymbol{y} = P(s)\boldsymbol{u}$ where $P(s) \in \mathbb{R}^{m\times m}$ ([11]): $P_2 = \lim_{s\to 0} s^2 P(s), P_1 = \lim_{s\to 0} s(P(s) - \frac{P_2}{s^2}), P_0 = \lim_{s\to 0}(P(s) - \frac{P_2}{s^2} - \frac{P_1}{s}$. It can be observed that $P_1 = 0, P_2 = 0$ means there is no free body dynamics, $P_1 \neq 0, P_2 = 0$ means there is free body dynamics with 1 pole at the origin, $P_2 \neq 0$ means there is free body dynamics with 2 poles at the origin. For the sake of page limitation, we will omit the calculating process which can be found in [11] to obtain the following useful matrices: $N_f = P_s(0) - P_s(0)F(F^T P_s(0)F)^{-1}F^T P_s(0)$ where $F = H_1\hat{V}_2 \in \mathbb{R}^{m\times\tilde{n}}$. When $P_2 \neq 0$, $N_2 = P_s(0) - P_s(0)J(J^T P_s(0)J)^{-1}J^T P_s(0)$ where J is a full column rank matrix factorising $P_2 \neq 0$ as $P_2 = JJ^T$. When $P_2 = 0, P_1 \neq 0$, $N_1 = P_s(0) - P_s(0)F_1(F_1^T P_s(0)F_1)^{-1}F_1^T P_s(0)$ where $F_1 = \tilde{U}_1 S_2$.

Next, the internal stability of $[P(s), P_s(s)]$ with free body dynamics can be summarised in the following lemma:

Lemma 8: ([11]) Let $P(s)$ be a strictly proper NI plant and $P_s(s)$ be an SNI controller:

1) Suppose $P_2 \neq 0$, N_f is sign definite and $F^T P_s(0)F$ is non-singular. Then, $[P(s), P_s(s)]$ is internally stable if and only if $F^T P_s(0)F < 0$ and either $I - N_f^{\frac{1}{2}} P_0 N_f^{\frac{1}{2}} - N_f^{\frac{1}{2}} P_1 J(J^T J)^{-2} J^T P_1^T N_f^{\frac{1}{2}} > 0$ when $N_f \geq 0$ or

$$
\det(I + \tilde{N}_f P_0 \tilde{N}_f + \tilde{N}_f P_1 J(J^T J)^{-2} J^T P_1^T \tilde{N}_f) \neq 0 \quad (7)
$$

when $N_f \leq 0$ and $\tilde{N}_f = (-N_f)^{\frac{1}{2}}$. If furthermore $P_1 = 0$, N_2 is sign definite and $J^T P_s(0)J$ is non-singular, the necessary and sufficient conditions for the internal stability of $[P(s), P_s(s)]$ reduce to $J^T P_s(0)J < 0$ and either $I - N_2^{\frac{1}{2}} P_0 N_2^{\frac{1}{2}} > 0$ when $N_2 \geq 0$ or $\det(I + \tilde{N}_2 P_0 \tilde{N}_2) \neq 0$ when $N_2 \leq 0$ and $\tilde{N}_2 = (-N_2)^{\frac{1}{2}}$. If additionally $\mathrm{Ker}(P_2) \subseteq \mathrm{Ker}(P_0^T)$, the necessary and sufficient condition for the internal stability of $[P(s), P_s(s)]$ reduces to $J^T P_s(0)J < 0$ and if furthermore $P_2 > 0$, the necessary and sufficient condition for the internal stability of $[P(s), P_s(s)]$ reduces to $P_s(0) < 0$.

2) Suppose $P_2 = 0$, $P_1 \neq 0$, N_1 is sign definite and $F_1^T P_s(0) F_1$ is non-singular. $[P(s), P_s(s)]$ is internally stable if and only if $F_1^T P_s(0) F_1 < 0$ and either $I - N_1^{\frac{1}{2}} P_0 N_1^{\frac{1}{2}} > 0$ when $N_1 \geq 0$ or $\det(I + \tilde{N}_1 P_0 \tilde{N}_1) \neq 0$ when $N_1 \leq 0$ and $\tilde{N}_1 = (-N_1)^{\frac{1}{2}}$. If furthermore $\mathrm{Ker}(P_1^T) \subseteq \mathrm{Ker}(P_0^T)$, the necessary and sufficient condition for internal stability of $[P(s), P_s(s)]$ reduces to $F_1^T P_s(0) F_1 < 0$ and if additionally P_1 is invertible, the necessary and sufficient condition for internal stability of $[P(s), P_s(s)]$ reduces to $P_s(0) < 0$.

Next we present the second main result of this paper with the following notation: $\bar{P}_2 = \lim_{s\to 0} s^2 \bar{P}(s)$, $\bar{P}_1 = \lim_{s\to 0} s(\bar{P}(s) - \frac{\bar{P}_2}{s^2})$, and $\bar{P}_0 = \lim_{s\to 0}(\bar{P}(s) - \frac{\bar{P}_2}{s^2} - \frac{\bar{P}_1}{s})$.

Theorem 2: Given a graph $\mathcal{G}$ with incidence matrix $\mathcal{Q}$, satisfying Assumption 1 and modelling the communication links among multiple strictly proper NI agents $\hat{P}_i(s)$ (allowing possible poles at the origin) which are appropriately extended to $P_i(s)$ as in Fig. 3, robust output feedback consensus is achieved via the feedback control law in (3) or (4) under any external disturbances $w_1 \in \mathrm{Im}_{\mathfrak{L}_2}(\mathcal{Q} \otimes I_m)$ and $w_2 \in \mathfrak{L}_2$ as well as under any model uncertainty $\Delta_i(s), i = 1, \cdots, n$ satisfying Assumption 2 if and only if the necessary and sufficient conditions in Lemma 8 are satisfied for $[\bar{P}(s), \bar{P}_s(s)]$.

Proof: Lemma 8 guarantees the internal stability of $[\bar{P}(s), \bar{P}_s(s)]$. Nominal output consensus is then achieved without considering the external disturbances w_1 and w_2 via internal stability as discussed in the proof of Theorem 1.

Then, similar analysis as in the proof of Theorem 1 guarantees robustness against both external disturbances as well as additive SNI model uncertainty. ■

Remark 2: When the SNI controllers are homogeneous, the consensus law (3) simplifies to $\boldsymbol{u} = (\mathcal{Q} \otimes I_m)(I_n \otimes P_s(s))(\mathcal{Q}^T \otimes I_m)\boldsymbol{y} = \mathcal{L}_n \otimes P_s(s)\boldsymbol{y}$, or in a distributed manner, $\boldsymbol{u}_i = P_s(s)\sum_{k=1}^n a_{ik}(\boldsymbol{y}_i - \boldsymbol{y}_k)$. It can be seen that this captures the main result of [18] in the homogeneous plant case but also generalises the results to the heterogeneous plant case. In the case of heterogeneous SNI controllers, the controller is given by $\boldsymbol{u} = (\mathcal{Q} \otimes I_m)\bar{P}_s(s)(\mathcal{Q}^T \otimes I_m)\boldsymbol{y} = \bar{\mathcal{L}}_e(s)\boldsymbol{y}$, which can be interpreted as a weighted graph $\mathcal{G}$ with the edges weighted by the controller transfer functions $P_{s,j}(s)$, $j = 1, \cdots, l$, or in a distributed manner: $\boldsymbol{u}_i = \sum_{k=1}^n a_{ik} P_{s,j}(s)(\boldsymbol{y}_i - \boldsymbol{y}_k)$, where j is the edge connected vertex i and k. The above facts give a nice intuitive interpretation and explain why we adopt the incidence matrix for the distributed property rather than the Laplacian matrix as indicated earlier.

IV. Illustrative Examples

This section gives two numerical examples to validate the main results of this paper, Theorems 1 and 2 respectively.

A. 2 lightly damped and 1 undamped flexible structures

2 lightly damped flexible structures with different parameters and 1 undamped flexible structure are considered to illustrate Theorem 1. The dynamics of the combined flexible structure can be represented by Fig.2 in [9]: $M_i\ddot{\boldsymbol{x}}_i + C_i\dot{\boldsymbol{x}}_i + K_i\boldsymbol{x}_i = \boldsymbol{u}_i$, $\boldsymbol{y}_i = \boldsymbol{x}_i$, $i = 1, \cdots, 3$, where $\boldsymbol{x}_i = \begin{bmatrix} x_{i,1} \\ x_{i,2} \end{bmatrix}$, $\boldsymbol{u}_i = \begin{bmatrix} u_{i,1} \\ u_{i,2} \end{bmatrix}$, $M_i = \begin{bmatrix} m_{i,1} & 0 \\ 0 & m_{i,2} \end{bmatrix}$, $C_i = \begin{bmatrix} c_{i,1} + c_i & -c_i \\ -c_i & c_{i,2} + c_i \end{bmatrix}$, $K_i = \begin{bmatrix} k_{i,1} + k_i & -k_i \\ -k_i & k_{i,2} + k_i \end{bmatrix}$. The undamped flexible structure is given by letting the damped term $C_i = 0$. The parameters are given as follows:

S1: $k_1 = k_{1,1} = k_{1,2} = 0.5$, $c_1 = c_{1,1} = c_{1,2} = 0.2$, $m_{1,1} = m_{1,2} = 1$ with initial condition of $\begin{bmatrix} 0.5 & 0.1 & 1 & 0.2 \end{bmatrix}^T$

S2: $k_2 = k_{2,1} = k_{2,2} = 1$, $c_2 = c_{2,1} = c_{2,2} = 0.1$, $m_{2,1} = 1, m_{2,2} = 0.5$ with initial condition of $\begin{bmatrix} 1 & 0.1 & 1.5 & 0.2 \end{bmatrix}^T$;

S3: $k_3 = k_{3,1} = k_{3,2} = 1$, $c_3 = c_{3,1} = c_{3,2} = 0$, $m_{3,1} = 1, m_{3,2} = 0.5$ with initial condition of $\begin{bmatrix} 1.5 & 0.1 & 2 & 0.2 \end{bmatrix}^T$.

The communication topology is given in Fig. 5 and thus $\mathcal{Q} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix}$ and $\mathcal{L}_n = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$. Both the SNI controllers are chosen as $\frac{1}{s+8}$ with an initial condition of -1 such that all the suppositions of Theorem 1 are satisfied. As shown in Fig. 6, the robust output feedback consensus can be achieved via controller in (3) and (4) without or even under external disturbances as well as SNI model uncertainties given by $\frac{1}{s+4}$.

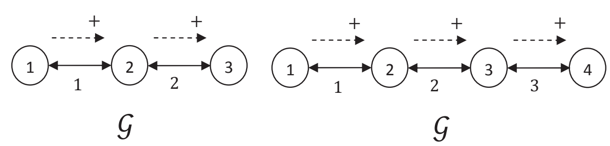

Fig. 5. Graph for 3 and 4 NI systems

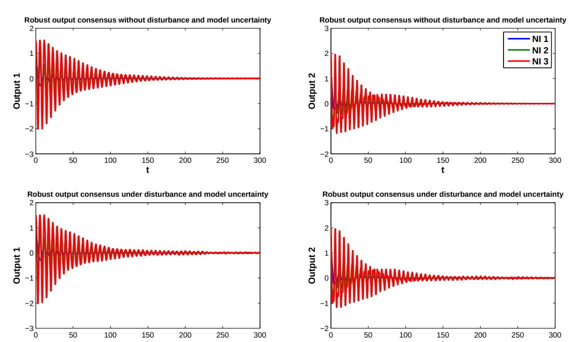

Fig. 6. Robust output consensus of heterogeneous NI systems

B. 1 single integrator, 1 double integrator, 1 undamped and 1 lightly damped flexible structure

A very complicated case containing 1 single integrator, 1 double integrator, 1 undamped and 1 lightly damped flexible structure is considered in this example. For consistency of dimension, the single integrator and the double integrator are extended as follows: $\begin{bmatrix} \frac{1}{s} & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{s^2} \end{bmatrix}$, which also means that the output of single integrator will be coordinated with first outputs of both the undamped and the lightly damped flexible structures, while the output of double integrator will be coordinated with second outputs of both the undamped and the lightly damped flexible structures. The parameters of all NI systems are as follows:

S1: $\frac{1}{s^2}$ with initial condition of $\begin{bmatrix} 1 & 0.1 \end{bmatrix}^T$;
S2: $\frac{1}{s}$ with initial condition of 2;
S3: $k_3 = k_{3,1} = k_{3,2} = 1$, $c_3 = c_{3,1} = c_{3,2} = 0$, $m_{3,1} = 1, m_{3,2} = 0.5$ with initial condition of $\begin{bmatrix} 3 & 0.1 & 3 & 0.2 \end{bmatrix}^T$;
S4: $k_4 = k_{4,1} = k_{4,2} = 1$, $c_4 = c_{4,1} = c_{4,2} = 0.1$, $m_{4,1} = 1, m_{4,2} = 0.5$ with initial condition of $\begin{bmatrix} 4 & 0.1 & 4 & 0.2 \end{bmatrix}^T$.

The communication topology is given in Fig. 5 and thus

$$\mathcal{Q} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \text{ and } \mathcal{L}_n = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

All three SNI controllers are chosen as $-\frac{s+1}{s+2}$ with an initial condition of 0.1. Therefore, the internal stability of $[\bar{P}(s), \bar{P}_s(s)]$ can be verified by calculating $\det(I + \tilde{N}_f \bar{P}_0 \tilde{N}_f + \tilde{N}_f \bar{P}_1 J (J^T J)^{-2} J^T \bar{P}_1^T \tilde{N}_f) = 3.7813 \neq 0$ as shown in Theorem 2. As shown in Fig. 7, it can be seen that robust output feedback consensus can be achieved via controller (3) or (4) without or even under external disturbances as well as model uncertainties.

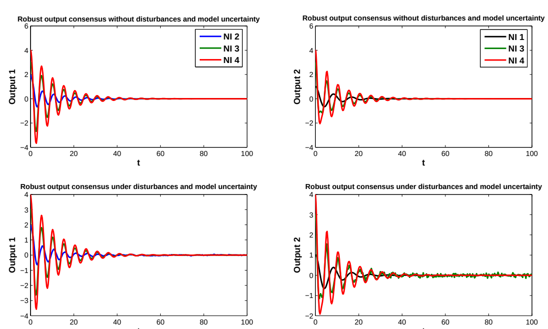

Fig. 7. Robust output consensus of heterogeneous NI systems

V. Conclusion and Future Directions

A robust output feedback consensus algorithm for heterogeneous NI systems against external disturbances and NI model uncertainty is proposed by using the incidence matrix rather than the Laplacian matrix as well as NI system theory. The key contributions of this paper can be summarised as: a) solving robust cooperative problems for general heterogeneous networks of MIMO NI systems under any undirected and connected graph; b) directly addressing robustness to exogenous disturbances and SNI model uncertainty; c) only exploiting output feedback information in contrast to full state information commonly used in the literature; d) providing a whole class of cooperative control laws, i.e. SNI controllers that can be tuned for performance and characterising conditions that can be easily checked for robust output feedback consensus; e) showing how consensus problems can exploit powerful internal stability and robust stability results available in the control theory literature.

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