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Revisiting robust stabilization of coprime factors: The general case

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Index Terms—**robust stability; uncertain linear systems; distance measures; observer-form controller.**

Abstract—**This article considers the robust stabilization problem of uncertain linear-time invariant plants with coprime factor uncertainty bounded in $\mathscr{RH}_\infty$. The problem considered here is a generalization of the normalized coprime factor robust stabilization problem. It is shown that the problem admits a simple and intuitive controller implementation parameterized in terms of a state-feedback matrix F and observer gain L. The choice of a state-feedback matrix F induces a metric in which distance between plants is measured. Subsequently, an observer gain L can be obtained to maximize robustness of the controller in this metric via the solution of a Riccati equation. This synthesis method results in a controller of the same order as the nominal plant. It is also shown that non-normalized coprime factorizations are a more suitable tool for obtaining robustly stabilizing controllers for uncertain lightly damped plants than normalized coprime factorizations, which only provide very limited robustness guarantees.**

I. Introduction

A key aspect of designing optimally robust controllers in the $\mathscr{H}_\infty$ setting is the choice of uncertainty structure. The main result on the robust stabilization of plants with uncertainty bounded in $\mathscr{RH}_\infty$ [1] assumes a generalized plant, based on which many different uncertainty structures can be formulated. It is well known that different uncertainty structures possess different properties [2]–[4]. The normalized coprime factor uncertainty structure [5] has been widely used, as it allows for the representation of an uncertain number of right half-plane poles and zeros. The $\mathscr{H}_\infty$ loop-shaping design procedure [6], [7] is based on normalized coprime factor uncertainty. Various complex experimental results for this method are reported e.g. in [8], [9].

Closely related to the robust stabilization problem is the question of which plants are similar in an $\mathscr{H}_\infty$ sense, i.e. of the metric connected to a particular uncertainty structure. The gap metric is a distance measure corresponding to normalized coprime factor uncertainty [10]–[13]. Plants which are close as measured by the gap metric are robustly stabilized by a controller designed for one plant with sufficiently large normalized coprime factor robust stability margin. The ν-gap is a less conservative measure of distance for normalized coprime factor uncertainty than the gap metric [14], [15]. More recently, distance measures have also been described for other uncertainty structures [4], [16], [17], including for coprime factor uncertainty that is not necessarily normalized (see also [18]). While normalized coprime factor uncertainty is extremely versatile, it has been known for some time that it is problematic for plants with uncertain lightly damped poles and zeros [4], [15], [18], [19]. Even optimally robust controllers can not be guaranteed to stabilize plants with small changes in the location of particular lightly damped poles/zeros. In this article, lower bounds for the ν-gap between multiple-input, multiple-output (MIMO) systems with such features are provided, showing that this is indeed a severe problem in the normalized coprime factor framework. Upper bounds on the robust stability margin in the presence of poles/zeros on the imaginary axis are also provided, showing that the problem compounds on both sides.

It was shown in [18] that coprime factor uncertainty (not normalized) provides robust stability guarantees which are less conservative than the normalized case. This was exploited in [18] to synthesize robustly stabilizing controllers for combinations of a nominal and one or multiple perturbed plants. This article provides a comprehensive and systematic interpretation of the general coprime factor robust stabilization problem based on a state-space approach. It is a well known fact that the central controller of the normalized coprime factor stabilization problem can be implemented in observer form, characterized by a state-feedback matrix F and an observer gain L [2], [20]. If one does not restrict the coprime factors to be normalized, one of these two matrices may be freely chosen by the designer (subject to a stability constraint), with the other being synthesized for optimal robustness. The approach taken herein is to allow the designer to choose a state-feedback F and to then synthesize an optimally robust observer gain L. It is shown that the choice of F induces a particular distance measure, and subsequently L determines how robust the resulting controller is in the metric induced by F. This provides enormous freedom to tailor the robustness optimization to particular uncertainty. This freedom is exploited in the final section of the article to obtain a state-feedback F which places the poles of a coprime factorization of the plant within a circular region in the left-half plain, inducing a metric in which uncertain lightly damped poles/zeros can be more easily stabilized.

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Consequently, this work also indicates that once a state feedback design is specified, the observer gain L should be designed to robustify the state feedback control law to coprime factor uncertainty, unlike the common practice of choosing a Kalman filter with Kalman gain L whenever an observer is required due to missing state measurements.

All plants in this article are assumed to be strictly proper, linear-time invariant systems. In the context of the $\mathscr{H}_\infty$ loop-shaping procedure, they are shaped plants, i.e. with performance weights included, and are therefore denoted P_s. The controller in this paper is C_∞, i.e. the controller before the wrapping around of the loopshaping weights.

In the following, Section II reviews state-space realizations of coprime factors, distance measures and robust stability margins. Subsequently, Section III describes the difficulties related to lightly damped uncertain systems in the ν-gap metric. Section IV is the key section of this article, containing the main synthesis theorem and a number of remarks concerning its interpretation. The final section describes a method for obtaining coprime factorizations which are more suitable for handling uncertainty in lightly damped systems.

A. Notation

Notation is standard. Denote by $\mathbb{C}$ the field of complex numbers, and by $\mathbb{C}^-$ the subset $\{s \in \mathbb{C} : \operatorname{Re}(s) < 0\}$. Let $\mathscr{R}$ denote the set of proper real-rational transfer functions. Denote by $\|\cdot\|_2$ the $\mathscr{H}_2$-norm of an operator, and by $\|\cdot\|_\infty$ the $\mathscr{H}_\infty$-norm of an operator. Also, let P^* denote the $\mathscr{L}_2$-adjoint of $P \in \mathscr{R}$ defined by $P^*(s) = P(-s)^T$. Let $\mathscr{RH}_\infty$ denote the space of proper real-rational functions bounded and analytic in the open right half complex plane. The ordered pair $\{N, M\}$, with $N \in \mathscr{RH}_\infty^{p\times q}$, $M \in \mathscr{RH}_\infty^{q\times q}$ is a right coprime factorization (rcf) of $P \in \mathscr{R}^{p\times q}$ if M is invertible in $\mathscr{R}^{q\times q}$, $P = NM^{-1}$ and N and M are right coprime over $\mathscr{RH}_\infty$.The ordered pair $\left\{\tilde{N}, \tilde{M}\right\}$, with $\tilde{N} \in \mathscr{RH}_\infty^{p\times q}$, $\tilde{M} \in \mathscr{RH}_\infty^{p\times p}$ is a left coprime factorization (lcf) of $P \in \mathscr{R}^{p\times q}$ if $\tilde{M}(s)$ is invertible in $\mathscr{R}^{p\times p}$, $P = \tilde{M}^{-1}\tilde{N}$ and $\tilde{N}$ and $\tilde{M}$ are left coprime over $\mathscr{RH}_\infty$. Also define the right and left graph symbols

$$G := \begin{bmatrix} N \\ M \end{bmatrix}, \quad \tilde{G} := \begin{bmatrix} -\tilde{M} & \tilde{N} \end{bmatrix}. \tag{1}$$

A right coprime factorization $\{N, M\}$ of P is called normalized if G as defined in (1) is inner. Similarly, a left coprime factorization $\left\{\tilde{N}, \tilde{M}\right\}$ of P is called normalized if $\tilde{G}$ as defined in (1) is co-inner. For a plant $P \in \mathscr{R}$ and a controller $C \in \mathscr{R}$, let $[P, C]$ denote the positive feedback interconnection displayed in Fig. 1 when $\Delta_\mathrm{N} = 0$, $\Delta_\mathrm{M} = 0$.

II. Coprime Factors, Distance Measures and Robust Stability Margins

This section recalls results on the state-space realizations of coprime factors of rational transfer function matrices, as well as distance measures and robust stability margins for plants with coprime factor uncertainty characterization. In contrast to [4], [18], where the distance measures and robust stability margins for general coprime factor uncertainty were previously defined in operator terms, the notation in this article is updated to reflect the state-space approach.

Strictly proper plants are assumed for mathematical convenience, but this assumption is not restrictive since pre- and post-compensator weights in loop-shaping are typically chosen such that the gain at high frequency approaches zero.

Lemma 1. *[2], [11] Given $P_\mathrm{s} \in \mathscr{R}^{p\times q}$ with a stabilizable and detectable state-space realization*

$$P_\mathrm{s} = \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right],$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times q}$, $C \in \mathbb{R}^{p\times n}$, let $F \in \mathbb{R}^{q\times n}$ be such that $A + BF$ is Hurwitz. Define

$$\begin{bmatrix} N_0 \\ M_0 \end{bmatrix} := \left[\begin{array}{c|c} A + BF & B \\ \hline C & 0 \\ F & I \end{array}\right]. \tag{2}$$

Then $\{N_0, M_0\}$ is a right coprime factorization of P_s.

The matrix F in (2) is a free parameter (subject to the stability constraint) that induces a specific rcf. A particular choice of F will result in the rcf being normalized, as can be seen from the following theorem.

Theorem 2. *Given $P_\mathrm{s} \in \mathscr{R}^{p\times q}$ with state and output equations given by $\dot{x} = Ax + Bu$, $y = Cx$, where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times q}$, $C \in \mathbb{R}^{p\times n}$, assume (A, B) is controllable and (C, A) has no unobservable modes on the imaginary axis. Let $F = -B^T X$, where $X \geq 0$ is the stabilizing solution to*

$$XA + A^T X - XBB^T X + C^T C = 0. \tag{3}$$

Then,

1) *the particular rcf $\{N, M\}$ of P_s induced by F via (2) is a normalized rcf; and*
2) *the unique state-feedback $u = r + Fx$ applied to P_s minimizes*

$$\|T_\mathrm{zr}\|_2 = \left\| \begin{bmatrix} N \\ M - I \end{bmatrix} \right\|_2,$$

over all $F \in \mathbb{R}^{q\times n}$ for $z = [y^* \quad u^*]^*$, with the resulting minimum cost given by $\|T_\mathrm{zr}\|_2^2 = \operatorname{trace}(B^* X B)$.

Proof: Item 1 corresponds to [2, Theorem 13.37, a)] and Item 2 corresponds to the standard $\mathscr{H}_2$ result for state-feedback, see e.g. [2, Section 14.8.1]. ■

Remark 1. *The connection between normalized coprime factors and $\mathscr{H}_2$ optimal control is well known [2], [21], see also [20, Section 10.5]. It is here restated formally to point out that the state-feedback matrix F which normalizes the rcf of P_s is essentially an $\mathscr{H}_2$-optimal state feedback. It will be shown that this particular choice of F may under certain conditions compromise the achievable robustness of controllers optimized with respect to coprime factor uncertainty.*

The corresponding results for normalized lcfs are omitted, as subsequent developments focus mostly on rcfs. Consider now a plant with right coprime factor uncertainty, as shown in Fig. 1. This setup corresponds to the following equation

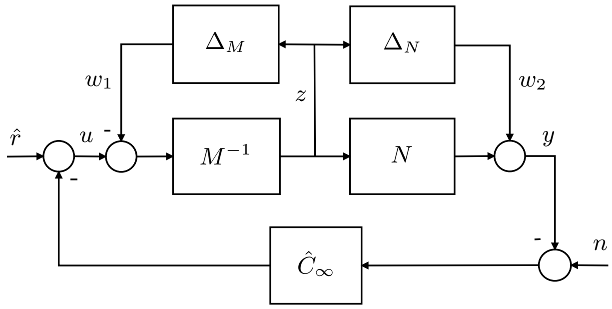

Fig. 1. A plant with right coprime factor uncertainty.

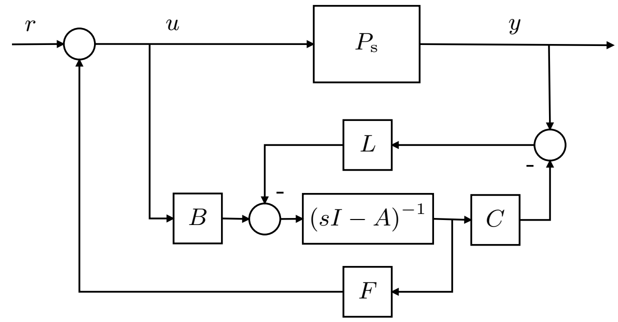

Fig. 2. Observer form implementation of the controller.

for a perturbed plant P_Δ, with $\{N_0, M_0\}$ being a rcf (not necessarily normalized) of the nominal plant P_s:

$$P_\Delta = (N_0 + \Delta_\mathrm{N})(M_0 + \Delta_\mathrm{M})^{-1}.$$

The following definition (see [4, Section VII] and also [18]) of a generalized right coprime factor distance measure (between a nominal plant P_s and a perturbed plant P_Δ) is parameterized in terms of the rcf of a nominal plant induced by the matrix F. The perturbed plant P_Δ enters the definition via a normalized lcf, for which there is no free parameter as is clear from Theorem 2.

Definition 1. *Given $P_\mathrm{s}, P_\Delta \in \mathscr{R}^{p\times q}$ and $F \in \mathbb{R}^{q\times n}$, let $\tilde{G}_\Delta$ be the normalized left graph symbol of P_Δ and G_0 the not necessarily normalized right graph symbol of P_s induced by F via* (2)*. Define the right coprime factor distance measure*

$$d_\mathrm{rcf}(P_\mathrm{s}, P_\Delta; F) := \left\| \tilde{G}_\Delta G_0 \right\|_\infty.$$

This distance measure reduces to the well known ν-gap metric [14], [15] if F is chosen to normalize the rcf of P_s as in Theorem 2. A distance measure is typically considered in conjunction with a robust stability margin which quantifies up to which distance from the nominal plant robust stability is guaranteed. See [4], [15]–[18] for further remarks and robust stability and performance theorems for various uncertainty structures. Robust stability is ensured for P_Δ's with a distance less than the robust stability margin, which also fulfill a winding number constraint (see [4, Theorem 7]). Consider a positive feedback interconnection of $P_\mathrm{s} \in \mathscr{R}^{p\times q}$ with state-space realization $P_\mathrm{s} = \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right]$ and a control law $u = C_\infty \begin{bmatrix} y \\ r \end{bmatrix}$, where C_∞ is implemented in observer form as in Fig. 2. The following state-space realization of C_∞ is illustrated in Fig. 2:

$$C_\infty := \left[\begin{array}{c|cc} A + BF + LC & -L & B \\ \hline F & 0 & I \end{array}\right]. \tag{4}$$

Denote by $\hat{C}_\infty$ the column of C_∞ corresponding to the transfer function from y to u. This corresponds to the controller $\hat{C}_\infty$ in Fig. 1. Full equivalence between Fig. 1 and Fig. 2 will be shown later (see Theorem 7).

With P_s given, C_∞ depends only on the choices of F and L. This implementation leads to the following definition of a generalized right coprime factor robust stability margin, for which only the $\hat{C}_\infty$ subpart of the controller is relevant.

Definition 2. *Given a positive feedback interconnection $[P_\mathrm{s}, C_\infty]$ of $P_\mathrm{s} \in \mathscr{R}^{p\times q}$ and a controller $C_\infty \in \mathscr{R}^{q\times p}$ in observer form induced by given matrices $F \in \mathbb{R}^{q\times n}$ and $L \in \mathbb{R}^{n\times p}$ via* (4)*, let $\{N_0, M_0\}$ be the not necessarily normalized rcf of P_s induced by F as in* (2)*. Define the right coprime factor robust stability margin of $[P_\mathrm{s}, C_\infty]$ as*

$$b_\mathrm{rcf}(P_\mathrm{s}; F, L) := \begin{cases} \left\| M_0^{-1} \left(I - \hat{C}_\infty P\right)^{-1} \begin{bmatrix} I & \hat{C}_\infty \end{bmatrix} \right\|_\infty^{-1} \\ \text{if } \begin{bmatrix} P_\mathrm{s}, \hat{C}_\infty \end{bmatrix} \text{ is internally stable;} \\ 0 \text{ otherwise.} \end{cases}$$

Remark 2. *The right coprime factor robust stability margin can also be defined for a generically structured C_∞, as in [4], [18]. The above formulation is chosen to highlight the impact of the state-feedback matrix F and observer-gain matrix L: F induces a rcf of P_s, and thereby the distance measure (via Definition 1). The choice of L will then impact the robust stabillity margin of the feedback interconnection in the metric induced by F (via Definition 2). There exists a controller in this implementation for any achievable robust stability margin, as will be shown subsequently.*

Remark 3. *If F is chosen such that the rcf of P_s is normalized, the right coprime factor robust stability margin reduces to the four-block/normalized coprime factor robust stability margin $b(P_\mathrm{s}, C_\infty)$ [13]–[15]. The observer-gain L induces a lcf. The definitions in this section can be mirrored for left coprime factor uncertainty [4], [18], but this paper deliberately opts for a right coprime formulation as induced by a state-feedback F, for reasons laid out in Remark 8. An analytical optimal robust stability margin $b_\mathrm{opt}(P_\mathrm{s})$ exists for normalized coprime factor uncertainty [5]. For generalized coprime factor uncertainty, the optimal stability margin can be computed via a line search.*

III. Lack of Robustness to Uncertain Lightly Damped Poles/Zeros

This section highlights the problems that uncertain lightly damped poles and zeros can cause for robust stability analysis and synthesis in a normalized coprime factor setting. The first two theorems describe bounds on the ν-gap when the nominal plant has a pair of zeros and poles, respectively, on the imaginary axis. The subsequent theorems show that these lightly damped features also impose constraints on the magnitude of the controller transfer function at the zero/pole frequency. To illustrate these difficulties, a simple benchmark example [15], [18], [19], [22] will be used throughout this section, which consists of a plant with uncertainty in the location of lightly damped zeros.

Example 1. *Consider the nominal plant*

$$P_{\mathrm{s}} = \frac{10\left(s^2+1\right)}{s^2\left(s^2+2\right)}. \tag{5}$$

A normalized rcf of P_{s} is given by

$$\begin{bmatrix} N \\ M \end{bmatrix} = \frac{1}{s^4+4.33s^3+11.4s^2+5.26s+10}\begin{bmatrix} 10\left(s^2+1\right) \\ s^2\left(s^2+2\right) \end{bmatrix}.$$

This rcf has two pairs of complex conjugate poles, one of them being extremely lightly damped (damping ratio $\zeta = 0.05$ at $\omega_{\mathrm{n}} = \pm 1.0025$). In terms of robustness, this is an undesirable effect of the requirement that the coprime factorization be normalized.

The following two theorems provide lower bounds on the ν-gap for plants with uncertain lightly damped zeros and poles, respectively. It will be seen that the ν-gap becomes very large even for small uncertainty if the uncertain zeros/poles are in particular frequency regions.

Theorem 3. *Given a nominal plant $P_{\mathrm{s}} \in \mathscr{R}^{p\times q}$ with a pair of transmission zeros at $s = \pm j\omega_0$, then for any $P_\Delta \in \mathscr{R}^{p\times q}$,*

$$\delta_\nu(P_{\mathrm{s}}, P_\Delta) \geq \sqrt{\frac{\underline{\sigma}^2\left(P_\Delta(j\omega_0)\right)}{1+\underline{\sigma}^2\left(P_\Delta(j\omega_0)\right)}}. \tag{6}$$

Proof: Assume, firstly, that P_{s} has full column normal rank. The case of full row normal rank is treated later. Let $\{N, M\}$ be a normalized rcf of P_{s}. Then

$$\exists 0 \neq u_0 \in \mathbb{C}^q \text{ s.t. } P_{\mathrm{s}}(j\omega_0)u_0 = 0.$$

Furthermore, $s = \pm j\omega_0$ must also be a transmission zero of N as it cannot be a pole of $M \in \mathscr{RH}_\infty$. Therefore,

$$\exists 0 \neq z_0 \in \mathbb{C}^q \text{ s.t. } N(j\omega_0)z_0 = 0.$$

The proof then follows via manipulation of the normalization equation

$$M(j\omega_0)^*M(j\omega_0) + N(j\omega_0)^*N(j\omega_0) = I,$$

and the lower bound on the ν-gap given by

$$\delta_\nu(P_{\mathrm{s}}, P_\Delta) \geq \overline{\sigma}\left(\tilde{N}_\Delta(j\omega_0)M(j\omega_0) - \tilde{M}_\Delta(j\omega_0)N(j\omega_0)\right).$$

The details are omitted for brevity and will be provided elsewhere. ■

Theorem 4. *Given a nominal plant $P_{\mathrm{s}} \in \mathscr{R}^{p\times q}$ with a pair of poles at $s = \pm j\omega_0$, then for any $P_\Delta \in \mathscr{R}^{p\times q}$,*

$$\delta_\nu(P_{\mathrm{s}}, P_\Delta) \geq \sqrt{\frac{1}{1+\overline{\sigma}^2\left(P_\Delta(j\omega_0)\right)}}. \tag{7}$$

Proof: Let $\{N, M\}$ be a normalized rcf of P_{s}. Since $N \in \mathscr{RH}_\infty$ and M has full normal rank,

$$\exists 0 \neq z_0 \in \mathbb{C}^q \text{ s.t. } M(j\omega_0)z_0 = 0.$$

Subsequent derivations follow the proof of Theorem 3 and are omitted here for brevity. ■

Remark 4. *These two theorems give lower bounds on the ν-gap for systems with a pole or zero on the imaginary axis. Theorem 3 implies that an uncertain undamped zero in an otherwise high-gain frequency range is problematic. If the zero occurs at a slightly different frequency in P_Δ, then $\underline{\sigma}\left(P_\Delta(j\omega_0)\right) >> 1$, and therefore $\delta_\nu(P_{\mathrm{s}}, P_\Delta) \approx 1$. A similar problem arises for uncertain undamped poles in an otherwise low-gain frequency range (Theorem 4). Therefore, while the state-feedback F that normalizes the rcf $\{N, M\}$ of P_{s} may allow a high robust stability margin $b(P_{\mathrm{s}}, C) \leq b_{\mathrm{opt}}(P_{\mathrm{s}})$, this section shows that this measure of robustness is deficient around lightly damped poles and zeros. The distance between P_{s} and any P_Δ with slightly differing lightly damped pole/zero locations will easily exceed any robust stability margin achieved by the controller for the nominal plant.*

Remark 5. *In the SISO case, these results simplify to the bounds given in [19] for the gap metric, which is itself bounded from below by the ν-gap metric [14], [15].*

Example 2. *Consider again the plant P_{s} given in* (5)*. Also let*

$$P_\Delta(s) = \frac{10\left(s^2+1.1\right)}{s^2\left(s^2+2\right)}. \tag{8}$$

The location of the zeros has been shifted slightly. From Theorem 3,

$$\delta_\nu(P_{\mathrm{s}}, P_\Delta) \geq \sqrt{\frac{\underline{\sigma}^2\left(P_\Delta(j)\right)^2}{1+\underline{\sigma}^2\left(P_\Delta(j)\right)^2}} = 0.7071.$$

This distance is large compared to the optimal normalized coprime factor robust stability margin $b_{\mathrm{opt}}(P_{\mathrm{s}}) = 0.3919$.The normalized coprime factor distance measure and robust stability margin do not provide any guarantee for the existence of a stabilizing controller for P_{s} that also robustly stabilizes P_Δ. Hence the ν-gap and the standard $\mathscr{H}_\infty$ loop-shaping theories abandon the designer here.

The following two results provide upper bounds on the achievable $b(P_{\mathrm{s}}, C)$ for plants with undamped zeros and poles, respectively.

Theorem 5. *Given a plant $P_{\mathrm{s}} \in \mathscr{R}^{p\times q}$ with transmission zeros at $s = \pm j\omega_0$, then for any controller $C \in \mathscr{R}^{q\times p}$,*

$$b(P_{\mathrm{s}}, C) \leq \min\left\{\sqrt{\frac{1}{1+\underline{\sigma}^2\left(C(j\omega_0)\right)}},\ b_{\mathrm{opt}}(P_{\mathrm{s}})\right\}. \tag{9}$$

Proof: If $[P_{\mathrm{s}}, C]$ is not internally stable, by definition $b(P_{\mathrm{s}}, C) = 0$ and (9) is automatically fulfilled. Otherwise, $b(P_{\mathrm{s}}, C) = \left\|\left(\tilde{G}K\right)^{-1}\right\|_\infty^{-1}$, where K is thr normalized right graph symbol of C. An obvious upper bound is given by $b_{\mathrm{opt}}(P_{\mathrm{s}})$. For a normalized lcf $\left\{\tilde{N}, \tilde{M}\right\}$ of P_{s}, a transmission zero at $s = j\omega_0$ implies that $\underline{\sigma}\left(\tilde{N}(j\omega_0)\right) = 0$. Assume that $\tilde{N}$ has full row normal rank. Then,

$$\exists 0 \neq \eta_0 \in \mathbb{C}^p \text{ s.t. } \eta_0^*\tilde{N}(j\omega_0) = 0.$$

Subsequent derivations follow from $b(P_{\mathrm{s}}, C) \leq \underline{\sigma}\left(\tilde{G}(j\omega_0)K(j\omega_0)\right)$. Details are omitted for brevity and will be reported elsewhere. ■

Theorem 6. *Given $P_{\mathrm{s}} \in \mathscr{R}^{p\times q}$ with poles at $s = \pm j\omega_0$, then for any $C \in \mathscr{R}^{q\times p}$,*

$$b(P_{\mathrm{s}}, C) \leq \min\left\{\sqrt{\frac{\overline{\sigma}^2\left(C(j\omega_0)\right)}{1+\overline{\sigma}^2\left(C(j\omega_0)\right)}},\ b_{\mathrm{opt}}(P_{\mathrm{s}})\right\}.$$

Proof: The proof mirrors the proof of Theorem 5 and details will be provided elsewhere. ■

Remark 6. *For good robustness in a normalized coprime factor sense, the controller should essentially "mimic" the behavior of P_{s}. At a lightly damped zero of the plant (Theorem 5), it is beneficial if $\underline{\sigma}\left(C(j\omega_0)\right)$ is also small, whereas at a lightly damped pole (Theorem 6), $\overline{\sigma}\left(C(j\omega_0)\right)$ should be large.*

IV. Right coprime factor synthesis

This section contains the key result for generalized (right) coprime factor synthesis. The main theorem describes an observer form controller as in Fig. 2 achieving a certain level of robust stability margin. Several subsequent observations provide a comprehensive interpretation of the synthesis theorem in view of the observer form implementation of the central controller. The crucial aspect is that the state-feedback matrix F may be chosen freely by the designer, resulting in coprime factorizations that are not necessarily normalized.

Theorem 7. *Given a plant $P_{\mathrm{s}} \in \mathscr{R}^{p\times q}$ with a stabilizable and detectable state-space realization*

$$P_{\mathrm{s}} = \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right],$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times q}$, $C \in \mathbb{R}^{p\times n}$, $\epsilon > 0$ and $F \in \mathbb{R}^{q\times n}$ such that $A + BF$ is Hurwitz, then there exists an observer gain $L_\infty \in \mathbb{R}^{n\times p}$ such that

$$b_{\mathrm{rcf}}\left(P_{\mathrm{s}}; F, L_\infty\right) > \epsilon \tag{10}$$

if and only if $\epsilon < 1$ and there exists a stabilizing solution $Y_\infty \geq 0$ solving

$$\left(A - \frac{\epsilon^2}{1-\epsilon^2}BF\right)Y_\infty + Y_\infty\left(A - \frac{\epsilon^2}{1-\epsilon^2}BF\right)^T + Y_\infty\left(\frac{\epsilon^2}{1-\epsilon^2}F^TF - C^TC\right)Y_\infty + \frac{1}{1-\epsilon^2}BB^T = 0.$$

If these conditions are satisfied, one such controller is given by

$$C_\infty(s) = \left[\begin{array}{c|cc} A+BF+L_\infty C & -L_\infty & B \\ \hline F & 0 & I \end{array}\right], \tag{11}$$

with $L_\infty = -Y_\infty C^*$.

Proof: This theorem is a dual to [2, Theorem 18.1]. The resulting central controller achieving $b_{\mathrm{rcf}}\left(P; F, L_\infty\right) > \epsilon$ is given by

$$\hat{C}_\infty = \left[\begin{array}{c|c} A+BF+L_\infty C & -L_\infty \\ \hline F & 0 \end{array}\right].$$

It is then possible to show that equivalence between the configuration in Fig. 2 and Fig. 1 is achieved by letting $r = \tilde{V}_0\hat{r}$, with $\left\{\tilde{U}_0, \tilde{V}_0\right\}$ being a lcf of $\hat{C}_\infty$ for which $\tilde{V}_0M_0 - \tilde{U}_0N_0 = I$, where existence of such $\left\{\tilde{U}, \tilde{V}\right\}$ is guaranteed by $\hat{C}_\infty$ being stabilizing. ■

Remark 7. *The controller C_∞ given in* (11) *is in observer form with observer gain matrix L_∞ and state-feedback gain matrix F. This implementation is shown in Fig. 2. The state-feedback matrix F may be freely chosen by the designer, under the obvious restriction that $A+BF$ be Hurwitz. One can therefore interpret the right coprime factor $\mathscr{H}_\infty$ optimization problem of maximizing ϵ in* (10) *as finding the* optimally robust *observer for a given state feedback gain matrix F. This is in contrast to the Kalman filter, which is optimal in an $\mathscr{H}_2$ sense, but provides no robustness guarantees when used as an observer for state-feedback control [2, Section 14.10], [23]. In fact, it is well known that L_∞ converges to the Kalman filter gain when $\epsilon \to 0$, i.e. when no robustness is required [2, Section 16.2], [20, Section 10.5]. As can be seen from the definitions of the distance measure $d_{\mathrm{rcf}}(P_{\mathrm{s}}, P_\Delta; F)$ and the robust stability margin $b_{\mathrm{rcf}}\left(P_{\mathrm{s}}; F, L\right)$, F and L have distinct roles in the robust control setting. Choosing the state-feedback matrix F induces a distance measure. Once F has been chosen, $b_{\mathrm{rcf}}\left(P_{\mathrm{s}}; F, L\right)$ depends only on the observer gain L. The above theorem shows that there exists a unique observer gain L_∞ which optimizes the robust stability margin for a given F, or a given metric (where optimization can be achieved via ϵ-iterations).*

Remark 8. *The robust stabilization problem of a plant with left coprime factor uncertainty [2, Theorem 18.1], conversely, can be interpreted as finding an optimally robust state-feedback gain matrix $F_\infty = -B^TX_\infty$ given a particular observer gain matrix L. While this is possible and indeed all results are just duals of the results given in this article, it seems fruitful, in light of this interpretation, to consider the right coprime factor synthesis problem, since a procedure that allows the designer to specify a state-feedback gain and then automatically synthesizes an optimal observer will be more intuitive to apply in practice.*

Remark 9. *The connection between coprime factorizations and state-space representions was established in [11], [21], [24], though its meaning in an optimization sense as described in Remark 7 seems to have remained little understood. With the increasing focus on synthesis for normalized coprime factorizations [5]–[7], the design freedom available via the variation of F seems to have become less noticed. Non-normalized coprime factor synthesis was also adressed in [18], but with a method that resulted in a controller of order larger than the order of the plant. This is avoided here.*

V. Ensuring well-dampedness via a circular pole-constraint

The state-feedback matrix F need not be chosen in such a way that the rcf of P_{s} induced by F is normalized. As shown in Section III, normalization may have undesired effects in terms of robustness with respect to uncertain lightly damped poles/zeros. This section describes one particular method for choosing F in a way that the poles of the coprime factorization of P_{s} induced by F are sufficiently well damped, which in turn leads to good robustness with respect to uncertain lightly damped pole/zero locations. Consider the following definition of a cirular region in the closed left-half complex plane with center at $s=-q$ and a radius of $\rho \leq q$.

Definition 3. *Let $q > 0,\ \rho > 0$ with $q \geq \rho$. Denote by $\mathcal{C}(q,\rho)$ the set $\left\{ s \in \mathbb{C}^- :\ \sqrt{\left(\Re(s)+q\right)^2 + \Im(s)^2} < \rho \right\}$.*

The following theorem describes a method for ensuring sufficient damping of the poles of a rcf induced by a state feedback F. As shown above, a coprime factorization with lightly damped poles induces a metric in which uncertainty in lightly damped poles/zeros of a plant can be problematic.

Theorem 8. *Given a system $P_{\mathrm{s}} \in \mathscr{R}^{p\times q}$ with state and output equation $\dot{x} = Ax + Bu$, $y = Cx$, where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times q}$, $C \in \mathbb{R}^{p\times n}$, assume (A,B) is controllable and (C,A) is observable. Also given $q > 0$, $\rho > 0$ s.t. $q \geq \rho$, define $A_{\mathrm{q},\rho} := \frac{1}{\rho}(A+qI)$ and $B_\rho := \frac{1}{\rho}B$. Let*

$$F := -\left(I + B_\rho^* X B_\rho\right)^{-1}\left(B_\rho^* X A_{\mathrm{q},\rho}\right),$$

where $X \geq 0$ is the stabilizing solution to the discrete-time Algebraic Riccati Equation

$$A_{\mathrm{q},\rho}^* X\left(I + B_\rho B_\rho^* X\right)^{-1} A_{\mathrm{q},\rho} - X + C^*C = 0. \qquad (12)$$

Then,

1) *the right coprime factorization $\{N_0, M_0\}$ of P_{s} induced by F via* (2) *has all its poles in $\mathcal{C}(q,\rho)$; and*
2) *the unique state-feedback $u = r + Fx$ minimizes*

$$\nu(X) = \frac{1}{\rho^2}\operatorname{trace}\left(B^* X B\right),$$

where $\nu(X)$ fulfills the following inequality:

$$\frac{1}{\rho}\left\|\begin{bmatrix} N_0 \\ M_0 - I\end{bmatrix}\right\|_2^2 \leq \nu(X),$$

where $\{N_0, M_0\}$ is the rcf of P induced by F.

Proof: This result follows from [25, Theorem 4.3]. ■

Remark 10. *For $q,\rho \to \infty$, the circular pole constraint region $\mathcal{C}(q,\rho)$ approaches the open left-half complex plane, and the results of Theorem 8 converge to those of Theorem 2, i.e. $\{N, M\}$ becomes a normalized rcf of P_{s}.*[1]

Example 3. *Consider again P_{s} and P_Δ given in* (5) *and* (8)*, respectively. It was noted in Example 2 that the existence of a robustly stabilzing controller for P_Δ can not be guaranteed on the basis of using a normalized coprime factorization of P_{s}. However, by choosing a state-feedback matrix F which ensures that the eigenvalues of $A+BF$ are more strongly damped (via a circular criterion) than those of the normalized rcf given in Example 1, a robustly stabilizing controller can be obtained via Theorem 7. Let $q = 2.9$, $\rho = 2.5$. Then, via Theorem 8, $F = [-1.5531 \quad -1.6026 \quad -3.9737 \quad -1.7731]$. To obtain an optimally robust observer gain, Theorem 7 is applied. Via a line search on ϵ, the optimal $b_{\mathrm{rcf}}(P_{\mathrm{s}}; F, L_\infty) = 0.0835$ is found. The corresponding observer gain is $L_\infty = [-2.0649 \quad 10.4187 \quad -4.8039 \quad -7.6397]^*$. While $b_{\mathrm{rcf}}(P_{\mathrm{s}}; F, L_\infty)$ is smaller than $b_{\mathrm{opt}}(P_{\mathrm{s}})$, the metric induced by F is much more benign for the uncertain lightly damped zeros of P_{s}: $d_{\mathrm{rcf}}(P_{\mathrm{s}}, P_\Delta; F) = 0.0804$. This is less than the robust stability margin, and hence (with a winding number condition also holding) robust stability of $[P_\Delta, C_\infty]$ is guaranteed via a dual to [4, Theorem 7].*

[1]Note that X in Theorem 8 is X in Theorem 2 scaled by r.

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