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LMI search for rational anticausal Zames–Falb multipliers

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Abstract— **Given a linear time-invariant plant, the search of a suitable multiplier over the class of Zames–Falb multiplier is a challenging problem which has been studied for several decades. Recently, a new linear matrix inequality search has been proposed over rational and causal Zames–Falb multipliers. This paper analyzes the conservatism of the restriction to causality on the multipliers and presents a complementary search for rational and anticausal multipliers.**

I. INTRODUCTION

The use of noncausal multipliers in absolute stability was widely studied in the Sixties, with particular attention to the class of slope-restricted nonlinearities. O'Shea [1], [2] was the first to propose a class of noncausal multipliers, see also [3]. Zames and Falb [4] propose a general framework for the use of noncausal multipliers in passivity theory and provide a formal proof for the results given in [2], since the validity of the results given by O'Shea in [2] was limited by "the a priori assumption that the solutions are bounded" [4]. Nowadays these multipliers are referred to as Zames–Falb multipliers. The rational Zames-Falb multipliers are defined as follows

$$\mathscr{M} = \{M(s) = 1 - H(s) : H(s) = \mathfrak{L}(h(t)) \ \int_{-\infty}^{\infty} |h(t)| dt < 1\}, \tag{1}$$

where $h : \mathbb{R} \to \mathbb{R}$ and $H(s)$ means the bilateral Laplace transform of $h(t)$, i.e. $H(s) = \int_{-\infty}^{\infty} h(t)e^{-st} dt$.

By use of a loop-transformation [5], the stability of a system $G \in \mathbf{RH}_\infty$ in feedback interconnection with any slope–restricted $S[0,k]$ and odd nonlinearity can be guaranteed if there exists a Zames–Falb multiplier M such that $M(G+1/k)$ is strictly positive, i.e.

$$\mathrm{Re}\left\{ M(j\omega)\left(G(j\omega) + \frac{1}{k}\right)\right\} > 0 \quad \forall \omega \in \mathbb{R}. \tag{2}$$

But given G, it is not straightforward to find such an M. The difficulty arises from the characterization of the Zames–Falb multipliers: their definition includes a time-domain response (1). The problem to be addressed is: given a system $G \in \mathbf{RH}_\infty$ and a constant $k > 0$, under what conditions is the existence of Zames–Falb multiplier $M \in \mathscr{M}$ ensured? To date, three partial solutions have been given.

1) In [6], [7], [8], a linear program is proposed to find a suitable irrational multiplier. This method requires the computation of the Nyquist plot over an infinitely dense frequency sweep, which is not computationally attractive. In general, the positiveness of the solution cannot be checked in an LMI framework.
2) In [9], a rational parametrization of a transfer function is proposed in such a way that its $\mathscr{L}_1$-norm can be bounded. A search over these set of parameters under the condition in (1) must be carried out. Once again, it can be optimized, for example, with a linear program using an infinitely dense frequency sweep [10] or using LMIs, see IQC toolbox [11]. If the discretization is used, the positiveness of the solution can be exactly checked after the search.
3) Recently, a linear matrix inequality (LMI) search over rational and causal Zames–Falb multipliers has been proposed in [12] (see also [13]), which is computationally more efficient, with some promising results (see [14], [15]). This search uses the multiobjective synthesis technique presented in [16].

The first two methods avoid any conservatism in the characterization of the multiplier when the nonlinearity is slope–restricted, since the parametrization is chosen in order to compute analytically the integral in (1). However, both include a serious conservatism when the nonlinearity is odd, since the integral can only be bounded using a triangular inequality. The practical implementation of these two methods requires an approximation via discretization and the solution is obtained by solving a linear program. The result depends on the skill of the user.

The last method is straightforward. The existence of a suitable multiplier can be guaranteed by checking the feasibility of a set of LMIs, but two main drawbacks can be stated:

- The search has an inherent conservativeness. For the check if a transfer function is a Zames–Falb multiplier, the integral in (1) is not computed, but bounded via an LMI. As commented in [16], this upper bound "can be fairly conservative".
- The multiplier is restricted to be causal and the same order of the plant.

The authors [12] justify the last assumption stating that other classes of multipliers, as used in the Circle and Popov criteria (see [17], [18]), and Park's method [19], are within this characterization. However, Park's method uses the following class of multipliers $M_p(s) = 1 + \frac{bs}{a^2 - s^2}$, where $a, b \in \mathbb{R}$. Hence causality is not required in Park's method [20].

Despite these sources of conservatism, the numerical results in [12], [13] are competitive with Park's method [19] for half of the examples. On the other hand, the search is worse

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than Park's method for the other half of the examples discussed, hinting that the causal restriction may be significant. The extension proposed in [14], [15] adds a Popov multiplier to the Zames–Falb multiplier and is able to improve some examples, but fails to reach the Nyquist value for Example 1. Since the Kalman conjecture is guaranteed for third order systems [21], this example shows some conservatism.

This paper analyzes the conservatism on limiting the search to causal Zames–Falb multipliers and proposes a complementary search for anticausal multipliers. Results indicate that a significant source of conservatism in [12] is the causal restriction. The best result of causal and anticausal searches provides at least competitive results for all examples. Due to space limitations, proofs are not included and the addition of Popov multipliers is not developed in this work, nevertheless it can be carried out in the same way as shown in [14], [15] and the corresponding results are shown in the paper.

II. Notation and Preliminary results

Part of the notation of this paper can be found in [20]. This paper focuses the stability of the feedback interconnection of a stable LTI system G and a slope-restricted nonlinearity ϕ_k, represented in Fig. 1 and given by

$$v = f + Gw, \qquad w = -\phi_k v \tag{3}$$

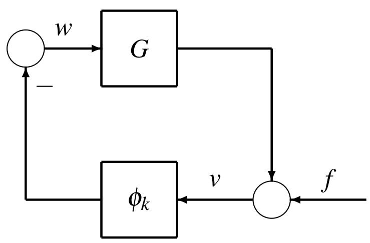

Fig. 1. Lur'e problem

Let $\bar{M}$ denote a linear time invariant operator mapping a time domain input signal to a time domain output signal and M denote the corresponding transfer function, for some particular region of convergence of the bilateral Laplace integral, mapping the bilateral Laplace transform of the time-domain input signal to the bilateral Laplace transform of the time domain output signal. To avoid ambiguity in impulse responses that correspond to transfer functions when the bilateral Laplace transform is used (see [22]), we insist on a causal $\bar{M}$ when $M \in \mathbf{RH}_\infty$ with the RHP contained in the region of convergence and an anticausal $\bar{M}$ when $M \in \mathbf{RH}_\infty^\perp$ with the LHP contained in the region of convergence. Since any $M \in \mathbf{RL}_\infty$ with a region of convergence that includes the imaginary axis can be split into the sum of two functions, one in $\mathbf{RH}_\infty$ and one in $\mathbf{RH}_\infty^\perp$, then the corresponding $\bar{M}$ is noncausal corresponding to the sum of a causal part and an anticausal part. Henceforward and with some abuse of notation, we will use the same notation for the operator and its transfer function.

Remark 2.1: In $\mathbf{H}_\infty$ theory a different terminology is used, e.g. [23], with stable, antistable and unstable transfer functions. We prefer the terminology of multiplier theory [5] as the term causal, anticausal, and noncausal describe more intuitively the behaviour of the impulse response.

The following theorem provides the absolute stability of system (3) subject to the search of an appropriate Zames–Falb multiplier.

Theorem 2.2 ([4], [5]): Consider the feedback system in Fig. 1 with $G \in \mathbf{RH}_\infty$, and ϕ_k a slope-restricted $S[0,k]$ and odd nonlinearity. Suppose that there exists a noncausal convolution operator $M : \mathscr{L}_2(-\infty,\infty) \to \mathscr{L}_2(-\infty,\infty)$ whose impulse response is of the form

$$m(t) = \delta(t) - \sum_{i=1}^{\infty} z_i \delta(t-t_i) - z_a(t), \tag{4}$$

where δ is the Dirac delta function and $\sum_{i=0}^{\infty} |z_i| < \infty$, $z_a \in \mathscr{L}_1$, and $t_i \in \mathbb{R}$ for all $i \in \mathbb{N}$. Assume that

$$\|z_a\|_1 + \sum_{i=1}^{\infty} |z_i| < 1 \tag{5}$$

and

$$\mathrm{Re}\{M(j\omega)(1+kG(j\omega))\} > 0 \quad \forall \omega \in \mathbb{R}. \tag{6}$$

Then the feedback interconnection (3) is $\mathscr{L}_2$-stable. ■

This theorem characterizes the class of Zames–Falb multipliers. In this paper, we restrict our attention to rational multipliers, i.e. $z_i = 0$ for all $i \in \mathbb{N}$.

Definition 2.3: Let $M \in \mathbf{RL}_\infty$ be a SISO rational transfer function in the form $M(s) = 1 + H(s)$, where $H(s)$ is a rational strictly proper transfer function. Then M belongs to the class of Zames–Falb multiplier, $\mathscr{M}$, if $\|H\|_1 < 1$.

In this paper the symbol $M^\sim$ means the $\mathscr{L}_2$-adjoint of M. This operator satisfies $\langle y, Mx \rangle = \langle M^\sim y, x \rangle$. for all $u \in \mathscr{L}_2(-\infty,\infty)$ and $y \in \mathscr{L}_2(-\infty,\infty)$. As a result, $M^\sim$ is anticausal if and only if M is causal [5]. In particular, the $\mathscr{L}_2$-adjoint of a rational transfer function $M(s)$ is given by $M^\top(-s)$. In the time domain, the impulse response is reflected with respect $t = 0$, i.e. given a linear operator M with a impulse response $m(t)$ then the impulse response of $M^\sim$ is $m^\top(-t)$. As a result, $M^\sim$ is an anticausal Zames–Falb multiplier if and only if M is a causal Zames–Falb multiplier.

The following lemma identifies when a transfer function is a Zames–Falb multiplier.

Lemma 2.4 ([13]): Let $M(s) \in \mathbf{RL}_\infty$ be a rational transfer function with $M(s) = M(\infty) + \hat{M}(s)$, where $\hat{M}(s)$ denotes its associated strictly proper transfer function. Then, $M(s)$ is a Zames–Falb multiplier if and only if $\|\hat{M}\|_1 < M(\infty)$. ■

Remark 2.5: The corresponding Lemma given in [13] is limited to $M(s) \in \mathbf{RH}_\infty$, but its extension to $M(s) \in \mathbf{RL}_\infty$ is straightforward.

Finally, the Nyquist value is defined and the Kalman conjecture are stated:

Definition 2.6: Given a stable linear plant $G \in \mathbf{RH}_\infty$, the Nyquist value, k_K is the supremum of the values k such that $KG(s)$ satisfies the Nyquist Criterion for all $K \in [0,k]$, i.e.

$$k_K = \sup\{k \in \mathbb{R}^+ : \inf_{\omega \in \mathbb{R}} \{|1 + KG(j\omega)|\} > 0 \; \forall K \in [0,k]\} \tag{7}$$

This value is used in other papers for searching Zames-Falb multiplier [6], [9], and it is straightforward to show that

Theorem 2.2 cannot be satisfied for $k > k_N$. As a result, given $G \in \mathbf{RH}_\infty$, the search of Zames-Falb multiplier must only be carried out for $0 < k < k_N$.

Conjecture 2.7 (Kalman Conjecture): The feedback interconnection of a strictly proper plant G and ϕ_k is stable for any $k < k_N$. ■

Remark 2.8: This conjecture has an important role in the development of absolute stability, and it is true for $n \leq 3$ [21], where n is the order of $G(s)$, but it is false, in general.

Lemma 2.9: Given a strictly proper plant G with order 3 or less, and $k < k_N$, there exists a first order Zames–Falb multiplier M such that $\mathrm{Re}\{M(j\omega)(1+kG(j\omega))\} > 0$, for all $\omega \in \mathbb{R}$.

III. Discussion on causal multipliers

In this section we show that causality can be a significant source of conservatism. Let us consider Example 1 in [12], which considers the plant $G(s) = \frac{s^2-0.2s-0.1}{s^3+2s^2+s+1}$, where a factor -1 has been applied to take into account negative feedback. A linear search shows that $k_N \in (4.5894, 4.5895)$; thus the search of suitable Zames–Falb multipliers satisfying Theorem 2.2 can be restricted to $k \in [0, 4.5894]$. Then there exists a first order Zames–Falb multiplier M such that $\mathrm{Re}\{M(1+4.5894G)\} > 0$ for all $\omega \in \mathbb{R}$ (see Lemma 2.9).

Fig. 2 shows the excessive phase lead defect of $(1+4.5894G)$ which must be shaped by the multiplier. After a simple trial and error procedure, a suitable noncausal Zames–Falb multiplier given by inspection is

$$M_{nc}(s) = \frac{s-0.0012}{s-1.09}. \quad (8)$$

We can check the following properties:

- it is a Zames–Falb multipliers since

$$\|\hat{M}_{nc}\|_1 = \left\| \frac{1.0888}{s-1.09} \right\|_1 = 0.9989 < 1 = M_{nc1}(\infty), \quad (9)$$

  where Lemma 2.4 has been used; and
- $\mathrm{Re}\{M_{nc}(j\omega)(1+4.5894G(j\omega))\} > 0$ for all $\omega \in \mathbb{R}$.

Techniques similar to those in [24] could be used here to improve the phase correction; however inspection suffices for this example

Thus, Theorem 2.2 ensures the absolute stability of G and $\phi \in S[0, 4.5894]$. This property has also been checked via LMI using the KYP lemma [25]. Some questions can be immediately asked: why is this multiplier anticausal? is it possible to find a causal first order Zames–Falb multiplier satisfying the above condition?

There exists a trade-off between phase and $\mathscr{L}_1$-norm [26], which is exacerbated when the multiplier is limited to be either causal or anticausal. Analytic results can be shown if we restrict our attention to first order Zames–Falb multipliers.

Lemma 3.1: If M_c is a causal first order Zames-Falb multiplier, then $\angle M_c(j\omega) > -\arctan(\sqrt{2}/4)$ for all $\omega \in \mathbb{R}$.

Lemma 3.2: Given $\varepsilon > 0$, there exists an anticausal Zames-Falb multiplier such that its phase lag is $90-\varepsilon$ at some frequency.

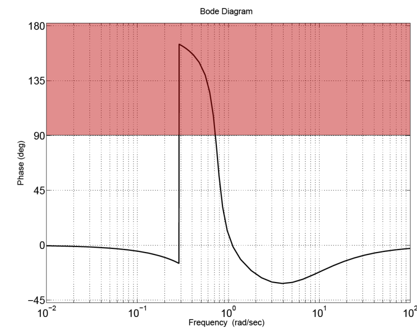

Fig. 2. Bode plot for $1+4.5894P_1$. The system has an excess of phase lead defect of 75 degrees at 0.2 rad/sec, thus the multiplier needs a phase lag of 75 degrees at this frequency.

Lemma 3.3: Given $\varepsilon > 0$, there exists an causal Zames-Falb multiplier such that its phase lead is $90-\varepsilon$ at some frequency. ■

Lemma 3.4: If M_{ac} is an anticausal first order Zames-Falb multiplier, then $\angle M_{ac}(j\omega) < \arctan(\sqrt{2}/4)$ for all $\omega \in \mathbb{R}$. ■

Summarizing, the above four lemmas state that the phase of a causal first order multiplier must be within $(-\arctan(\sqrt{2}/4), 90)$ and the phase of an anticausal first order multiplier must be within $(-90, \arctan(\sqrt{2}/4))$.

We can now investigate the dependence of the phase of $1+kG(j\omega)$ with respect to k. For $0 \leq k \leq 1.2431$, the phase will be with the interval $(-90, 90)$ and the circle criterion can be applied. For $1.2431 < k \leq 4.5894$, the phase lead defect increases from 0 at $k = 1.2431$ up to 75 degrees at $k = 4.5894$, as shown Fig. 2. Therefore, for a causal first order Zames-Falb multiplier, a theoretical limitation can be set when $(1+kG)$ has a phase lead larger than $90+\arctan(\sqrt{2}/4)$, approximately, 109.47. This limit is crossed at $k = 1.805$.

Now we can answer the two questions given at the beginning of the section. Since the system $(1+4.5894G)$ has a phase lead larger than 109.47, there exists no causal first order Zames-Falb multiplier M_c satisfying (6). Anticausal first order Zames-Falb multipliers must be used since their phase lag is not constrained by the $\mathscr{L}_1$-norm condition.

If we consider causal third order Zames-Falb multiplier by using the search in [12], [13], we find that the maximum slope is 2.2428, improving the value of the causal first order Zames–Falb multiplier. Thus, one could postulate that the theoretical limitations given by restricting the search over the set of causal first order Zames–Falb multipliers may be avoided by using higher order or irrational causal Zames-Falb multipliers. Table I shows the results obtained with different methods for searches for Zames–Falb multiplier proposed in the literature. The other methods [6], [9] consider noncausal multipliers, so they have been modified to search over causal multipliers only. Although they have been optimized with more powerful tools than the above inspection method, these causal multipliers remain conservative. In conclusion, all searches proposed in the literature are conservative if the search is restricted to causal multipliers for this example.

Even the more advanced method using causal Zames–Falb multipliers plus Popov multipliers [15], is not able to reach the Nyquist value.

Multiplier	Maximum slope k
Causal high order [9]	1.624
Causal irrational [6]	1.775
Causal order 1	1.8 (approx)
Causal order 3, Turner method [12]	2.2428
Causal order 3 plus Popov multiplier [15]	3.5026
Noncausal order 1, equation (8)	4.5894
Nyquist value	4.5894

TABLE I

MAXIMUM SLOPE FOR DIFFERENT CLASSES OF MULTIPLIERS

IV. LMI SEARCH FOR ANTICAUSAL MULTIPLIERS

This section presents a modification to causal method [12] able to search for an anticausal Zames–Falb multiplier. This modification must be considered as a complementary method. It is known that any rational Zames–Falb multiplier $M(s)$ has a canonical factorization [4], i.e.$M = M_{-}M_{+}$, where $M_{-}^{\sim} \in \mathbf{RH}_\infty$, $(M_{-}^{\sim})^{-1} \in \mathbf{RH}_\infty$, $M_{+} \in \mathbf{RH}_\infty$, and $M_{+}^{-1} \in \mathbf{RH}_\infty$. Loosely speaking, in [12] M_{-} is taken as the identity, whereas we propose a equivalent synthesis taking M_{+} as the identity.

We present two independent methods:

Causal search The first method inverts the system generated by the loop transformation $\tilde{G} = (1+kG)$. Then the existence of a causal multiplier M for $\tilde{G}^{-1}$ implies the existence of a suitable anticausal multiplier $M^{\sim}$ for $\tilde{G}$. Some modifications in the LMIs must be performed: Proposition 2 in [12] uses the slope restriction to perform a loop transformation over the IQC multiplier (see [27] for more details), whereas we need to invert $\tilde{G}^{-1}$, i.e. to use the slope restriction to perform a loop transformation over the plant G.

Anticausal search A complementary technique to [12] which was originally restricted to causal multiplier, is developed. A prior result is needed to find an LMI condition to bound the $\mathscr{L}_1$-norm of an anticausal transfer function. Despite that the first method will be able to find an anticausal multiplier, the importance of a pure anticausal search is due to the possibility of adding a Popov multiplier [14], which can improve the parametrization of the causal or anticausal Zames–Falb multiplier.

Both results lead to very similar numerical results, but show different insights on the Zames–Falb multipliers.

A. LMI modifications

Since the original search proposed in [12] uses the IQC theorem, results are obtained for positive feedback. We have preferred the use of negative feedback for a clearer explanation on phase-shaping. Both techniques are equivalent for this problem [28].

Following [12], given a plant $G \in \mathbf{RH}_\infty$ with a minimal representation (A_p, B_p, C_p, D_p) and a nonlinearity within $S[0,k]$, consider that the set of LMIs given by

$$\begin{bmatrix} -\mathbf{A}_u - \mathbf{A}_u + \lambda(P_{11} - S_{11}) & \mathbf{B}_u \\ \star & -\mu \end{bmatrix} < 0, \qquad (10)$$

$$\begin{bmatrix} \lambda(P_{11} - S_{11}) & 0 & \mathbf{C}_u^\top \\ \star & (1-\mu) & 0 \\ \star & \star & 1 \end{bmatrix} > 0, \qquad (11)$$

and (12), which is the negative feedback version of the (9) in [12], is feasible for some matrices $P_{11} > 0$, $S_{11} > 0$, unstructured matrices $\mathbf{A}_u$, $\mathbf{B}_u$, and $\mathbf{C}_u$ and constants $\mu > 0$ and $\lambda > 0$. Then there exists $M(s) \in \mathscr{M}$ such that

$$Re\{M(j\omega)(1+kG(j\omega))\} > 0 \quad \forall \omega \in \mathbb{R}. \qquad (14)$$

Hence $\mathscr{L}_2$-stability is ensured by using Theorem 2.2. D_u has been set to zero (see [13], [15] for a discussion).

In order to implement the negative feedback, C_g and D_g have incurred a change of sign. Moreover, we carry out the loop transformation in order to check the positivity of $Re(M(j\omega)\tilde{G}(j\omega)) > 0$ for all $\omega \in \mathbb{R}$. This is implemented by replacing the state space representation of $\tilde{G} = (1+kG)$, i.e., $A_{\tilde{g}} = A_g$, $B_{\tilde{g}} = B_g$, $C_{\tilde{g}} = kC_g$, $D_{\tilde{g}} = kD_g + 1$. Hence the LMIs in Proposition 2 [12] are equivalent to (11), (13), and (10).

B. Causal search

Once we have modified the system to be tested, the inversion method can be stated as follows:

Proposition 4.1: Let $G \in \mathbf{RH}_\infty$ represented in the state space by A_g, B_g, C_g, and D_g.. Let ϕ be a slope restricted $S[0,k]$ and odd nonlinearity. Without loss of generality, assume that the feedback interconnection of G and linear gain k is stable. Let us define four matrices as follows:

$$A_{\tilde{g}} = A_g - B_g(kD_g+1)^{-1}C_g, \qquad (15)$$

$$B_{\tilde{g}} = -B_g(kD_g-1)^{-1}, \qquad (16)$$

$$C_{\tilde{g}} = (kD_g+1)^{-1}C_g, \qquad (17)$$

$$D_{\tilde{g}} = (kD_g+1)^{-1}. \qquad (18)$$

Assume that there exist positive definite symmetric matrices $S_{11} > 0$, $P_{11} > 0$, unstructured matrices $\mathbf{A}_u$, $\mathbf{B}_u$, and $\mathbf{C}_u$, and positive constant $\mu > 0$ and $\lambda > 0$, such that LMIs (13), (10), and (11) are satisfied. Then the feedback interconnection is $\mathscr{L}_2$-stable

The reconstruction of the causal multiplier M can be carried out as suggested in [13] and the anticausal multiplier which satisfies Theorem 2.2 will be given by $M^{\sim} = 1 + C_u(sI + A_u)^{-1}B_u$.

C. Anticausal search

The method proposed in [12] is based in the multiobjective synthesis developed in [16]. In our complementary search, we substitute the condition $P < 0$ (for $P > 0$ in [12]) and P nonsingular ensures that the change of variable is feasible. A prior lemma is needed to bound the $\mathscr{L}_1$-norm of an anticausal transfer function.

$$\begin{bmatrix} S_{11}A_g + A_g^\top S_{11} & S_{11}A_g + A_g^\top P_{11} + kC_g^\top \mathbf{B}_u^\top + \mathbf{A}_u^\top & S_{11}B_g - kC_g^\top + \mathbf{C}_u^\top \\ \star & P_{11}A_g + A_g^\top P_{11} + \mathbf{B}_u kC_g + kC_g^\top \mathbf{B}_u^\top & P_{11}B_g + \mathbf{B}_u(1+kD_g) - kC_g^\top \\ \star & \star & -(I+kD_g) - (I+kD_g)^\top \end{bmatrix} < 0, \quad (12)$$

$$\begin{bmatrix} S_{11}A_{\tilde{g}} + A_{\tilde{g}}^\top S_{11} & S_{11}A_{\tilde{g}} + A_{\tilde{g}}^\top P_{11} + C_{\tilde{g}}^\top \mathbf{B}_u^\top + \mathbf{A}_u^\top & S_{11}B_{\tilde{g}} - C_{\tilde{g}}^\top + \mathbf{C}_u^\top \\ \star & P_{11}A_{\tilde{g}} + A_{\tilde{g}}^\top P_{11} + \mathbf{B}_u C_{\tilde{g}} + C_{\tilde{g}}^\top \mathbf{B}_u^\top & P_{11}B_{\tilde{g}} + \mathbf{B}_u D_{\tilde{g}} - C_{\tilde{g}}^\top \\ \star & \star & -D_{\tilde{g}} - D_{\tilde{g}}^\top \end{bmatrix} < 0, \quad (13)$$

Lemma 4.2: Given a strictly proper transfer function $H(s) \in \mathbf{RH}_\infty^\perp$ given by $H(s) = C(sI-A)^{-1}B$, where $-A$ is Hurwitz. Assume that there exist $Y < 0$, $\mu > 0$, and $\lambda > 0$ such that

$$\begin{bmatrix} A^\top Y + YA - \lambda Y & B_u \\ \star & -\mu \end{bmatrix} < 0, \quad (19)$$

$$\begin{bmatrix} -\lambda Y & 0 & C^\top \\ \star & (\xi - \mu) & 0 \\ \star & \star & \xi \end{bmatrix} > 0, \quad (20)$$

then $\|H\|_1 < \xi$.

Using the above lemma, the result for anticausal multiplier can be stated as follows.

Proposition 4.3: Let $G \in \mathbf{RH}_\infty$ represented in the state space by A_g, B_g, C_g, and D_g. Let ϕ be a slope restricted $[0,k]$ and odd nonlinearity. Assume that there exist positive definite symmetric matrices $S_{11} > 0$, $P_{11} > 0$, unstructured matrices $\mathbf{A}_u$, $\mathbf{B}_u$, and $\mathbf{C}_u$, and positive constant $\mu > 0$ and $\lambda > 0$, such that the LMIs (12),

$$\begin{bmatrix} -\mathbf{A}_u - \mathbf{A}_u - \lambda(P_{11} - S_{11}) & \mathbf{B}_u \\ \star & -\mu \end{bmatrix} < 0, \quad (21)$$

$$\begin{bmatrix} -\lambda(P_{11} - S_{11}) & 0 & \mathbf{C}_u^\top \\ \star & (1-\mu)I & 0 \\ \star & \star & 1 \end{bmatrix} > 0. \quad (22)$$

Then the feedback interconnection (3) is $\mathscr{L}_2$-stable.

The addition of Popov multiplier to anticausal Zames–Falb multipliers will be given in a longer version of this paper.

D. Numerical examples

Table II shows the examples discussed in [12], [13], and two more examples, the examples 7 and 8 are given in [29] and [8], respectively. Results are obtained using the MATLAB LMI Toolbox. For some examples, results are obtained using $1/k + G(j\omega)$ rather than $1 + kG(j\omega)$ as the numerical results sometimes differ, which is interpreted as numerical issues on the LMI solver. A line search over λ is carried out with 50 points in the interval $(10^{-3}, 10^3)$.

Table III gives the result for the systems in Table I. As expected, results for the anticausal method improves the maximum slope for the plants where Park's method is better than the causal method [12] (Examples 1, 3, and 6). Park's multipliers are competitive for slightly damped plants, since they carry no conservativeness in the bound of the $\mathscr{L}_1$-norm. Nevertheless, the addition of the Popov multiplier in [14], [15] and its implementation for the anticausal method generates the best results in literature if they are combined.

TABLE II
EXAMPLES

Ex.	$G(s)$
1	$G_1(s) = \frac{s^2-0.2s-0.1}{s^3+2s^2+s+1}$
2	$G_2(s) = -G_1(s)$
3	$G_3(s) = \frac{s^2}{s^4+0.2s^3+6s^2+s+1}$
4	$G_4(s) = -G_3(s)$
5	$G_5(s) = \frac{s^2}{s^4+0.0003s^3+10s^2+0.0021s+9}$
6	$G_6(s) = -G_5(s)$
7	$G_7(s) = \frac{s^2}{s^3+2s^2+2s+1}$
8	$G_8(s) = \frac{(s^2+15.6s+147.8)(s^2+2.374s+56.23)}{(s^2+2.588s+90.9)(s^2+11.79s+113.7)} \frac{(s^2+0.332s+26.15)}{(s^2+14.84s+84.05)(s+8.83)}$

V. Discussion on the methods

The LMI search over causal Zames–Falb multipliers [12], and especially its extension by adding a Popov multiplier [14], [15], provide good results. The addition of a Popov multiplier generates a noncausal Zames–Falb multiplier but with limited flexibility in the anticausal component; and the causal constraint is partially overcome. However, some conservatism remains for some plants, such as example 1. The complementary search given by Proposition 4.3 is able to generate good result in the plants where the causal method is weak. A Popov multiplier can similarly be added, and the result are equally improved since the equivalent Zames–Falb is once again noncausal. An additional advantage of the Popov multiplier is that it can be included without further conservatism in the $\mathscr{L}_1$-norm. As a complementary search, it will have the same limitation as discussed in [12], [13], [14], [15]. The search is suboptimal, since a search over λ must be carried out; and it has the same conservatism in the bound of the $\mathscr{L}_1$-norm.

The numerical results can be interpreted following the theoretical development of the Section III for first order Zames–Falb multipliers. The causal search gives better results when $(1+kG)$ has excessive phase lag, whereas the anticausal search gives better results $(1+kG)$ has excessive phase lead. As the searches are complementary, we recommend checking for stability with both of them.

An efficient convex method for exploiting the generality of noncausal Zames–Falb multipliers remains open.

VI. Conclusion

This paper has analysed the consequences of restricting the set of Zames–Falb multiplier to causal multipliers. For

TABLE III

SECTOR/SLOPE BOUNDS OBTAINABLE USING VARIOUS STABILITY CRITERIA

Criteria	Ex. 1	Ex. 2	Ex. 3	Ex. 4	Ex. 5	Ex. 6	Ex. 7	Ex. 8
Circle	1.2431	0.7640	0.3263	0.3081	0.00040	0.00039	8.1235	4.3159
Park's method [19]	4.5894	1.0894	0.7883	0.7083	0.00183	0.00183	10,000+	62.5691
Causal method [12]	2.2428	1.0894	0.7049	0.8526	0.00181	0.00095	17.605	87.3854
Anticausal method	4.5894	1.0745	0.9846	0.6135	0.00095	0.00182	10,000+	21.6190
Causal+Popov method [14], [15]	3.2897	1.0894	0.7760	1.0792	0.00333	0.00318	17.724	87.3854
Anticausal method+Popov method	4.5894	1.0745	1.4513	0.7222	0.00319	0.00333	10,000+	22.4304
Nyquist value	4.5894	1.0894	∞	3.5000	∞	1.7142	∞	87.3854

first order Zames–Falb multiplier, theoretical results have shown that causal Zames–Falb multipliers have a strong constraint on their phase lag and anticausal Zames–Falb multiplier have the same constraint on their phase lead. An example given in the literature has been used to show that a noncausal multiplier obtained by inspection beats all the convex searches if they are restricted to causal Zames–Falb multipliers.

Using the method developed in [12], a search of anticausal multipliers has been implemented, which is a complementary solution to the search of causal multipliers. The new search has been tested and it improves the results given by Turner's method in the examples where this method is not competitive. The anticausal search developed in this paper confirms that a major source of conservatism in [12] is the restriction to causal multipliers for some examples. However, the combination of causal and anticausal methods generates the best results in literature, showing the significance of [12]. Finally, the extension of the results of this paper to MIMO systems can be developed following [30].

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