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Equivalence between classes of multipliers for slope-restricted nonlinearities

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Abstract— **Different classes of multipliers have been proposed in the literature for obtaining stability criteria using passivity theory, integral quadratic constraint (IQC) theory or Lyapunov theory. Some of these classes of multipliers can be applied with slope-restricted nonlinearities. In this paper the concept of phase-containment is defined and it is shown that several classes are phase-contained within the class of Zames–Falb multipliers. There are two main consequences: firstly it follows that the class of Zames–Falb multipliers remains, to date, the widest class of available multipliers for slope-restricted nonlinearities; secondly further restrictions may be avoided when applying some of the other classes of multipliers.**

I. Introduction

The investigation of absolute stability for the system in Fig. 1, where G is a linear time-invariant (LTI) system and ϕ is a nonlinearity within a given class, is known as the Lur'e problem. For example, if one would like to investigate the stability of a feedback control system with saturation in the actuator, the closed-loop could be expressed as Fig. 1. Because the saturation belongs to the class of so-called sector bounded nonlinearities, simple analysis conditions based upon the LTI part of the system can be derived, i.e. strictly positive realness [1], and applied to anti-windup synthesis [2]. However the conditions are inherently conservative. In order to reduce such conservatism, saturation can be more efficiently described as a slope-restricted and odd nonlinearity; multiplier techniques may then be used.

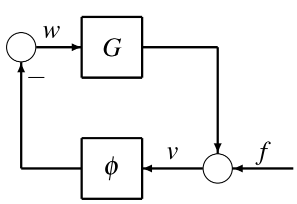

Fig. 1. Lur'e problem

In this paper, we are concerned with static nonlinearities that are slope-restricted. In particular, for SISO slope-restricted and odd nonlinearities several classes of multipliers were proposed in the Sixties, summarized by Barabanov [3]. The most celebrated are the so-called "Zames–Falb multipliers" [4]. However more conservative graphical criteria, such as the circle [5], Popov [6] or off-axis circle criteria [7], were developed due to the lack of tools for searching for Zames-Falb multipliers.

In [4], [8], the classical multiplier approach is developed for any general application. Loosely speaking, if M has a canonical factorization, then positivity of MG and $M^*\phi$ is enough to prove stability. The multiplier technique can be used for applying either passivity theory [4] or integral quadratic constraint (IQC) theory [9]. Although the multiplier is not directly used when Lyapunov theory is used, the results can also be interpreted in terms of multipliers [10] by using the path integral approach developed by Brockett (see [11], [12]). In summary, given a class of nonlinearities, a class of multipliers preserving the positivity of the class of nonlinearities must be defined; then for a particular linear system, stability of the feedback interconnection of Fig. 1 is ensured if it can be found an element of the class of multiplier such that MG is positive.

As appropriate computation tools have become available, several works [13], [14], [15], [16], [17], [18] have proposed different searches within the class of Zames–Falb multipliers. Meanwhile several authors propose extensions to the Zames-Falb class. [19] extends the class of Zames–Falb multipliers by adding a Popov term, and [20] extends the class of Zames–Falb multipliers by adding a quadratic term. [21] proposes an LMI search for certain MIMO multipliers and discusses their relation to the Zames-Falb multipliers in the SISO case; the SISO version of [21] is also given by [3], and was introduced by [22] via a Lyapunov approach and by [23] via an input-output approach.

Dynamic multipliers such as Zames–Falb multipliers have been used as an analysis tool [24], [25], [26]. Although their use in synthesis is proposed by, for example, [27], [28], [29], it is not yet understood how to exploit their full generality; the above graphical criteria are still in use for antiwindup techniques.

The aim of this paper is to demonstrate that the Zames–Falb class is the widest available class of multipliers for slope–restricted nonlinearities. Firstly, we define notions of *phase-containment* and *phase-equivalence*. Secondly, we show that Popov multipliers, Park's multipliers [21], and the extensions of the Zames–Falb multipliers are all phase-contained within the class of Zames–Falb multipliers. Our treatment is consistent with that of Falb and Zames [30] who showed a similar relation for RC and RL multipliers. Note that Popov and RL and RC classes of multipliers have been described as belonging to (limiting) subsets of the class of Zames–Falb multipliers [31]. But such statements have not been rigorously proven; nor are they necessarily correct in

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all cases.

The most important consequences of this paper are as follows. Firstly, Zames–Falb multipliers are the widest class for analysing the stability of the class of slope-restricted nonlinearities. If a new class of multipliers is proposed containing multipliers which are not Zames–Falb multipliers, it should not necessarily be concluded that this new class is wider. Secondly, stability result for slope-restricted nonlinearities using the multipliers under discussion can be stated as Corollaries of the Zames–Falb theorem. Therefore no extra conditions are needed and $\mathscr{L}_2$ stability is obtained. Thirdly, since the passivity theory and IQC theory are equivalent for Zames–Falb multipliers [32]; it follows that the theories are also equivalent for slope-restricted nonlinearities using any of the multipliers under discussion.

II. Notation and Preliminary results

Let $\mathscr{L}_2^m[0,\infty)$ be the Hilbert space of all square integrable and Lebesgue measurable functions $f:[0,\infty)\to\mathbb{R}^m$. Similarly, $\mathscr{L}_2^m(-\infty,\infty)$ can be defined for $f:[-\infty,\infty]\to\mathbb{R}^m$. A truncation of the function f at T is given by $f_T(t)=f(t)\ \forall t\le T$ and $f_T(t)=0\ \forall t>T$. The function f belongs to the extended space $\mathscr{L}_{2e}^m[0,\infty)$ if $f_T\in\mathscr{L}_2^m[0,\infty]$ for all $T>0$. In addition, $\mathscr{L}_1[-\infty,\infty]$ (henceforward $\mathscr{L}_1$) is the space of all absolute integrable functions; given a function $h:\mathbb{R}\to\mathbb{R}$ such that $h\in\mathscr{L}_1$, its $\mathscr{L}_1$-norm is given by

$$\|h\|_1=\int_{-\infty}^{\infty}|h(t)|dt. \tag{1}$$

A nonlinearity $\phi:\mathscr{L}_{2e}[0,\infty]\to\mathscr{L}_{2e}[0,\infty]$ is said to be memoryless if there exists $N:\mathbb{R}\to\mathbb{R}$ such $(\phi v)(t)=N(v(t))$ for all $t\in\mathbb{R}$. Henceforward we assume that $N(0)=0$. A memoryless nonlinearity ϕ is said to be bounded if there exists a positive constant C such that $|N(x)|<C|x|$ for all $x\in\mathbb{R}$. The nonlinearity ϕ is said monotone if for any two real numbers x_1 and x_2 we have

$$0\le\frac{N(x_1)-N(x_2)}{x_1-x_2} \tag{2}$$

Moreover, ϕ is said to be slope-restricted or incrementally bounded in the sector $S[0,k]$, (henceforward we write ϕ_k), if

$$0\le\frac{N(x_1)-N(x_2)}{x_1-x_2}\le k \tag{3}$$

for all $x_1\neq x_2$. The nonlinearity ϕ is said to be odd if $N(x)=-N(-x)$ for all $x\in\mathbb{R}$.

This paper focuses the stability of the feedback interconnection of a stable LTI system G and a slope-restricted nonlinearity ϕ_k, represented in Fig. 1 and given by

$$\begin{cases} v=f+Gw,\\ w=-\phi_k v.\end{cases} \tag{4}$$

Since G is a causal, i.e. $(Gu)_T=(Gu_T)_T$, and stable LTI system, the exogenous input in this part of the loop can be taken as the zero signal without loss of generality. It is well–posed if the map $(v,w)\mapsto(0,f)$ has a causal inverse on $\mathscr{L}_2^2[0,\infty)$, and this interconnection is stable if for any $f\in\mathscr{L}_2[0,\infty)$, and it is absolutely stable if it is $\mathscr{L}_2$-stable for all ϕ_k within the class of nonlinearities. In addition, $G(j\omega)$ means the transfer function of the LTI system G. Finally, given an operator M, then M^* means its adjoint (see [8] for a definition). For LTI systems, $M^*(s)=M^\top(-s)$, where $^\top$ means transpose.

The standard notation $\mathbf{L}_\infty$ ($\mathbf{RL}_\infty$) is used for the space of all (proper real rational) transfer functions bounded on the imaginary axis; $\mathbf{RH}_\infty$ is used for the space of all proper real rational transfer functions such that all their poles have strictly negative real parts. The H_∞-norm of a SISO transfer function G is defined as

$$\|G\|_\infty=\sup_{\omega\in\mathbb{R}}(|G(j\omega)|) \tag{5}$$

With some acceptable abuse of notation, given a rational strictly proper transfer function $H(s)$ bounded on the imaginary axis, $\|H\|_1$ means the $\mathscr{L}_1$-norm of the impulse response of $H(s)$.

A. Zames–Falb theorem

The following theorem provides the absolute stability of system (4) subject to the existence of an appropriate Zames–Falb multiplier.

Theorem 2.1 ([4], [8]): Consider the feedback system in Fig. 1 with G a stable LTI system, i.e. $G(s)\in\mathbf{RH}_\infty$, and a nonlinearity $\phi_{k-\varepsilon}$ slope-restricted in $S[0,k-\varepsilon]$ for some $\varepsilon>0$. Assume that the feedback interconnection is well-posed. Then suppose that there exists a noncausal convolution operator $M:\mathscr{L}_2(-\infty,\infty)\to\mathscr{L}_2(-\infty,\infty)$ whose impulse response is of the form

$$m(t)=\delta(t)-\sum_{i=0}^{\infty}z_i\delta(t-t_i)-z_a(t), \tag{6}$$

where δ is the Dirac delta function and

$$\sum_{i=0}^{\infty}|z_i|<\infty,\quad z_a\in\mathscr{L}_1,\quad\text{and}\quad t_i\in\mathbb{R}\ \forall i\in\mathbb{N}. \tag{7}$$

Assume that:

(*i*)
$$\|z_a\|_1+\sum_{i=0}^{\infty}|z_i|<1 \tag{8}$$

(*ii*) either $z_a(t)\ge0$ for all $t\in\mathbb{R}$ and $z_i\ge0$ for all $i\in\mathbb{N}$, or $\phi_{k-\varepsilon}$ is odd; and

(*iii*) there exist $\delta>0$ such that
$$\mathrm{Re}\{M(j\omega)(1+kG(j\omega))\}\ge\delta\quad\forall\omega\in\mathbb{R}. \tag{9}$$

Then the feedback interconnection (4) is $\mathscr{L}_2$-stable. ■

B. Zames–Falb multipliers

Equations (6), (7) and (8) in Theorem 2.1 provide the class of Zames–Falb multipliers. It is a subset of $\mathbf{L}_\infty$. i.e. it is not limited to rational transfer functions. However in many parts of this paper, we restrict our attention to such rational multipliers, i.e. we set $z_i=0$ for all $i\in\mathbb{N}$.

Definition 2.2: The class of SISO rational Zames–Falb multipliers $\mathscr{M}$ contains all SISO rational transfer functions $M \in \mathbf{RL}_\infty$ such that $M(s) = 1 + H(s)$, where $H(s)$ is a rational strictly proper transfer function and $\|H\|_1 < 1$.

Lemma 2.3: [17] Let $M(s) \in \mathbf{RL}_\infty$ be a rational transfer function with $M(s) = M(\infty) + \widehat{M}(s)$, where $\widehat{M}(s)$ denotes its associated strictly proper transfer function. Then, $M(s)$ is a Zames–Falb multiplier if and only if $\|\widehat{M}\|_1 < M(\infty)$. ∎

Remark 2.4: The corresponding Lemma given in [17] is limited to $M(s) \in \mathbf{RH}_\infty$, but its extension to $M(s) \in \mathbf{RL}_\infty$ is straightforward.

In this paper, we will often consider first order Zames–Falb multipliers. These are by implication rational. It is worth noting that the impulse response of a first order Zames–Falb multiplier is always positive. Hence in this case condition (ii) in Theorem 2.1 is always satisfied and the odd condition on $\phi_{k-\varepsilon}$ is not needed.

C. Practicalities and absolute stability

In the original theorem [4], when the linear condition (9) holds for some constant k, the nonlinearity is required to belong to the sector $S[0, k-\varepsilon]$, where ε can be arbitrarily small but strictly positive. Nevertheless, its extension to the sector $S[0,k]$ is trivial since $\|M\|_\infty < 1$ (see [33], [17]). A prior lemma is needed.

Lemma 2.5: Let $M \in \mathbf{L}_\infty$ be a Zames–Falb multiplier satisfying equations (6), (7), and (8). Assume that (9) is satisfied for $G \in \mathbf{RH}_\infty$, M, $k > 0$, and $\delta > 0$. Then there exist $\xi > 0$ and $\delta_1 > 0$ such that

$$\mathrm{Re}\{M(j\omega)(1 + (k+\xi)G(j\omega))\} \geq \delta_1 > 0 \quad \forall \omega \in \mathbb{R}. \quad (10)$$

∎

The significance of Theorem 2.1 is that it can be applied when the characterization of the nonlinearity is that it is memoryless, slope-restricted, and odd (in some cases). An absolute stability result can be stated as follows:

Corollary 2.6: Consider the feedback system in Fig. 1 with $G \in \mathbf{RH}_\infty$ and any nonlinearity ϕ_k slope-restricted in $S[0,k]$. Assume that the system is well-posed. Then suppose that there exists $M \in \mathscr{M}$ such that:

(i) either ϕ_k is odd or the inverse Laplace transform of $H(s) = M(s) - 1$, $h(t)$, is negative for all $t \in \mathbb{R}$; and

(ii) there exists $\delta > 0$ such that

$$\mathrm{Re}\{M(j\omega)(1 + kG(j\omega))\} \geq \delta \quad \forall \omega \in \mathbb{R}. \quad (11)$$

Then the feedback interconnection (4) is absolutely stable. ∎

In this paper, we compare different criteria for absolute stability. If a criterion guarantees the stability of feedback of $G \in \mathbf{RH}_\infty$ and any nonlinearity ϕ_k slope-restricted in $S[0,k]$, then the linear feedback interconnection of $G(s)$ and any linear gain $0 \leq K \leq k$ must be stable. The following definition is used in works that focus on stability criteria, e.g. [13], [15].

Definition 2.7: Given $G \in \mathbf{RH}_\infty$, the Nyquist value k_N is the supremum of the values k such that $KG(s)$ satisfies the Nyquist Criterion for all $K \in [0,k]$, i.e.

$$k_N = \sup\{k \in \mathbb{R}^+ : \inf_\omega\{|1 + KG(j\omega)|\} > 0 \; \forall K \in [0,k]\}. \quad (12)$$

As a result, we can restrict our attention to a subset of $\mathbf{RH}_\infty$ without loss of generality, which will be essential to prove the relationship between Zames-Falb multipliers and Popov multipliers.

Definition 2.8: The subset $\mathscr{G} \subset \mathbf{RH}_\infty$ is defined as follows

$$\mathscr{G} = \{\widehat{G} \in \mathbf{RH}_\infty : \widehat{G}^{-1} \in \mathbf{RH}_\infty \text{ and } \widehat{G}(\infty) > 0\}. \quad (13)$$

Lemma 2.9: Assume that the feedback interconnection in Fig. 1 with $G \in \mathbf{RH}_\infty$ and any nonlinearity ϕ_k slope-restricted $S[0,k]$ is $\mathscr{L}_2$-stable. Then $(1 + kG) \in \mathscr{G}$. ∎

It follows that if $1 + kG \notin \mathscr{G}$, with $G \in \mathbf{RH}_\infty$ and $k > 0$, the feedback interconnection of G and the class of nonlinearities slope-restricted in $S[0,k]$ cannot be absolutely $\mathscr{L}_2$-stable.

III. Equivalence of multipliers

In the literature of SISO bounded, monotone and odd nonlinearities, several classes of multipliers have been defined. The equivalence between specific classes is discussed in [30] and [31] but in neither is a general concept of equivalence rigorously defined. Similarly alternative definitions of the Popov multiplier in the early literature [10] implicitly assume such equivalence. In the following, we define the terms "phase-contained" and "phase-equivalent" with respect to classes of multipliers.

Following our discussion in Section II.D, we restrict our attention to the subset $\mathbf{G}$ without loss of generality for the case of slope-restricted nonlinearities.

Definition 3.1: Let $\mathscr{M}_A$ and $\mathscr{M}_B$ be two classes of multipliers. The class $\mathscr{M}_A$ is *phase-contained* within the class $\mathscr{M}_B$ if given a multiplier $M_a \in \mathscr{M}_A$ such that

$$\mathrm{Re}\left\{M_a(j\omega)\widehat{G}(j\omega)\right\} \geq \delta_1 \quad \forall \omega \in \mathbb{R} \quad (14)$$

for some $\delta_1 > 0$ and $\widehat{G} \in \mathbf{G}$, then there exists $M_b \in \mathscr{M}_B$ such that

$$\mathrm{Re}\left\{M_b(j\omega)\widehat{G}(j\omega)\right\} \geq \delta_2 \quad \forall \omega \in \mathbb{R} \quad (15)$$

for some $\delta_2 > 0$.

Definition 3.2: Two classes of multipliers, $\mathscr{M}_A$ and $\mathscr{M}_B$, are *phase-equivalent* if $\mathscr{M}_A$ is phase-contained within $\mathscr{M}_B$ and $\mathscr{M}_B$ is phase-contained within $\mathscr{M}_A$.

In the rest of this paper, we will show relationships between different classes of multipliers and the Zames–Falb multipliers:

- The class of Popov multipliers is phase-contained within the class of first order Zames–Falb multipliers, as suggested in [31]. Here we confirm the relation with mathematical rigour. In particular we show the class of Popov multipliers with positive constant is phase-contained within the class of causal first order Zames–Falb multipliers while the class of Popov multipliers with negative constant is phase contained within the class of anti-causal first order Zames-Falb multipliers.
- The classes of RC and RL multipliers are respectively phase-contained within the classes of anti-causal and causal Zames–Falb multipliers, as shown in [30]. We include the result for completeness.

- We show the class of multipliers proposed by Park [21] is phase-equivalent to the class of first order Zames–Falb multipliers.
- We show the classes of multipliers generated by the extensions of the Zames–Falb multipliers given in [19], [20], [35] are all phase-contained within the class of Zames–Falb multipliers.

IV. Popov multipliers

Popov multipliers were the first multipliers proposed in the literature [6]. Moreover, the cited paper gives the first general solution to the Lur'e problem when the nonlinearity is sector-bounded and time invariant. However, the use of this class of multiplier carries the restriction that the LTI system must be strictly proper and the derivative of the input (depicted f in Fig 1) must belong to $\mathscr{L}_2$ (see Section 6.6.2 in [34]).

Definition 4.1: The class of Popov multipliers $\mathscr{M}_P$ is given by $M(s) = 1+qs$ where $q \in \mathbb{R}$.

Remark 4.2: An alternative definition is given in [10]: $M(s) = (1+qs)^{\pm 1}$, where $q > 0$. This gives a phase–equivalent class of multiplier.

Since they are not biproper, these are not Zames–Falb multipliers. They have been identified as a limiting case of the Zames-Falb multiplier in [31] as follows:

$$1+qs = \lim_{\varepsilon\to 0^+} \frac{1+qs}{1+\varepsilon s}, \tag{16}$$

$$\frac{1}{1+qs} = \lim_{\varepsilon\to 0^+} \frac{1+\varepsilon s}{1+qs}. \tag{17}$$

A detailed analysis of both limits shows that the transfer function on the right in (16) is a Zames–Falb multiplier when ε is sufficiently small. However, the transfer function on the right in (17) is not a Zames–Falb multiplier for small ε. Moreover the equivalence is not well-defined: at high frequency the Popov multiplier in (16) is unbounded.

Let us first characterize the class of first order Zames–Falb multipliers.

Corollary 4.3: Let $M(s)$ be a first order transfer function given by

$$M(s) = \frac{1+\nu s}{1+\kappa s}. \tag{18}$$

Then, $M(s) \in \mathscr{M}$ if and only if $\nu\kappa > 0$ and $\left|1-\frac{\nu}{\kappa}\right| < \frac{\nu}{\kappa}$. ■
It is clear that the limit in (16) is a Zames–Falb multiplier for all $\varepsilon > 0$ since $\frac{q}{\varepsilon} - 1 < \frac{q}{\varepsilon}$. The relation between multipliers indicated by (16) can be formalized as follows:

Lemma 4.4: The class of Popov multipliers with positive constant q is phase-contained within the class of causal first order Zames–Falb multipliers. ■

However if $\nu < \kappa$, Corollary 4.3 requires $\nu < \kappa < 2\nu$. Thus, the limit in (17) is not a Zames-Falb multiplier as soon as $2\varepsilon \leq q$. Nevertheless, an appropriate limit with $\varepsilon < 0$ can be stated:

$$1+qs = \lim_{\varepsilon\to 0^-} \frac{1+qs}{1+\varepsilon s}, \qquad q<0. \tag{19}$$

Lemma 4.5: The class of Popov multipliers with negative constant q is phase-contained within the class of anti-causal first order Zames–Falb multipliers. ■

Remark 4.6: We may think of a Popov multiplier as a first order Zames-Falb multiplier but with its pole at infinity.

As a result, a new version of the Popov Theorem can be given as a corollary of Theorem 2.1. We require that the nonlinearity be slope-restricted with k finite, whereas the classical Circle and Popov criteria only require sector-bounded. However the LTI system may be biproper and input-output stability is established without further restriction on the derivative of the input:

Corollary 4.7 (Popov Theorem): Let $G \in \mathbf{RH}_\infty$ and let ϕ_k be a slope-restricted $S[0,k]$ nonlinearity. If there exists $q \in \mathbb{R}$ such that

$$\mathrm{Re}\{(1+qj\omega)(1+kG(j\omega))\} \geq \delta \quad \forall \omega \in \mathbb{R}, \tag{20}$$

for some $\delta > 0$ then the feedback interconnection (4) is $\mathscr{L}_2$-stable. ■

V. RL and RC multipliers

The RL and RC multipliers are obtained in [10] as an extension of the Popov multipliers for slope-restricted nonlinearities using the Lyapunov theorem. They can be defined as follows:

Definition 5.1: The class of RL multipliers is given by

$$M_{RL}(s) = \frac{(s+\alpha_1)(s+\alpha_2)\dots(s+\alpha_n)}{(s+\beta_1)(s+\beta_2)\dots(s+\beta_n)}, \tag{21}$$

where $0 < \alpha_1 \leq \beta_1 < \alpha_2 < \beta_2 < \cdots < \alpha_n < \beta_n$.

Definition 5.2: The class of RC multipliers is given by

$$M_{RC}(s) = \frac{(s+\alpha_1)(s+\alpha_2)\dots(s+\alpha_n)}{(s+\beta_1)(s+\beta_2)\dots(s+\beta_n)}, \tag{22}$$

where $0 < \beta_1 \leq \alpha_1 < \beta_2 < \alpha_2 < \cdots < \beta_n < \alpha_n$.

Definition 5.3: The class of RL and RC multipliers is given by $M(s) = M_{RL}(s)M_{RC}(s)$, where $M_{RL}(s)$ is an RL multiplier and $M_{RC}(s)$ is an RC multiplier.

In [31], it is commented that RL and RC multipliers are examples of Zames–Falb multipliers. This statement is not completely correct: RL multipliers are Zames–Falb multipliers; however there exist RC multipliers which are not Zames–Falb multipliers. For example, consider the multiplier given by:

$$M(s) = \frac{s+3}{s+1} = 1 + \frac{2}{s+1}, \qquad \left\|\frac{2}{s+1}\right\|_1 = 2 > 1, \tag{23}$$

This is an RC multiplier but it is not a Zames–Falb multiplier. Nevertheless, in [30], the following result is given which states (in our terminology) that the class of RC and RL multipliers is phase-contained within a specific sub-class of Zames–Falb multipliers:

Lemma 5.4 ([30]): Let M be a RL and RC multiplier. Then there exists an RC multiplier M_{RC} such that if M is written in the form $M(j\omega) = (1-Z(j\omega))|M_{RC}(j\omega)|^2$, with $Z(j\omega)$ the Fourier transform of $z(t)$, then $z(t) \geq 0$ and $\|Z\|_1 < 1$. ■

Remark 5.5: Given the multiplier $M_{RL}(j\omega)M_{RC}(j\omega)$, the object $M_{RL}(j\omega)(M^*_{RC}(j\omega))^{-1}$ has the same phase and is a Zames–Falb multiplier.

Corollary 5.6: The class of RL and RC multipliers is phase-contained within the class of Zames–Falb multipliers for which the corresponding impulse response (i.e. z) is non-negative. Then the feedback interconnection (4) is $\mathscr{L}_2$-stable. ■

Remark 5.7: The sub-class of Zames-Falb multipliers corresponds to that for which the nonlinearity need not be odd. A more precise classification also follows immediately.

Corollary 5.8: The class of RL multipliers is phase-contained within the class of causal Zames–Falb multipliers for which the corresponding impulse response (i.e. z) is non-negative. The class of RC multipliers is phase-contained within the class of anti-causal Zames–Falb multipliers for which the corresponding impulse response (i.e. z) is non-negative. ■

VI. Park's multipliers

Park [21] proposes a class of multipliers which corresponds to a stability condition that can be tested by a convex search. It is easy to show the phase-equivalence between this class of multipliers and the class of first order Zames-Falb multipliers. The multipliers are given by

$$M(s) = -s^2 + a^2 + bs, \qquad a \in \mathbb{R}, b \in \mathbb{R}. \tag{24}$$

The quadratic term was introduced by Yakubovich [22][1] where the frequency condition was obtained by using Lur'e-Postnikov type Lyapunov function. A similar result was independently developed by Dewey and Jury [23] within an embryonic passivity framework. In [3] the same multiplier class is used to demonstrate the Kalman conjecture for third order systems.

As commented in [21], a proper multiplier with the same phase can be defined as follows:

Definition 6.1: The class of Park's multipliers is given by $M_P(s) = 1 + \frac{bs}{-s^2+a^2}$, where a and b are real numbers.

Not all multipliers in this class are Zames-Falb multipliers. However the following result provides the equivalence to a first order Zames–Falb multipliers.

Lemma 6.2: The class of Park's multipliers is phase-equivalent to the class of first order Zames–Falb multipliers. ■

Remark 6.3: If $b > 0$ the phase-equivalent Zames–Falb multiplier is causal, whereas if $b < 0$, the phase-equivalent Zames–Falb multiplier is anticausal.

The stability result proposed in [21] now follows as a corollary of Theorem 2.1 and is stated in terms of $\mathscr{L}_2$-stability without a requirement that the LTI plant be strictly proper.

Corollary 6.4: Let $G \in \mathbf{RH}_\infty$ and let ϕ_k be a slope-restricted $S[0,k]$ nonlinearity. If there exists a multiplier M_p such that $\text{Re}\left\{M_p(j\omega)(1+kG(j\omega))\right\} \geq \delta$ for all $\omega \in \mathbb{R}$ and for some $\delta > 0$, then the feedback interconnection (4) is $\mathscr{L}_2$-stable. ■

[1] Other authors [3], [20] cite a conference paper in 1962 now unavailable.

In summary, the result given in [21] can be understood as an LMI search over the whole class of first order rational Zames–Falb multipliers.

VII. Extension of Zames–Falb multiplier

Two different extensions of the class of Zames–Falb multipliers have been proposed in the literature: adding a "Popov term", i.e qs, [19], [35] and adding a "Yakubovich term" i.e. $-\kappa^2 s^2$ [20]. For SISO systems, we show that both extensions are phase-contained within the original class, and hence the additional conditions associated with the extra terms are not needed.

Two ways of extending the class of Zames-Falb multipliers by adding a Popov term are considered in the literature. In [19] the term qs (henceforward, the "Popov term") is added to the Zames–Falb multiplier. Similarly, in [35] the term $1+qs$ is added to the Zames-Falb multiplier.

A. Extension adding the Popov term

Definition 7.1: The class of Popov-extended Zames–Falb multipliers is given by $M_{\text{pZF}}(s) = qs + M_1(s)$ where $q \in \mathbb{R}$ and where $M_1(s)$ belongs to the class of Zames–Falb multipliers.

Lemma 7.2: The class of Popov-extended Zames–Falb multipliers is phase-contained within the class of Zames–Falb multiplier. ■

We can state the result given in [19] as a corollary of Theorem 2.1, avoiding further conditions normally imposed by the use of a Popov multiplier.

Corollary 7.3: Let $G \in \mathbf{RH}_\infty$ and let ϕ_k be a slope-restricted $S[0,k]$ nonlinearity. Assume that there exists a Popov-extended Zames–Falb multiplier M_{pZF} such that

$$\text{Re}\left\{M_{\text{pZF}}(j\omega)(1+kG(j\omega))\right\} \geq \delta \quad \forall \omega \in \mathbb{R}, \tag{25}$$

for some $\delta > 0$. Then the feedback interconnection (4) is $\mathscr{L}_2$-stable. ■

The extension in [35] is addressed in a longer version of this paper. As result similar to Corollary 7.3 can be shown provided sector and slope restrictions coincide. This does not contradict results in [35]. The original search of [16] is carried out within the class of causal Zames–Falb multipliers, whereas the search over the class of Popov plus Zames–Falb multipliers can result in a noncausal Zames-Falb multiplier if $q < 0$, as shown in Lemma 4.5.

B. Extension with "Yakubovich term" [20]

Using Theorem 3 in [20] for SISO systems, the Zames–Falb multipliers can be extended as follows:

Definition 7.4: The class of Yakubovich-extended Zames–Falb multipliers is given by $M_{\text{yZF}}(s) = -\kappa^2 s^2 + M(s)$, where $\kappa \in \mathbb{R}$ and $M(s) = 1 + H(s)$ is a Zames–Falb multiplier.

Lemma 7.5: The class of Yakubovich-extended Zames–Falb multiplier is phase-contained with the class of Zames–Falb multipliers. ■

Finally, the following corollary of Theorem 2.1 is less restrictive that the SISO version of the result given in [20].

Corollary 7.6: Let $G \in \mathbf{RH}_\infty$ and let ϕ_k be a slope-restricted $S[0,k]$ nonlinearity. Assume that there exists a Yakubovich-extended Zames–Falb multiplier M_{yZF} such that

$$\mathrm{Re}\left\{M_{\mathrm{yZF}}(j\omega)(1+kG(j\omega))\right\} \geq \delta \quad \forall \omega \in \mathbb{R}, \tag{26}$$

for some $\delta > 0$. Then the feedback interconnection (4) is $\mathscr{L}_2$-stable. ■

VIII. Conclusion

In order to analyze the relationships between different classes of multipliers, notions of phase-containment and phase-equivalence have been defined. Most of the classes of multipliers defined in the literature for slope-restricted nonlinearities, such as Popov multipliers, RC and RL multipliers, Park's multipliers Zames–Falb multipliers and their extension, are discussed in this paper. We have shown that all these classes of multipliers are phase-contained within the class of Zames–Falb multipliers. This provides new $\mathscr{L}_2$-stability results as corollaries of Theorem 2.1. The only conditions required are those which are given for the Zames–Falb multipliers [4]. Corollary 4.7, Corollary 6.4, Corollary 7.3, and Corollary 7.6 are all believed to be novel.

In principle, there is no need to search over any class of multipliers other than those of the class of Zames–Falb multiplier. However, no convex search over the whole class of Zames-Falb multipliers has yet been found. This can be largely ascribed to difficulties associated with the bound on the $\mathscr{L}_1$-norm in the original definition. Hence from a practical point of view, the classes of multipliers discussed in this paper remain useful. The analysis of this paper indicates that any improvement from their use should be interpreted as arising from a convenient parameterization within the class of Zames-Falb multipliers.

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