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On State-Space Characterization for Strict Negative-Imaginariness of LTI Systems

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***Abstract*— Negative-imaginary systems appear quite often in engineering applications, for example, in flexible structures with collocated position sensors and force actuators, in electrical circuits, in system biology, etc. In this paper, a strongly strict negative-imaginary lemma is proposed to ensure the strict negative-imaginary property of an LTI system. This result will facilitate both robustness analysis and controller synthesis for interconnected negative-imaginary systems. In the proposed characterization, numerical advantages are achieved by avoiding a minimality assumption, a non-convex rank constraint and a non-strict inequality condition present in previous literature. Two numerical examples are provided to illustrate the effectiveness of the proposed results.**

I. INTRODUCTION

Negative-imaginary (NI) systems are found in many engineering applications, for example, the transfer function from force actuator to collocated position sensor (for instance, piezoelectric sensor) in mechanical systems [1]–[6], in electrical filters [7], or in system biology [8], etc. Also, there are some uncertain systems that can equivalently be presented into systems with the uncertain part being NI [1]–[3]. An intuitive definition of (strictly) NI systems lies in the fact that, in an SISO setting, the imaginary part of the frequency response in the positive frequency interval is (negative) non-positive. Formally (both for SISO and MIMO), the definitions of NI systems and SNI systems are given as follows:

Definition 1: (NI Systems) [5] A real-rational proper transfer function matrix $R(s) \in \mathcal{R}^{m\times m}$ is said to be NI if
1) $R(\infty) = R^T(\infty)$;
2) $R(s)$ has no poles at the origin and in $\mathbf{Re}[s] > 0$;
3) $j[R(j\omega) - R(j\omega)^*] \geq 0$ for all $\omega \in (0,\infty)$ except the values of ω where $j\omega$ is a pole of $R(s)$;
4) If $j\omega_0$ is a pole of $R(s)$, it is at most a simple pole and the residue matrix $K_0 \triangleq \lim_{s\to j\omega_0}(s-j\omega_0)s(R(s)-R(\infty))$ is positive-semidefinite Hermitian;
where $\mathcal{R}^{m\times m}$ denotes the set of all proper real-rational transfer function matrices of dimension $m\times m$, $\mathbf{Re}[\cdot]$ denotes the real part of a complex number, and $(\cdot)^*$ denotes the complex conjugate transpose of a complex matrix.

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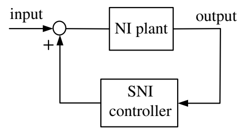

Fig. 1. Robust control of an uncertain NI plant

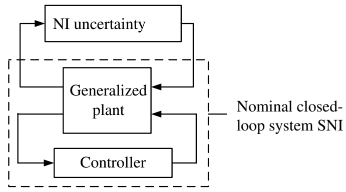

Fig. 2. Robust stabilization in the presence of SNI uncertainty

Definition 2: (SNI Systems) [1], [5] A real-rational proper transfer function matrix $R(s) \in \mathcal{R}^{m\times m}$ is said to be SNI if
1) $R(\infty) = R^T(\infty)$;
2) $R(s)$ has no poles in $\mathbf{Re}[s] \geq 0$;
3) $j[R(j\omega) - R(j\omega)^*] > 0$ for all $\omega \in (0,\infty)$.
The concept of NI systems is similar to that of positive-real (PR) systems, where the frequency response is constrained in one half of the complex plane. However, NI systems can have a maximum relative degree of two, whereas PR systems cannot have more than unity. Most importantly, the frequency dependent condition for NI systems is fulfilled on the punctured $j\omega$-axis; i.e., it excludes zero frequency whereas the PR condition is satisfied for all frequencies [9].

In practice, one may want to synthesize an SNI controller interconnected via positive feedback with an NI plant as shown in Fig. 1, since a recent stability analysis result shows that internal stability of a positive feedback interconnection of an NI system and an SNI system is established provided the DC gain of the loop is contractive [1], [5]. Most importantly, robust stability is retained for arbitrary plant variations as long as the plant satisfies the NI property and the DC loop gain condition [1], [5]. Also, in the LFT framework as shown in Fig. 2, it is desirable to render the nominal closed-loop system to be SNI in order to robustly stabilize systems with NI uncertainties. For robust stability, the DC loop gain should be strictly less than unity.

To this end, ensuring an SNI property is hence essential in both robustness analysis and controller synthesis related to NI systems. Note that, a state-space characterization for SNI systems is proposed in [2], [5], which is referred to as the "Weakly Strict Negative-Imaginary (WSNI) Lemma" as it is derived via an underpinning weakly strict positive-real (WSPR) property of the system [10]. This WSNI lemma in [2], [5] is difficult to apply for NI controller synthesis as it requires a minimality assumption and a non-convex rank condition to be fulfilled on a punctured $j\omega$-axis. By circumventing these difficulties, this paper gives a new state-space characterization for strictly negative-imaginary systems (SNI). The proposed SNI lemma is referred to as the "Strongly Strict Negative-Imaginary (SSNI) lemma" as it is developed via an underpinning strongly strict positive-real (SSPR) result [11].

II. NI LEMMA AND STABILITY OF INTERCONNECTED LOOP

This section presents some background material that aids the understanding of the key components of this paper.

First, we recall the Negative-Imaginary Lemma, which gives a necessary and sufficient state-space characterization for NI systems. It is restated as follows:

Lemma 1: (NI Lemma) [1], [5] Let (A, B, C, D) be a minimal state-space realization of an $m \times m$ real-rational transfer function matrix $R(s)$, where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{m\times n}$ and $D \in \mathbb{R}^{m\times m}$. Then, $R(s)$ is NI if and only if
(i) $\det(A) \neq 0$, $D = D^T$;
(ii) there exists a matrix $Y = Y^T > 0$, $Y \in \mathbb{R}^{n\times n}$, such that

$$AY + YA^T \leq 0, \text{ and } B = -AYC^T,$$

where $\det(A)$ denotes the determinant of matrix A.

The next result gives a characterization for the strictly negative-imaginariness of a system. We will refer to it as WSNI Lemma throughout this paper.

Lemma 2: (WSNI Lemma) [5] Let (A, B, C, D) be a minimal state-space realization of an $m \times m$ real-rational transfer function matrix $R(s)$, where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{m\times n}$ and $D \in \mathbb{R}^{m\times m}$. Then, $R(s)$ is SNI if and only if
(i) A is Hurwitz, $D = D^T$, $\text{rank}(B) = \text{rank}(C) = m$;
(ii) there exists a matrix $Y = Y^T > 0$, $Y \in \mathbb{R}^{n\times n}$, such that

$$AY + YA^T \leq 0, \text{ and } B = -AYC^T;$$

(iii) the transfer function matrix $M(s) \sim \left[\begin{array}{c|c} A & B \\ \hline LY^{-1}A^{-1} & 0 \end{array}\right]$ has full column rank at $s = j\omega$ for any $\omega \in (0, \infty)$. Here, $L^T L = -AY - YA^T$. That is, $\text{rank}(M(j\omega)) = m$ for any $\omega \in (0, \infty)$.

The above versions of NI lemma and WSNI lemma are built on the requirement that the state-space representation of the system is minimal. However, from controller synthesis point of view, the minimality assumption cannot be computed 'a priori' for a synthesized closed-loop system as it is involved with unknown controller matrices. Moreover, the state-space characterizations posed in Lemma 1 and Lemma 2 involve non-convex conditions and those are the determinant condition in Lemma 1 and the rank constraint for the punctured $j\omega$-axis in Lemma 2. These non-convex conditions are hard to implement numerically in controller synthesis, especially the latter rank condition of Lemma 2 makes it difficult to apply for robust control of NI systems.

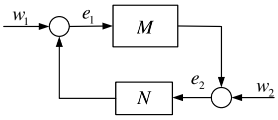

Fig. 3. Positive-feedback interconnection.

The next theorem gives a robustness and stability analysis result for NI systems. It states that a positive-feedback interconnection of two NI systems is internally stable if and only if the DC loop gain is contractive and at least one of the systems in the interconnected loop is SNI.

Theorem 3: [1], [5] Given that $M(s)$ is NI and $N(s)$ is SNI, and suppose that $M(\infty)N(\infty) = 0$ and $N(\infty) \geq 0$. Then, the positive-feedback interconnection of these two systems illustrated in Fig. 3 is internally stable if and only if the maximum eigenvalue of the matrix $M(0)N(0)$, denoted by $\bar{\lambda}(M(0)N(0))$, satisfies

$$\bar{\lambda}(M(0)N(0)) < 1. \quad (1)$$

This stability result captures control schemes such as positive position feedback control [12] and resonant control [13] in a systematic framework. These methods typically rely on NI controllers to robustly stabilize uncertain SNI systems. Also, via this result, for systems with SNI uncertainties, if a controller is designed such that the nominal closed-loop linear fractional transformation (LFT) system is NI with a proper DC gain, then the resulting closed-loop system is robustly stable. This idea is incorporated in [2], [3] to robustly stabilize systems with SNI uncertainties.

Note that existing results on robust control for uncertain NI systems typically only enforce a (non-strict) NI property on the feedback interconnection of the nominal plant and controller and can thereby only handle SNI uncertainty [2], [3]. However, the uncertainties do not always satisfy the SNI property, see, e.g., the example in [1]. For systems with non-strict NI uncertainties, it is desirable to swap the system property in the loop, i.e., the nominal closed-loop system needs to satisfy the SNI property for robustness against NI uncertainties. This is because the stability of interconnected NI systems requires at least one of the systems to be SNI. In practice, one might also want to synthesize an SNI controller to stabilize an NI plant. Hence, a numerically attractive characterization for the SNI property is important in analysis and synthesis for the robust control of NI systems. In the following section, a result for characterizing such a system property is presented.

III. MAIN RESULTS

In this section, a new state-space characterization is given to check the SNI property of an LTI system. This result relaxes the minimality assumption required in Lemma 1 or Lemma 2. This relaxation facilitates controller synthesis as the minimality assumption cannot be computed 'a priori' in controller synthesis to satisfy the SNI property of the synthesized loop, which is necessary for robust stability of the closed-loop system. By avoiding the non-convex rank constraint and the non-strict inequality which are present in Lemma 2, the proposed characterization also gives numerical advantages. This result is derived based on the strongly strict positive-real (SSPR) property of a transformed system. Before stating the main result, the following definition of SSPR systems is needed.

Definition 3: [14] A real-rational proper transfer function matrix $G(s) \in \mathcal{R}^{m\times m}$ is SSPR if
1) $G(s)$ has no poles in $\mathbf{Re}[s] \geq 0$,
2) $G(j\omega) + G(j\omega)^* > 0$ for all $\omega \in \mathbb{R}$,
3) $\lim_{\omega\to\infty} \omega^{2\rho}\det(G(j\omega) + G(j\omega)^*) > 0$, where ρ is the dimension of the null space of $G(\infty) + G(\infty)^T$.

Remark 1: [14] For strictly proper transfer functions, condition 3) in *Definition* 3 reduces to $\lim_{\omega\to\infty} \omega^2(G(j\omega) + G(j\omega)^*) > 0$, which coincides with the condition previously presented in the literature (see [9], [14] for details).

Next, a state-space characterization for SSPR property of a system is given. The standard Strictly Positive-Real Lemma is given for minimal systems [9], however, the following lemma is given for non-minimal systems. This lemma will be invoked later to derive the main results of this paper.

Lemma 4: (SSPR Lemma) Let $G(s) = C(sI - A)^{-1}B$ be a strictly proper $m \times m$ transfer function matrix. Suppose $G(s) + G(-s)^T$ has normal rank m.
(i) If there exists a matrix $P = P^T > 0$ that satisfies

$$PA + A^T P < 0, \text{and } PB = C^T, \tag{2}$$

then A is Hurwitz and $G(s)$ is SSPR.
(ii) Suppose (C, A) is observable. If A is Hurwitz and $G(s) = C(sI - A)^{-1}B$ is SSPR, then there exists a matrix $P = P^T > 0$ that satisfies the conditions in (2).
(iii) Suppose the state-space realization (A, B, C) has no observable uncontrollable modes. If A is Hurwitz and $G(s) = C(sI - A)^{-1}B$ is SSPR, then there exists a matrix $P = P^T > 0$ that satisfies the conditions in (2).

Proof: This proof is omitted due to the space constraint and will be published elsewhere. ■

Lemma 4 (i) states that the algebraic conditions in (2) are sufficient to guarantee the SSPR property of a strictly proper system, while Lemma 4 (ii) and Lemma 4 (iii) are given to obtain necessary and sufficient conditions for the existence of solution to the LMI conditions posed in (2) under the mildest possible system-theoretic assumptions.

Now, the proposed characterization for the SNI property is stated in the following theorem. In contrast to the Weakly Strict Negative-Imaginary Lemma, we refer to this theorem as the Strongly Strict Negative-Imaginary Lemma.

Theorem 5: (SSNI Lemma I) Given a square transfer function matrix $R(s) \in \mathcal{R}^{m\times m}$ with a state-space realization (A, B, C, D), where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{m\times n}$ and $D \in \mathbb{R}^{m\times m}$. Suppose $R(s) + R(-s)^T$ has normal rank m and (C, A) is observable. Then, A is Hurwitz and $R(s)$ is SNI with

$$\begin{aligned} &\lim_{\omega\to 0} j\frac{1}{\omega}\left[R(j\omega) - R(j\omega)^*\right] > 0 \\ \text{and } &\lim_{\omega\to\infty} j\omega\left[R(j\omega) - R(j\omega)^*\right] > 0 \end{aligned} \tag{3}$$

if and only if $D = D^T$ and there exists a matrix $Y = Y^T > 0$ such that

$$AY + YA^T < 0 \text{ and } B = -AYC^T. \tag{4}$$

Proof: ($\Rightarrow$) First, note that $R(s)$ is SNI. It then follows from Lemma 2 of [1] that $R(0) = R(0)^T$. Now, letting $G(s) = -\frac{1}{s}\left[R(s) - R(0)\right]$, we have

$$\begin{aligned} & G(j\omega) + G(j\omega)^* \\ = & \left(-\frac{1}{j\omega}[R(j\omega) - R(0)]\right) + \left(\frac{1}{j\omega}[R(j\omega)^* - R(0)^T]\right) \\ = & j\frac{1}{\omega}\left[R(j\omega) - R(j\omega)^*\right]. \end{aligned} \tag{5}$$

Also, since $j\left[R(j\omega) - R(j\omega)^*\right] > 0$ for all $\omega \in (0, \infty)$ by noting that $R(s)$ is SNI, it follows that $G(j\omega) + G(j\omega)^* > 0$ for all $\omega \in (0, \infty)$. Similarly,

$$\begin{aligned} G(0) + G(0)^T &= \lim_{\omega\to 0}(G(j\omega) + G(j\omega)^*) \\ &= \lim_{\omega\to 0} j\frac{1}{\omega}\left[R(j\omega) - R(j\omega)^*\right] > 0, \end{aligned} \tag{6}$$

and

$$\begin{aligned} & \lim_{\omega\to\infty} \omega^2(G(j\omega) + G(j\omega)^*) \\ = & \lim_{\omega\to\infty} j\omega\left[R(j\omega) - R(j\omega)^*\right] > 0. \end{aligned} \tag{7}$$

Hence, via Definition 3, $G(s) = -\frac{1}{s}(R(s) - R(0))$ is SSPR by noting that $G(s)$ is strictly proper. Also, since

$$\begin{aligned} G(s) &= -\frac{1}{s}(R(s) - R(0)) \\ &= -\frac{1}{s}\left(C(sI - A)^{-1}B + D - (D - CA^{-1}B)\right) \\ &\quad \text{noting that A is Hurwitz and hence nonsingular} \\ &= -\frac{1}{s}C\left[(sA^{-1} - I)^{-1}sA^{-1}\right]B \\ &= -C(sI - A)^{-1}A^{-1}B, \end{aligned} \tag{8}$$

it follows that $(A, A^{-1}B, -C, 0)$ is a state-space realization for $G(s)$. Also, note that A is Hurwitz and $(-C, A)$ is observable since (C, A) is observable. Furthermore, the fact that $R(s) + R(-s)^T$ has normal rank m implies that $G(s) + G(-s)^T$ has normal rank m. Then, it follows from Lemma 4 (ii) that there exists a positive definite matrix $P = P^T$ such that

$$PA + A^T P < 0 \text{ and } P(A^{-1}B) = -C^T. \tag{9}$$

Finally, letting $Y = P^{-1}$, it follows via a algebraic computation that conditions in (9) are equivalent to those in (4).
($\Leftarrow$) Since there exists a matrix $Y = Y^T > 0$ such that the conditions in (4) are satisfied, it follows that there exists a $P = Y^{-1} > 0$ such that the conditions in (9) are satisfied. Also, note that (4) implies that A is Hurwitz and hence nonsingular. Furthermore, since $(A, A^{-1}B, -C, 0)$ is a state-space realization for $G(s)$ via (8), then it follows from Lemma 4 (i) that $G(s)$ is SSPR. Hence, we have $G(j\omega) + G(j\omega)^* > 0$ for all $\omega \in \mathbb{R}$ and $\lim_{\omega\to\infty} \omega^2 [G(j\omega) + G(j\omega)^*] > 0$ by noting that $G(s)$ is strictly proper. Also, since $D = D^T$, it follows that

$$R(0) = D - CA^{-1}B = D + CA^{-1}AYC^T = R(0)^T$$

via (4). Hence, $G(j\omega) + G(j\omega)^* > 0$ for all $\omega \in \mathbb{R}$ implies that $j[R(j\omega) - R(j\omega)^*] > 0$ for all $\omega \in (0, \infty)$ and $\lim_{\omega\to 0} j\frac{1}{\omega}[R(j\omega) - R(j\omega)^*] > 0$ via (5) and (6), and $\lim_{\omega\to\infty} \omega^2 [G(j\omega) + G(j\omega)^*] > 0$ implies $\lim_{\omega\to\infty} j\omega [R(j\omega) - R(j\omega)^*] >$ via (7). Finally, $R(s)$ has no poles in $\mathbf{Re}[s] > 0$ since A is Hurwitz, hence $R(s)$ is SNI which satisfies (3) and thereby completes the proof. ■

Theorem 6: (SSNI Lemma II) Given a square transfer function matrix $R(s) \in \mathcal{R}^{m\times m}$ with a state-space realization (A, B, C, D), where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{m\times n}$ and $D \in \mathbb{R}^{m\times m}$. Suppose $R(s) + R(-s)^T$ has normal rank m and (A, B, C, D) has no observable uncontrollable modes. Then, A is Hurwitz and $R(s)$ is SNI satisfying (3) if and only if $D = D^T$ and there exists a matrix $Y = Y^T > 0$ that satisfies the conditions in (4).

Proof: Letting $G(s) = -\frac{1}{s}[R(s) - R(\infty)]$, it follows from (8) that $(A, A^{-1}B, -C, 0)$ is a state-space realization for $G(s)$ noting that A is nonsingular. Note that the assumption that (A, B, C, D) has no observable uncontrollable modes implies that the state-space realization of $G(s)$: $(A, A^{-1}B, -C, 0)$ has no observable uncontrollable modes when A is nonsingular. Then, the results follow along the same lines as in the proof of Theorem 5, where Lemma 4 (i) and Lemma 4 (iii) will be invoked instead of Lemma 4 (i) and Lemma 4 (ii). ■

Remark 2: The assumption that (C, A) is observable in Theorem 5 is only needed to prove necessity part of the theorem. The assumption that (A, B, C, D) has no observable uncontrollable modes is another necessary requirement to show the SNI property as posed in Theorem 6.

Theorem 5 and Theorem 6 imply that the symmetry of D and the existence of a positive definite solution to the algebraic conditions in (4) are sufficient to guarantee the SNI property of a system. The earlier (non-strict) NI lemma (Lemma 1) [1], [2], [5] gives a complete state-space characterization of (non-strict) NI systems, which invokes a non-strict Lyapunov inequality in (4). When the Lyapunov inequality in (4) becomes strict as in Theorem 5 (Theorem 6), then we get a complete state-space characterization of SNI systems but we also enforce a departure condition from and an arrival condition to the real axis as described by the limiting conditions in (3).

Theorems 5 and 6 will enable robust control synthesis for uncertain NI systems. Via this result, an SNI controller can be synthesized by considering the simple algebraic conditions given in (4) to stabilize an NI plant interconnected via positive feedback in a closed-loop as shown in Fig. 1; or we can design a controller such that an LFT closed-loop system satisfies (4) to ensure the SNI property that facilitates robust stability against NI uncertainties, as shown in Fig. 2. For robust stability, the DC loop gain should be contractive [1]. As a consequence, this result will facilitate robust synthesis methods to handle non-strict NI uncertainties. It also helps avoid numerical issues, for example, the determinant condition and the rank constraint in a frequency interval posed, in Lemma 1 and Lemma 2, respectively.

Next, we give some physical interpretations of the mathematical conditions in (3).

Lemma 7: Given a proper scalar SNI transfer function $R(s)$ with $R(\infty) \geq 0$, then

$$\lim_{\omega\to 0} j\frac{1}{\omega}(R(j\omega) - R(j\omega)^*) > 0 \quad \Leftrightarrow \quad \lim_{\omega\to 0} \frac{d\phi(\omega)}{d\omega} < 0,$$

where $\phi(\omega)$ denotes the phase of $R(j\omega)$.

Proof: This proof is omitted due to the space constraint and will be published elsewhere. ■

The above lemma states that for a proper scalar transfer function $R(s)$ with the SNI property and $R(\infty) \geq 0$, $\lim_{\omega\to 0} j\frac{1}{\omega}(R(j\omega) - R(j\omega)^*) > 0$ means that the phase of $R(j\omega)$ strictly decreases as frequency increases from $\omega = 0$.

Remark 3: For strictly proper scalar transfer functions, $\lim_{\omega\to\infty} j\omega(R(j\omega) - R(j\omega)^*) > 0$ implies that the imaginary part of $R(j\omega)]$ cannot go to zero faster than ω^{-1} when $|\omega| \to \infty$. This implies that the relative degree of $R(j\omega)$ must be zero or one.

Remark 3 implies that if one uses the conditions in (4) to design an SNI controller, systems with relative degree of two cannot be captured.

IV. ILLUSTRATIVE EXAMPLE

In this section, two examples are given to demonstrate the applicability of our main results.

Example 1: Let

$$A = \begin{bmatrix} -5 & 4 & 0 & 0 & 0 & 0 & 0 \\ -1.75 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 4 & 0 & 0 \\ 0 & 0 & 0 & -1.75 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -8 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0.5 \\ 1.5 & 0.75 \\ 2 & 1 \\ 0.5 & 1 \\ 0.75 & 1.5 \\ 1 & 2 \\ 0 & 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 & 0 & 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \tag{10}$$

We shall now determine whether $R(s) = C(sI - A)^{-1}B + D$ is SNI. Note that the above state-space realization for $R(s)$ is not minimal since there is one uncontrollable mode {-8}. Hence, neither Lemma 1 nor Lemma 2 can be applied to analyze the (strict) negative-imaginariness of this system with the above given state-space realization.

Note that $R(s)+R(-s)^T$ has normal rank 2. Also, (C, A) in (10) is observable and $D = \mathbf{0} = D^T$. Hence, the assumptions in Theorem 5 are satisfied. We can now use Theorem 5 to analyze the SNI property of $R(s)$.

We use YALMIP and SeDuMi to solve the conditions in (4) and the following solution is obtained

$$Y = \begin{bmatrix} 1.03 & 1.13 & 0.99 & 0.53 & 0.57 & 0.49 & -0.06 \\ 1.13 & 1.64 & 1.42 & 0.57 & 0.81 & 0.72 & -0.02 \\ 0.99 & 1.42 & 2.73 & 0.49 & 0.72 & 1.16 & 0.02 \\ 0.53 & 0.57 & 0.49 & 1.03 & 1.13 & 0.99 & -0.06 \\ 0.57 & 0.81 & 0.72 & 1.13 & 1.64 & 1.42 & -0.02 \\ 0.49 & 0.72 & 1.16 & 0.99 & 1.42 & 2.73 & 0.02 \\ -0.06 & -0.02 & 0.02 & -0.06 & -0.02 & 0.02 & 0.12 \end{bmatrix} > 0.$$

Thus, via Theorem 5, $R(s)$ is SNI that satisfies the conditions in (3).

Via simple computation, we obtain

$$R(s) = \frac{s^2+6s+8}{s^3+5s^2+7s+4}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}. \tag{11}$$

$R(s)$ has no poles in $\mathbf{Re}[s] \geq 0$ since the poles of $R(s)$ are located at -3.2056,-3.2056, $-0.8972 \pm 0.6655j$, $-0.8972 \pm 0.6655j$. Also, note that

$$\begin{aligned} j[R(j\omega) - R(j\omega)^*] \;&=\; \frac{52\omega + 37\omega^3 + \omega^5}{(4-5\omega^2)^2 + (8\omega - \omega^3)^2}\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix} \\ &> \; 0 \quad \text{for all } \omega \in (0, \infty). \end{aligned}$$

Hence, $R(s)$ is SNI via Definition 2. Also, it is easy to verify by calculation that

$$\lim_{\omega \to 0} j\frac{1}{\omega}(R(j\omega) - R(j\omega)^*) = \begin{bmatrix} 13 & 6.5 \\ 6.5 & 13 \end{bmatrix} > 0,$$

and

$$\lim_{\omega \to \infty} j\omega(R(j\omega) - R(j\omega)^*) = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix} > 0,$$

which coincide with the statement of Theorem 5.

Now, suppose $R(s)$ in (11) is with the another state-space realization which is shown below (note that $\bar{C}$ is different from C),

$$\bar{A} = \begin{bmatrix} -5 & 4 & 0 & 0 & 0 & 0 & 0 \\ -1.75 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 4 & 0 & 0 \\ 0 & 0 & 0 & -1.75 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -8 \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} 1 & 0.5 \\ 1.5 & 0.75 \\ 2 & 1 \\ 0.5 & 1 \\ 0.75 & 1.5 \\ 1 & 2 \\ 0 & 0 \end{bmatrix},$$

$$\bar{C} = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 \end{bmatrix}, \quad \bar{D} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \tag{12}$$

The above state-space realization is not minimal, since there is an uncontrollable unobservable mode $\{-8\}$. However, all the observable modes of the system in (12) are controllable, hence the assumptions in Theorem 6 are satisfied. Also, since we have already known that $R(s)$ is SNI with the conditions in (3) satisfied, then using Theorem 6, there should exist a positive definite solution $Y = Y^T > 0$ that satisfies the conditions in (4).

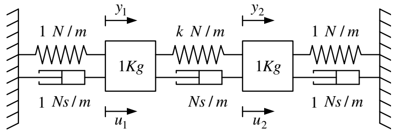

Fig. 4. Lightly damped uncertain mechanical plant

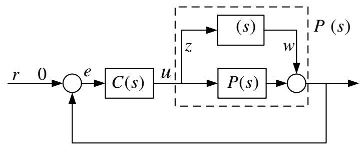

Fig. 5. Controlled closed-loop system

Again, YALMIP and SeDuMi are used to solve the conditions in (4) with $(\bar{A}, \bar{B}, \bar{C})$ shown in (12) and we obtain the following solution

$$Y = \begin{bmatrix} 0.36 & -0.17 & -0.06 & -0.04 & -0.03 & 0.04 & 0.00 \\ -0.17 & 1.14 & -0.77 & -0.03 & -0.10 & 0.09 & 0.00 \\ -0.06 & -0.77 & 1.94 & 0.04 & 0.09 & -0.76 & -0.00 \\ -0.04 & -0.03 & 0.04 & 0.36 & -0.17 & -0.06 & -0.00 \\ -0.03 & -0.10 & 0.09 & -0.17 & 1.14 & -0.77 & -0.00 \\ 0.04 & 0.09 & -0.76 & -0.06 & -0.77 & 1.94 & 0.00 \\ 0.00 & 0.00 & -0.00 & -0.00 & -0.00 & 0.00 & 0.19 \end{bmatrix} > 0,$$

which coincides with the statement of Theorem 6.

Example 2: Consider the lightly damped mechanical plant [1] depicted in Fig. 4, which consists of two unit masses constrained to slide rectilinearly on a frictionless table. Each mass is attached to a fixed wall via a spring of known unit stiffness and via a damper of known unit viscous resistance. Furthermore, the two unit masses are coupled together via a spring of uncertain stiffness k (N/m) and via a damper of uncertain viscous resistance α (N-s/m). A force is applied to each mass (denoted by u_1 and u_2, respectively) and the displacement of each mass is measured (denoted by y_1 and y_2 , respectively).

Here, we will show how the proposed results in this paper can facilitate the robust control of NI systems.

The transfer function matrix from the input $u := \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ to the output $y := \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ is described by $y = P_\triangle(s)u$, where

$$\begin{aligned} P_\triangle(s) :=& p(s)\delta(s) \\ &\times \begin{bmatrix} s^2 + (\alpha+1)s + (k+1) & (\alpha s + k) \\ (\alpha s + k) & s^2 + (\alpha+1)s + (k+1) \end{bmatrix}, \\ p(s) \; :=& \frac{1}{s^2+s+1}, \text{ and } \delta(s) \; := \frac{1}{s^2 + (2\alpha+1)s + (2k+1)}. \end{aligned}$$

This plant is uncertain since α and k are unknown.

For robust control, the closed-loop system in Fig. 5 is rearranged in a standard LFT interconnection shown in Fig. 6, where the generalized plant Σ, the nominal plant P and

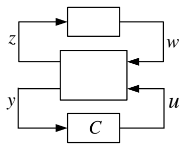

Fig. 6. Rearranged LFT interconnection

the uncertainty Δ are given respectively by

$$\Sigma = \begin{bmatrix} 0 & I \\ -I & -P \end{bmatrix}, \; P(s) = \begin{bmatrix} 0.5p(s) & 0.5p(s) \\ 0.5p(s) & 0.5p(s) \end{bmatrix},$$

$$\text{and } \Delta(s) = \begin{bmatrix} 0.5\delta(s) & -0.5\delta(s) \\ -0.5\delta(s) & 0.5\delta(s) \end{bmatrix}. \qquad (13)$$

It can be verified that the uncertainty $\Delta(s)$ is (non-strict) NI, hence earlier works for SNI uncertainties [2], [3] cannot be used to tackle this robust control problem.

Let us consider a controller $C(s)$ (note robust controller synthesis for NI system is an area for future work):

$$C(s) := \left[\begin{array}{cccccc|cc} -0.01 & -0.48 & 0.30 & -1.73 & 1.78 & -0.69 & -1.14 & 0.23 \\ 0.09 & -3.04 & -1.92 & -0.10 & -1.43 & -0.21 & 0.15 & 3.76 \\ -0.57 & 2.20 & 0.56 & -1.71 & 1.07 & -0.27 & 0.32 & 0.49 \\ 0.52 & 0.41 & 1.84 & -1.17 & 1.73 & -0.84 & -2.85 & -0.03 \\ 0.06 & -0.20 & 0.19 & 1.84 & -1.34 & 0.37 & 2.30 & 0.18 \\ 0.02 & 0.88 & -0.31 & 0.84 & 0.05 & 0.26 & -0.32 & -0.35 \\ \hline -0.51 & 1.03 & 0.03 & 1.26 & -0.78 & 0.84 & 0 & 0 \\ 0.37 & -1.34 & -0.53 & -1.16 & 0.22 & -0.74 & 0 & 0 \end{array}\right], \qquad (14)$$

where the corresponding closed-loop system $F_\ell(\Sigma, C)$ is given by $F_\ell(\Sigma, C) := \left[\begin{array}{c|c} A_{cl} & B_{cl} \\ \hline C_{cl} & D_{cl} \end{array}\right]$, where

$$Acl = \begin{bmatrix} -1.0 & -0.93 & 0.71 & 0.71 & 0 & 0 & 0 & 0 \\ 1.01 & -0.00 & -0.05 & -0.05 & 0 & 0 & 0 & 0 \\ 0.08 & 1.13 & -1.00 & 0 & 0.36 & -0.14 & -0.47 & -0.14 \\ 0.08 & 1.13 & 0 & -1.00 & 0.35 & 0.13 & -0.48 & -0.16 \\ -0.078 & -1.11 & 0 & 0 & -0.63 & 0.00 & 1.15 & 0.48 \\ 0.0 & 0.00 & 0 & 0 & 0.01 & -1.08 & 0.00 & 0.01 \\ -0.13 & -1.96 & 0 & 0 & -1.26 & 0.01 & 0.22 & -0.16 \\ 0.02 & 0.27 & 0 & 0 & -0.49 & -0.01 & 0.48 & -1.25 \end{bmatrix},$$

$$B_{cl} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 4.91 & -3.31 \\ -3.26 & 4.86 \\ -0.78 & -0.79 \\ 7.78 & -7.77 \\ -1.43 & -1.35 \\ 0.24 & 0.14 \end{bmatrix}, \; C_{cl} = \begin{bmatrix} 0\,0\,1\,0\,0\,0\,0\,0 \\ 0\,0\,0\,1\,0\,0\,0\,0 \end{bmatrix}, \; D_{cl} = \begin{bmatrix} 0\,0 \\ 0\,0 \end{bmatrix}.$$

Note that this state-space realization for $F_\ell(\Sigma, C)$ is minimal, and $D_{cl} = \mathbf{0} = D_{cl}^T$. Now we use YALMIP and SeDuMi to solve the conditions in (4) with the state-space realization for $F_\ell(\Sigma, C)$, namely $(A_{cl}, B_{cl}, C_{cl}, D_{cl})$ and we obtain the following solution

$$Y = \begin{bmatrix} 5.07 & -1.97 & 0.17 & 0.17 & -2.74 & -0.02 & -4.60 & -2.45 \\ -1.97 & 9.44 & 2.45 & 2.45 & -13.96 & -0.01 & 4.17 & 6.31 \\ 0.17 & 2.45 & 4.86 & -1.40 & -5.40 & 7.20 & -0.94 & 2.46 \\ 0.17 & 2.45 & -1.408 & 4.88 & -5.42 & -7.22 & -0.93 & 2.47 \\ -2.74 & -13.96 & -5.40 & -5.42 & 31.85 & 0.08 & 0.28 & -10.00 \\ -0.02 & -0.01 & 7.20 & -7.22 & 0.08 & 24.51 & 0.03 & -0.02 \\ -4.60 & 4.173 & -0.94 & -0.93 & 0.28 & 0.032 & 11.85 & 2.60 \\ -2.45 & 6.31 & 2.46 & 2.47 & -10.00 & -0.02 & 2.60 & 15.15 \end{bmatrix}$$

$$= Y^T > 0.$$

Hence, via Theorem 5 (or Theorem 6), $F_\ell(\Sigma, C)$ is SNI. Also, note that $F_\ell(\Sigma, C)(\infty) = \mathbf{0}$, consequently, this controller in (14) guarantees robust stability for all non-strict NI perturbations as long as the DC loop gain condition is also satisfied [Theorem 3].

V. Conclusions

A state-space characterization for the SSNI property is proposed to facilitate robust controller synthesis and analysis for uncertain systems where non-strict NI uncertainties are present. It offers important advantages by avoiding the rank condition and the minimality assumption required in the results of [1], [5]. Using this result, the NI robust analysis and synthesis frameworks can be extended to deal with both non-strict and strict NI systems. This work also clarifies the relationship between the strict Lyapunov inequality (see (4)) and the SNI property of the system.

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