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Distance Measures for Linear Systems with Multiplicative and Inverse Multiplicative Uncertainty Characterisation

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***Abstract*—Input multiplicative and output inverse multiplicative uncertainty characterisations are common in the robust control literature. We present specialised distance measures and robust stability margins for these types of uncertainty, and derive both robust stability and robust performance theorems, based on the generic distance measure framework of [1]. The uncertainty is allowed to belong to the space $\mathscr{RL}_\infty$, and hence includes unstable systems. This constitutes a significant advance over previous robust stability results for such uncertainty structures, which were valid only for systems in $\mathscr{RH}_\infty$.**

I. Introduction

Multiplicative and inverse multiplicative uncertainty characterisations have been used in $\mathscr{H}_\infty$ robust control since the early days of this field [2]. They are conceptually simple yet allow capturing a large set of possible uncertainties in a plant. Input multiplicative uncertainty has often been used to model uncertain high frequency dynamics and uncertain right half plane zeros, while output inverse multiplicative uncertainty was used to model uncertain low frequency parameter errors and uncertain right half plane poles (see e.g. [3, Chapter 9]). The robust stability results for these structures are based on the small gain theorem [4], which holds for systems in the space $\mathscr{RH}_\infty$, i.e. systems without poles in the open right half complex plane. As a consequence, uncertain right half plane zeros (resp. poles) can not be modelled by output inverse multiplicative uncertainty (resp. input multiplicative uncertainty) in the traditional $\mathscr{RH}_\infty$ setting.

This paper presents robust stability and robust performance theorems for systems with output inverse multiplicative uncertainty and input multiplicative uncertainty for systems in the space $\mathscr{RL}_\infty$ (i.e. bounded on the imaginary axis), based on distance measures for such uncertainty structures. Generic distance measures for uncertain systems have been developed in [1]. The framework of [1] captures systems in $\mathscr{RL}_\infty$, and can be applied to a large number of different uncertainty structures via specialisations of a generic four-block plant model. Specialised results have been obtained for non-normalised coprime factor uncertainty [1] and normalised coprime factor uncertainty [5]. The robust stability and robust performance theorems for the generic theory are not easily applicable for specific uncertainty structures without significant linear algebra manipulation, especially to solve the consistency equation and to ensure well-posedness of the uncertain plant description. The specialised results theoretically validate distance measure approaches to robust stability analysis already used by the practising community (see e.g. [6] for an example constructing a minimal-size multiplicative uncertainty), by giving well-defined bounds and insights into the structure of the problem. Furthermore, there is great educational value in considering specific cases, rather than the generic theory only.

Distance measures for uncertain systems were first proposed in [7], [8] as a tool for quantifying difference between plants in a closed-loop sense. It was shown in [9] that optimising the robust stability margin in the gap metric [7] or graph metric [8] corresponds to optimising stability in a four-block or normalised coprime factor uncertainty setting. Subsequent research further developed the distance measures concept [10], [11], [12], but the uncertainty structure remained fixed to the normalised coprime factor setting. While normalised coprime factor uncertainty captures a large set of perturbed plants, it can be shown that the associated robust stability results are conservative vis-a-vis those results associated with other, less general uncertainty structures in specific situations, due to the differently-shaped regions of robust stability (as visualized e.g. on a Nyquist plot).

The main contribution of the paper is the following: A specialisation of the generic distance measure, robust stability margin and the robust stability and robust performance theorems of [1] for the case of output inverse multiplicative uncertainty, and brief summary of the results for the case of input multiplicative uncertainty. These results are readily applicable given descriptions of the nominal plant $P \in \mathscr{RL}_\infty$ and of a perturbed plant $P_\Delta \in \mathscr{RL}_\infty$ (under perturbations $\Delta \in \mathscr{RL}_\infty$), signifying a huge extension of the allowable plant and uncertainty spaces, which were previously restricted to $\mathscr{RH}_\infty$. As a consequence of also allowing unstable uncertainties, input multiplicative and output inverse multiplicative uncertainty become more flexible and can also be used to model right half plane zeros and poles, respectively.

A. Notation

Notation is standard. Let $\mathscr{R}$ denote the set of proper real-rational transfer functions. Also, let $P^*(s)$ denote the adjoint

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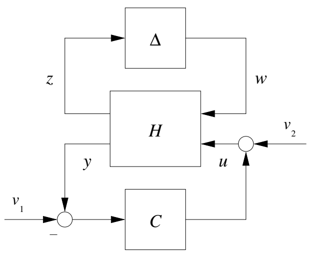

Fig. 1. The closed-loop system when a perturbation Δ is present.

of $P(s) \in \mathscr{R}$ defined by $P^*(s) = P(-s)^T$. For a constant matrix $A \in \mathbb{C}^{m\times n}$, denote by $A^\dagger \in \mathbb{C}^{n\times m}$ its Moore-Penrose pseudoinverse. Let $\mathscr{RL}_\infty$ denote the space of proper real-rational functions bounded on $j\mathbb{R}$ including ∞, and $\mathscr{RH}_\infty$ denote the space of proper real-rational functions bounded and analytic in the open right half complex plane. Denote the space of functions that are units in $\mathscr{RH}_\infty$ by $\mathscr{GH}_\infty$ (that is, $f \in \mathscr{GH}_\infty \Leftrightarrow f, f^{-1} \in \mathscr{RH}_\infty$). Let $\mathscr{F}_l(\cdot,\cdot)$ (resp. $\mathscr{F}_u(\cdot,\cdot)$) denote a lower (resp. upper) linear fractional transformation (LFT). For a scalar $p(s) \in \mathscr{R}$, its winding number $\operatorname{wno} p(s)$ is defined as the number of encirclements of the origin made by $p(s)$ as s follows the standard Nyquist D-contour, indented into the right half plane around any imaginary axis poles or zeros of $p(s)$. Furthermore, let $\eta(P)$ and $z(P)$ denote, respectively, the number of open right half plane poles and zeros of $P \in \mathscr{R}$. For a plant $P \in \mathscr{R}$ and a controller $C \in \mathscr{R}$, let $[P,C]$ denote the nominal feedback interconnection obtained by setting $\Delta = 0$ in Fig. 1, and let $\langle H,C\rangle$ denote the linear fractional interconnection of H and C with input w, v_1, v_2 and output y, z as displayed in Fig. 1.

II. Generic Distance Measures

This section reviews the distance measure for generic uncertainty representations, the associated stability margin as well as the robust stability and robust performance theorems of [1]. Given a family of perturbed plants, described by a generalised plant $H = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22}\end{bmatrix} \in \mathscr{R}$ and the uncertainty representation $\Delta \in \mathscr{R}$, connected through an upper LFT as displayed in Fig. 1 such that (assuming $(I - H_{11}\Delta)^{-1} \in \mathscr{R}$) the perturbed plant is given by

$$P_\Delta = \mathscr{F}_{\mathrm{u}}(H,\Delta) = H_{22} + H_{21}\Delta(I - H_{11}\Delta)^{-1}H_{12}. \qquad (1)$$

When $\Delta = 0$, eqn. (1) reduces to $P_\Delta = P$, i.e. the nominal plant $P \in \mathscr{R}^{p\times q}$. From eqn. (1) we can also observe that there exists a set of uncertainties $\mathbf{\Delta}$ yielding a perturbed plant P_Δ from a nominal plant P whenever a well-posedness condition ($(I - H_{11}\Delta)^{-1} \in \mathscr{R}$) and the consistency equation (1) are fulfilled for one or several Δ's. We introduce a distance measure based on the sizes of such allowable uncertainties.

Definition 1. *[1, Section II] Given a plant $P \in \mathscr{R}^{p\times q}$, a generalized plant $H \in \mathscr{R}$ with $H_{22} = P$, and a perturbed plant $P_\Delta \in \mathscr{R}^{p\times q}$. Let the set of all admissible perturbations be given by* $\mathbf{\Delta} = \left\{\Delta \in \mathscr{RL}_\infty : (I - H_{11}\Delta)^{-1} \in \mathscr{R}, P_\Delta = \mathscr{F}_u(H,\Delta)\right\}$. *Define the distance measure $d^H(P,P_\Delta)$ between plants P and P_Δ for the uncertainty structure implied by H as:*

$$d^H(P,P_\Delta) := \begin{cases} \inf_{\Delta\in\mathbf{\Delta}} \|\Delta\|_\infty, & \text{if } \mathbf{\Delta} \neq \emptyset \\ \infty, & \text{otherwise.} \end{cases}$$

Also define a set of minimal-size admissible uncertainties for ease of notation.

Definition 2. *[1, Section II] Given a plant $P \in \mathscr{R}^{p\times q}$, a generalized plant $H \in \mathscr{R}$ with $H_{22} = P$, and a perturbed plant $P_\Delta \in \mathscr{R}^{p\times q}$. Define*

$$\mathbf{\Delta}^{\min} := \left\{\Delta \in \mathbf{\Delta} : \|\Delta\|_\infty = d^H(P,P_\Delta)\right\}.$$

Let us now define a small-gain type stability margin.

Definition 3. *[1, Section II] Given a plant $P \in \mathscr{R}^{p\times q}$, a generalized plant $H \in \mathscr{R}$ with $H_{22} = P$, and a controller $C \in \mathscr{R}^{q\times p}$. Define the stability margin $b^H(P,C)$ of the feedback interconnection $\langle H,C\rangle$ as:*

$$b^H(P,C) := \begin{cases} \|\mathscr{F}_l(H,C)\|_\infty^{-1} & \text{if } 0 \neq \mathscr{F}_l(H,C) \in \mathscr{RL}_\infty, \\ & [P,C] \text{ is internally stable,} \\ 0 & \text{otherwise,} \end{cases}$$

where $\mathscr{F}_l(H,C) = H_{11} + H_{12}C(I - H_{22}C)^{-1}H_{21}$.

These two concepts, distance measure and stability margin, are used in the following theorem to obtain robust stability guarantees for systems in $\mathscr{RL}_\infty$. Note that in contrast to systems in $\mathscr{RH}_\infty$, a small-gain type condition is not enough to guarantee stability, and that a winding number condition must be additionally introduced [13], [12].

Theorem 1 (Robust Stability)**.** *[1, Section III] Given a plant $P \in \mathscr{R}^{p\times q}$, a stabilizable generalized plant $H \in \mathscr{R}$ with $H_{22} = P$, a perturbed plant $P_\Delta \in \mathscr{R}^{p\times q}$ and a controller $C \in \mathscr{R}^{q\times p}$ such that $d^H(P,P_\Delta) < b^H(P,C)$ and $\mathbf{\Delta}^{\min} \neq \emptyset$, then the following statements are equivalent:*

(*a*) $[P_\Delta, C]$ *is internally stable*;
(*b*) $\forall \Delta \in \mathbf{\Delta}^{\min}$, $\eta(P_\Delta) = \eta(P) + \operatorname{wno}\det(I - H_{11}\Delta)$;
(*c*) $\exists \Delta \in \mathbf{\Delta}^{\min} : \eta(P_\Delta) = \eta(P) + \operatorname{wno}\det(I - H_{11}\Delta)$. (2)

The proof is omitted here for the sake of brevity; it can be found in [1, Section III]. In the following theorem, we concretize the structure of the generalized plant H to that of a left four-block structure, i.e.

$$H = \left[\begin{array}{c|c} S_z & \\ \hline & I \end{array}\right] \left[\begin{array}{cc|c} I & -P & P \\ 0 & 0 & I \\ \hline I & -P & P \end{array}\right] \left[\begin{array}{c|c} S_w & \\ \hline & I \end{array}\right],$$

where S_w, $S_z \in \mathscr{R}$ are matrices used to make the generalized plant H represent any uncertainty structure that is typically important in engineering applications.[1] The following theorem gives robust performance guarantees for perturbed plants.

[1] S_w and S_z for the output inverse mutltiplicative and input multiplicative case are detailed in Sections III and IV.

Theorem 2 (Robust Performance)**.** *[1, Section V] Given a nominal plant $P \in \mathscr{R}^{p\times q}$, a stabilizable generalized plant*

$$H = \left[\begin{array}{c:c} S_z & \\ \hdashline & I \end{array}\right] \left[\begin{array}{cc:c} I & -P & P \\ 0 & 0 & I \\ \hdashline I & -P & P \end{array}\right] \left[\begin{array}{c:c} S_w & \\ \hdashline & I \end{array}\right]$$

where $S_w, S_z \in \mathscr{R}$, a perturbed plant $P_\Delta \in \mathscr{R}^{p\times q}$ and a controller $C \in \mathscr{R}^{q\times p}$ such that $d^H(P,P_\Delta) < b^H(P,C)$ and $\mathbf{\Delta}^{\min} \neq \emptyset$. Assume furthermore that there exists a $\Delta \in \mathbf{\Delta}^{\min}$ that satisfies $\eta(P_\Delta) = \eta(P) + \operatorname{wno}\det(I - H_{11}\Delta)$, where the winding number is evaluated on a contour indented to the right around any imaginary axis poles of P and P_Δ. Suppose furthermore that $S_{w_\Delta} = S_w(I - k\Delta S_z S_w)^{-1} \in \mathscr{R}$ for a given $k \in \{0,1\}$, $S = (1-k)S_z S_w$ and

$$H_\Delta = \left[\begin{array}{c:c} S_z & \\ \hdashline & I \end{array}\right] \left[\begin{array}{cc:c} I & -P_\Delta & P_\Delta \\ 0 & 0 & I \\ \hdashline I & -P_\Delta & P_\Delta \end{array}\right] \left[\begin{array}{c:c} S_{w_\Delta} & \\ \hdashline & I \end{array}\right].$$

Then the following results hold when $S \in \mathscr{RL}_\infty$ and $(I - \Delta S)^{-1} \in \mathscr{R}$:

(*a*) $0 \neq \mathscr{F}_l(H_\Delta, C) \in \mathscr{RL}_\infty$ *and* $[P_\Delta, C]$ *is internally stable*;
(*b*) $\left|b^{H_\Delta}(P_\Delta,C) - b^H(P,C)\right| \leq$
$\|\mathscr{F}_l(H_\Delta,C) - S\|_\infty b^{H_\Delta}(P_\Delta,C)\, d^H(P,P_\Delta)$; *and*
(*c*) $\|\mathscr{F}_l(H_\Delta,C) - \mathscr{F}_l(H,C)\|_\infty \leq$
$\dfrac{\|\mathscr{F}_l(H_\Delta,C) - S\|_\infty d^H(P,P_\Delta)}{b^H(P,C)}$.

Again, the proof for this theorem is omitted here, but can be found in [1, Section V]. The above theorem allows us to make several statements about the robust performance of the perturbed system: From result (b), it is clear that the change in robust stability margin between nominal and perturbed plant is bounded from above. Similarly, in result (c), the worst case discrepancy between the transfer functions involving P and P_Δ is bounded from above. Both bounds are proportional to $d^H(P,P_\Delta)$ and hence, intuitively, a small distance results in tight bounds on the performance degradation.

III. Output Inverse Multiplicative Uncertainty

In this section, the stability margin, distance measure, robust stability theorem and robust performance theorem are specialised for output inverse multiplicative uncertainty, i.e.

$$P_\Delta = (I - \Delta)^{-1} P.$$

This corresponds to choosing $S_w = \begin{bmatrix} I & 0 \end{bmatrix}^T$ and $S_z = \begin{bmatrix} I & 0 \end{bmatrix}$ in the four-block structure described in Theorem 2. It is common engineering practice to use this type of uncertainty for modelling low frequency parameter errors and uncertain right-half plane poles. When the uncertainty $\Delta \in \mathscr{RL}_\infty$, as is the case here, it can also be used to model uncertain right-half plane zeros, which would not be possible when it is restricted to $\mathscr{RH}_\infty$. The subsequent derivations follow a procedure suggested in [1, Section VI] for characterising the generic concepts for specific uncertainty structures.

1) Define the stability margin $b_{\text{oim}}(P,C)$: Straight from Definition 3, the stability margin $b_{\text{oim}}(P,C)$ for an output inverse multiplicative uncertainty characterisation is:

$$b_{\text{oim}}(P,C) := \begin{cases} \left\|(I-PC)^{-1}\right\|_\infty^{-1} & \text{if } [P,C] \text{ int. stable,} \\ 0 & \text{otherwise.} \end{cases} \quad (3)$$

2) Solve consistency equation for all $\Delta \in \mathscr{RL}_\infty$: In this specific case, eqn. (1) reduces to

$$P_\Delta = (I-\Delta)^{-1} P \;\Leftrightarrow\; P_\Delta - P = \Delta P_\Delta, \quad (4)$$

We shall assume in this section that $P_\Delta(\infty)$ has full rank, which is imposed for mathematical convenience.[2] In the following, the derivations are split into square, tall and fat plant cases, as each case requires a slightly different approach and yields slightly different consistency conditions.

Square Plants: Assume that $P,\ P_\Delta \in \mathscr{R}^{p\times q}$ with $p = q$, and that $P_\Delta(\infty)$ has full rank. Then eqn. (4) can be solved for Δ:

$$\Delta = (P_\Delta - P)P_\Delta^{-1}. \quad (5)$$

Hence, a necessary and sufficient condition for the existence of a $\Delta \in \mathscr{RL}_\infty$ that satisfies the consistency equation (4) is $PP_\Delta^{-1} \in \mathscr{RL}_\infty$. An obvious sufficient condition is $P,\ P_\Delta^{-1} \in \mathscr{RL}_\infty$. Given any $P,\ P_\Delta$ that satisfy $PP_\Delta^{-1} \in \mathscr{RL}_\infty$ there exists one unique solution for $\Delta \in \mathscr{RL}_\infty$ given by eqn. (5).

Tall Plants: Assume now that $P,\ P_\Delta \in \mathscr{R}^{p\times q}$ with $p > q$ and that $P_\Delta(\infty)$ has full rank. Let P_Δ have the state-space realisation $P_\Delta = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ with D having full column rank, and define the state-space system

$$\breve{P}_\Delta = \left[\begin{array}{c|c} A - BD^\dagger C & -BD^\dagger \\ \hline D_\perp^* C & D_\perp^* \end{array}\right] \in \mathscr{R}^{(p-q)\times p}, \quad (6)$$

where $D_\perp$ satisfies $\begin{bmatrix} D^\dagger \\ D_\perp^* \end{bmatrix} \begin{bmatrix} D & D_\perp \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$. Find a $U \in \mathscr{R}^{q\times q}$ satisfying $U^*U = P_\Delta^* P_\Delta$ and a $V \in \mathscr{R}^{(p-q)\times(p-q)}$ satisfying

$$VV^* = \breve{P}_\Delta \breve{P}_\Delta^{\,*}. \quad (7)$$

Note that since U (resp. V) is square and D^*D (resp. $D_\perp^* D_\perp$) is nonsingular, it follows that $U^*U = P_\Delta^* P_\Delta$ (resp. $VV^* = \breve{P}_\Delta \breve{P}_\Delta^{\,*}$) implicitly implies that $U^{-1} \in \mathscr{R}^{q\times q}$ (resp. $V^{-1} \in \mathscr{R}^{(p-q)\times(p-q)}$). Define

$$\Psi = \begin{bmatrix} U^{-*} P_\Delta^* \\ V^{-1} \breve{P}_\Delta \end{bmatrix} \in \mathscr{R}^{p\times p} \quad (8)$$

and note that $\Psi\Psi^* = I$ since $\breve{P}_\Delta^{\,*} P_\Delta = 0$. Since Ψ is also square, we have $\Psi^{-1} = \Psi^*$. Now eqn. (4) can be rearranged:

$$P_\Delta - P = \Delta P_\Delta \;\Leftrightarrow\; P_\Delta - P = \Delta \Psi^{-1} \begin{bmatrix} U \\ 0 \end{bmatrix}$$
$$\Leftrightarrow \quad \Delta = \begin{bmatrix} (P_\Delta - P)U^{-1} & Q \end{bmatrix} \Psi \text{ for any } Q \in \mathscr{R}^{p\times(p-q)}.$$

Consequently, for this specific case, since Ψ is a unit in $\mathscr{RL}_\infty$, a necessary and sufficient condition for there to

[2] If the perturbed plant P_Δ does not satisfy this assumption, one can always negligibly perturb P_Δ at infinite frequency so as to satisfy this assumption.

exist a $\Delta \in \mathscr{RL}_\infty$ that satisfies consistency of equations is $PU^{-1} \in \mathscr{RL}_\infty$, with a simple sufficient condition being P_Δ having no transmission zeros on $j(\mathbb{R}\cup\{\infty\})$ and $P \in \mathscr{RL}_\infty$. Then, given any P, P_Δ pair that satisfy $PU^{-1} \in \mathscr{RL}_\infty$, there always exist multiple solutions for $\Delta \in \mathscr{RL}_\infty$ given by

$$\Delta = \begin{bmatrix}(P_\Delta - P)U^{-1} & Q\end{bmatrix}\Psi \text{ for any } Q \in \mathscr{RL}_\infty^{p\times(p-q)}.$$

Fat Plants: Assume in this case that $P,\ P_\Delta \in \mathscr{R}^{p\times q}$ such that $p<q$ and that $P_\Delta(\infty)$ has full rank. Let P_Δ have the state-space realisation $P_\Delta = \left[\begin{array}{c|c} A & B \\ \hline C & D\end{array}\right]$ with D having full row rank, and define $\bar{P_\Delta} = \left[\begin{array}{c|c} A-BD^\dagger C & -BD_\perp^* \\ \hline D^\dagger C & D_\perp^*\end{array}\right] \in \mathscr{R}^{q\times(q-p)}$ where $D_\perp$ satisfies $\begin{bmatrix} D \\ D_\perp\end{bmatrix}\begin{bmatrix}D^\dagger & D_\perp^*\end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I\end{bmatrix}$. Find an $X \in \mathscr{R}^{p\times p}$ satisfying $XX^* = P_\Delta P_\Delta^*$ and a $Y \in \mathscr{R}^{(q-p)\times(q-p)}$ satisfying $Y^*Y = \bar{P_\Delta}^*\bar{P_\Delta}$. Note that since X (resp. Y) is square and D^*D (resp. $D_\perp^* D_\perp$) is nonsingular, it follows that $XX^* = P_\Delta P_\Delta^*$ (resp. $Y^*Y = \bar{P_\Delta}^*\bar{P_\Delta}$) implicitly implies that $X^{-1} \in \mathscr{R}^{p\times p}$ (resp. $Y^{-1} \in \mathscr{R}^{(q-p)\times(q-p)}$). Define $\Phi = \begin{bmatrix}P_\Delta^* X^{-*} & \bar{P_\Delta} Y^{-1}\end{bmatrix} \in \mathscr{R}^{q\times q}$ and note that $\Phi^*\Phi = I$ since $P_\Delta \bar{P_\Delta} = 0$. Since Φ is also square, we have $\Phi^{-1} = \Phi^*$. Now eqn. (4) can be rearranged:

$$\begin{aligned} P - P_\Delta = \Delta P_\Delta \ &\Leftrightarrow\ (P-P_\Delta)\Phi = \Delta P_\Delta \Phi \\ \Leftrightarrow\ \begin{bmatrix}(P-P_\Delta)P_\Delta^* X^{-*}X^{-1} & P\bar{P_\Delta}\end{bmatrix} &= \begin{bmatrix}\Delta & 0\end{bmatrix}. \end{aligned}$$

Consequently, necessary and sufficient conditions for there to exist a $\Delta \in \mathscr{RL}_\infty$ that satisfies consistency of equations are

$$PP_\Delta^*(P_\Delta P_\Delta^*)^{-1} \in \mathscr{RL}_\infty \text{ and } P\bar{P_\Delta} = 0, \tag{9}$$

with a simple sufficient condition being P_Δ having no transmission zeros on $j(\mathbb{R}\cup\{\infty\})$, $P \in \mathscr{RL}_\infty$ and $P\bar{P_\Delta} = 0$. Then, given any P, P_Δ pair that satisfy condition (9), there exists only a uniqe solution for $\Delta \in \mathscr{RL}_\infty$ given by

$$\Delta = (P-P_\Delta)P_\Delta^*(P_\Delta P_\Delta^*)^{-1}. \tag{10}$$

3) Derive conditions to guarantee well-posedness of $\mathscr{F}_u(H,\Delta)$: We now wish to make a connection between consistency equation (4) and the uncertainty characterisation $P_\Delta = \mathscr{F}_u(H,\Delta)$. Since we define $P_\Delta = \mathscr{F}_u(H,\Delta)$ to be well-posed when $\det(I - H_{11}\Delta)(\infty) \neq 0$, we first need to express $\det(I - H_{11}\Delta)(\infty)$ independently of Δ. Since $H_{11} = I$ (a very important difference from the multiplicative case), it follows (after some simple algebra) that:

$$\begin{aligned} &\det(I - H_{11}\Delta)(\infty) \neq 0 \\ \Leftrightarrow &\begin{cases} \det(P(\infty)) \neq 0 & \text{when } p = q, \\ \det\begin{bmatrix}P & (\check{P}_\Delta^* V^{-*} - Q)\end{bmatrix}(\infty) \neq 0 & \text{when } p > q, \\ \det(PP_\Delta^*)(\infty) \neq 0 & \text{when } p < q. \end{cases} \end{aligned} \tag{11}$$

In equivalence (11), $Q \in \mathscr{RL}_\infty^{p\times(p-q)}$ is arbitrary, $V \in \mathscr{R}^{(p-q)\times(p-q)}$ satisfies (7) and $\check{P}_\Delta$ is as defined in eqn. (6).

The inequalities in (11) restrict the allowable $P(\infty)$, $P_\Delta(\infty)$ data and $Q(\infty)$ for well-posedness of the linear fractional transformation $\mathscr{F}_u(H,\Delta)$. The following technical lemma is needed to simplify condition (11).

Lemma 3. 1) *When $p = q$, condition (11) is equivalent to $P(\infty)$ having full rank;*

2) *When $p > q$, $\exists Q \in \mathscr{RL}_\infty$ so that condition (11) is fulfilled if and only if $P(\infty)$ has full rank;*

3) *When $p < q$, condition (11) is equivalent to $P(\infty)$ having full rank under the supposition$P\bar{P}_\Delta = 0$.*

Proof:

1) Trivial.
2) Since $\check{P}_\Delta^* V^{-*} \in \mathscr{RL}_\infty$, $\exists Q \in \mathscr{RL}_\infty$ so that condition (11) is fulfilled if and only if $\exists \hat{Q} \in \mathscr{RL}_\infty$ so that $\det\begin{bmatrix}P & \hat{Q}\end{bmatrix}(\infty) \neq 0$ if and only if $P(\infty)$ has full rank.
3) Since $\operatorname{rank}(P(\infty)) = \operatorname{rank}\left(P(\infty)\begin{bmatrix}P_\Delta(\infty)^* & \bar{P}_\Delta(\infty)\end{bmatrix}\right) = \operatorname{rank}\left(\begin{bmatrix}(PP_\Delta^*)(\infty) & 0\end{bmatrix}\right)$, it easily follows that $P(\infty)$ has full rank if and only if $\det(PP_\Delta^*)(\infty) \neq 0$. ■

Under the restrictions imposed by (11),

$$P_\Delta - P = \Delta P_\Delta \qquad \Leftrightarrow \qquad P_\Delta = \mathscr{F}_u(H,\Delta),$$

as shown above in eqn. (1). Consequently, given a nominal plant P and a perturbed plant P_Δ, we have shown above that one of the following three conditions is a necessary and sufficient condition for there to exist a $\Delta \in \mathscr{RL}_\infty$ satisfying $P_\Delta = \mathscr{F}_u(H,\Delta)$:

- *Condition I* means $P, P_\Delta \in \mathscr{R}^{p\times q}$ with $p = q$ satisfying $P(\infty)$, $P_\Delta(\infty)$ having full rank and $PP_\Delta^{-1} \in \mathscr{RL}_\infty$;
- *Condition II* means $P, P_\Delta \in \mathscr{R}^{p\times q}$ with $p > q$ satisfying $P(\infty)$, $P_\Delta(\infty)$ having full rank and $PU^{-1} \in \mathscr{RL}_\infty$ (where $U \in \mathscr{R}^{q\times q}$ satisfies $U^*U = P_\Delta^* P_\Delta$);
- *Condition III* means $P, P_\Delta \in \mathscr{R}^{p\times q}$ with $p < q$ satisfying $P(\infty)$, $P_\Delta(\infty)$ having full rank, $PP_\Delta^*(P_\Delta P_\Delta^*)^{-1} \in \mathscr{RL}_\infty$ and $P\bar{P}_\Delta = 0$.

Also, for the equation $P_\Delta = \mathscr{F}_u(H,\Delta)$, when:

- *Condition I* is satisfied (square plant case), there only exists a unique solution $\Delta \in \mathscr{RL}_\infty$ given by eqn. (5).
- *Condition II* is satisfied (tall plant case), there always exist multiple solutions $\Delta \in \mathscr{RL}_\infty$ given by

$$\Delta = \begin{bmatrix}(P_\Delta - P)U^{-1} & Q\end{bmatrix}\Psi \tag{12}$$

  for any $Q \in \mathscr{RL}_\infty^{p\times(p-q)}$ that satisfies (11) (where Ψ is defined in eqn. (8)) ;
- *Condition III* is satisfied (fat plant case), there only exists a unique solution $\Delta \in \mathscr{RL}_\infty$ given by eqn. (10).

4) Define the solution set $\mathbf{\Delta}$ and distance measure $d_{\text{oim}}(P,P_\Delta)$: From Definition 1, the solution set $\mathbf{\Delta}$ is characterised as follows:

$$\mathbf{\Delta} = \begin{cases} \{(P_\Delta - P)P_\Delta^{-1}\} & \text{when } \textit{Cond. I} \text{ holds}, \\ \{\begin{bmatrix}(P_\Delta - P)U^{-1} \ Q\end{bmatrix}\Psi : \\ Q \in \mathscr{RL}_\infty^{p\times(p-q)}, \\ \det\begin{bmatrix}P \ (\check{P}_\Delta^* V^{-*} - Q)\end{bmatrix}(\infty) \neq 0\} & \text{when } \textit{Cond. II} \text{ holds}, \\ \{(P_\Delta - P)P_\Delta^*(P_\Delta P_\Delta^*)^{-1}\} & \text{when } \textit{Cond. III} \text{ holds}, \\ \emptyset & \text{otherwise} \end{cases}$$

in this specific case. The definition of the distance measure $d_{\text{oim}}(P,P_\Delta)$ when Condition I or Condition III holds is trivial since there is only one element in the set $\mathbf{\Delta}$. When Condition II holds, and since Ψ is allpass, $\inf_{\Delta\in\mathbf{\Delta}}\|\Delta\|_\infty = \left\|(P_\Delta - P)U^{-1}\right\|_\infty$. The following technical lemma states under what conditions can $Q(\infty)=0$ be chosen.

Lemma 4. *The choice $Q(\infty)=0$ in the solution $\Delta = \begin{bmatrix}(P_\Delta - P)U^{-1} & Q\end{bmatrix}\Psi$ for the $p>q$ case gives a $\Delta(\infty)$ that satisfies $\det(I-H_{11}\Delta)(\infty)\neq 0$ if and only if $\det(P_\Delta^* P)(\infty)\neq 0$.*

Proof: Choose $Q(\infty)=0$ in (11) for $p>q$. Then

$$\begin{aligned}\det(I-H_{11}\Delta)(\infty)\neq 0 &\Leftrightarrow \det\begin{bmatrix}P & \check{P}_\Delta^*\end{bmatrix}(\infty)\neq 0\\ &\Leftrightarrow \det\left(\begin{bmatrix}P_\Delta^*\\ \check{P}_\Delta\end{bmatrix}\begin{bmatrix}P & \check{P}_\Delta^*\end{bmatrix}\right)(\infty)\neq 0\\ &\Leftrightarrow \det(P_\Delta^* P)(\infty)\neq 0 \text{ since } \check{P}_\Delta P_\Delta = 0.\end{aligned}$$

■

Then, it follows from Definition 1 that—for a $U\in\mathscr{R}^{q\times q}$ that satisfies $U^*U = P_\Delta^* P_\Delta$—the distance measure $d_{\text{oim}}(P,P_\Delta)$ for output inverse multiplicative uncertainty characterisations is given by:

$$d_{\text{oim}}(P,P_\Delta) := \begin{cases}\left\|(P_\Delta - P)U^{-1}\right\|_\infty & \text{when } \textit{Cond. I/II} \text{ holds,}\\ \left\|(P_\Delta - P)P_\Delta^*(P_\Delta P_\Delta^*)^{-1}\right\|_\infty & \text{when } \textit{Cond. III} \text{ holds,}\\ \infty & \text{otherwise.}\end{cases} \quad (13)$$

5) Write the winding number condition independent of Δ: The problem needs to be split again into three cases: square, tall and fat plants.

Square Plants: When Condition I is satisfied, using $\Delta\in\mathbf{\Delta}^{\min}$ given by equation (5) in winding number condition eqn. (2) gives

$$\eta(P_\Delta)-\eta(P) = \text{wno}\det(PP_\Delta^{-1}) = \text{wno}\det(PP_\Delta^*). \quad (14)$$

Note that equation (14) can be simplified to $z(P_\Delta)=z(P)$, but for consistency with the tall/fat plant cases we choose not to use this simpler formulation.

Tall Plants: Before tackling this case, note that an immediate corollary to Lemma 4 is as follows:

Corollary 5. *Choosing $Q=0$ in equation (12) for the $p>q$ case gives a $\Delta\in\mathbf{\Delta}^{\min}$ if and only if $\det(P_\Delta^* P)(\infty)\neq 0$.*

Consequently, when Condition II and $\det(P_\Delta^* P)(\infty)\neq 0$ are satisfied, using $\Delta\in\mathbf{\Delta}^{\min}$ given by eqn. (12) with $Q=0$ in winding number condition eqn. (2) gives

$$\begin{aligned}\eta(P_\Delta)-\eta(P) &= \text{wno}\det(I-(P_\Delta - P)(P_\Delta^* P_\Delta)^{-1}P_\Delta^*)\\ &= \text{wno}\det(P_\Delta^* P).\end{aligned}$$

Fat Plants: When Condition III is satisfied, using $\Delta\in\mathbf{\Delta}^{\min}$ given by eqn. (10) in winding number condition eqn. (2) gives

$$\begin{aligned}\eta(P_\Delta)-\eta(P) &= \text{wno}\det(PP_\Delta^*(P_\Delta P_\Delta^*)^{-1})\\ &= \text{wno}\det(PP_\Delta^*).\end{aligned}$$

6) State robust stability and robust performance theorems:

Theorem 6 (Robust Stability — Output Inverse Multiplicative)**.** *Given a plant $P\in\mathscr{R}^{p\times q}$, a perturbed plant $P_\Delta\in\mathscr{R}^{p\times q}$ and a controller $C\in\mathscr{R}^{q\times p}$. Define a stability margin $b_{\text{oim}}(P,C)$ as in (3), a distance measure $d_{\text{oim}}(P,P_\Delta)$ as in (13), and an object $\Xi = \begin{cases}PP_\Delta^* & \text{when } p\le q\\ P_\Delta^* P & \text{otherwise}\end{cases}$.*

Furthermore, suppose $d_{\text{oim}}(P,P_\Delta) < b_{\text{oim}}(P,C)$ and when $p>q$, suppose also $\det(P_\Delta^* P)(\infty)\neq 0$. Then

$$[P_\Delta, C] \textit{ is internally stable} \Leftrightarrow \text{wno}\det(\Xi) = \eta(P_\Delta)-\eta(P),$$

where the winding number is evaluated on a contour indented to the right around any imaginary axis poles of P and P_Δ.

Proof: This theorem specialises Theorem 1 using formulae derived in the above subsection. The supposition $d_{\text{oim}}(P,P_\Delta) < b_{\text{oim}}(P,C)$ implies that either Condition I or II or III must hold since $d_{\text{oim}}(P,P_\Delta) < b_{\text{oim}}(P,C)\le\infty$. Note also that the supposition that "H is stabilizable" is automatically fulfilled in this specific design case. ■

Theorem 7 (Robust Performance — Output Inverse Multiplicative)**.** *Given the suppositions of Theorem 6 and furthermore assuming $\text{wno}\det(\Xi) = \eta(P_\Delta)-\eta(P)$, where the winding number is evaluated on a contour indented to the right around any imaginary axis poles of P and P_Δ. Then*

$$\left|1-\frac{b_{\text{oim}}(P,C)}{b_{\text{oim}}(P_\Delta,C)}\right| \le \|P_\Delta(I-CP_\Delta)^{-1}C\|_\infty\, d_{\text{oim}}(P,P_\Delta) \quad (15)$$

and

$$\frac{\|\mathscr{F}_l(H_\Delta,C)-\mathscr{F}_l(H,C)\|_\infty}{\|\mathscr{F}_l(H,C)\|_\infty} \le \|P_\Delta(I-CP_\Delta)^{-1}C\|_\infty\, d_{\text{oim}}(P,P_\Delta), \quad (16)$$

where $H = \left[\begin{array}{c|c} I & P\\ \hline I & P\end{array}\right]$ and $H_\Delta = \left[\begin{array}{c|c} I & P_\Delta\\ \hline I & P_\Delta\end{array}\right]$.

Proof: This theorem specialises Theorem 2 using formulae derived in the above subsection. The result follows on choosing $k=0$ and noting that $S=S_zS_w=I\in\mathscr{RL}_\infty$ thereby giving $\|\mathscr{F}_l(H_\Delta,C)-S\|_\infty = \|P_\Delta(I-CP_\Delta)^{-1}C\|_\infty$. ■

Note that the object $\left\|P_\Delta(I-CP_\Delta)^{-1}C\right\|_\infty$ corrupts the distance measure $d_{\text{oim}}(P,P_\Delta)$ on the right side of inequalities (15) and (16). For systems with large gain at low frequencies, good stability margin and a large roll-off frequency, this quantity is very close to unity in the pass-band, very small in the stop-band and not too big around crossover. Hence, it is a factor that assists in tightening the inequalities in the stopband. The discrepancy between nominal and perturbed stability margin and closed-loop transfer function given in inequalities (15) and (16) appear naturally in multiplicative form.

IV. Input Multiplicative Uncertainty

This section briefly states the main results for robust stability and performance for the input multiplicative case. In this input multiplicative case,

$$P_\Delta = P(I-\Delta).$$

The stability margin $b_{\rm im}(P,C)$ for an input multiplicative uncertainty characterisation can be computed directly from Definition 3 upon noting that inverse multiplicative uncertainty is captured by the four-block structure on choosing $S_w = \begin{bmatrix} 0 & I \end{bmatrix}^T$, $S_z = \begin{bmatrix} 0 & I \end{bmatrix}$, and hence:

$$b_{\rm im}(P,C) := \begin{cases} \left\|C(I-PC)^{-1}P\right\|_\infty^{-1} & \text{if } [P,C] \text{ int. stable,} \\ 0 & \text{otherwise.} \end{cases} \tag{17}$$

The consistency equation for this case,

$$P_\Delta = P(I-\Delta) \;\Leftrightarrow\; P_\Delta - P = -P\Delta, \tag{18}$$

will hold under different well-posedness conditions in the case of square, fat or tall plants. For the tall plants case, we require an auxiliary matrix $\check{P}$. Let P have the state-space realisation $P = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ with D having full column rank, and define

$$\check{P} = \left[\begin{array}{c|c} A - BD^\dagger C & -BD^\dagger \\ \hline D_\perp^* C & D_\perp^* \end{array}\right] \in \mathscr{R}^{(p-q)\times p} \tag{19}$$

where $D_\perp$ satisfies $\begin{bmatrix} D^\dagger \\ D_\perp^* \end{bmatrix} \begin{bmatrix} D & D_\perp \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$. For the square, fat and tall plants case, the consistency equation (18) is fulfilled under Condition I, II or III, respectively, where

- *Condition I* means $P, P_\Delta \in \mathscr{R}^{p\times q}$ with $p = q$ satisfying $P(\infty)$ having full rank and $P^{-1}P_\Delta \in \mathscr{RL}_\infty$;
- *Condition II* means $P, P_\Delta \in \mathscr{R}^{p\times q}$ with $p < q$ satisfying $P(\infty)$ having full rank and $X^{-1}P_\Delta \in \mathscr{RL}_\infty$ (where $X \in \mathscr{R}^{p\times p}$ satisfies $XX^* = PP^*$);
- *Condition III* means $P, P_\Delta \in \mathscr{R}^{p\times q}$ with $p > q$ satisfying $P(\infty)$ having full rank, $(P^*P)^{-1}P^*P_\Delta \in \mathscr{RL}_\infty$ and $\check{P}P_\Delta = 0$ (where $\check{P}$ is defined in equation (19)).

We can now define the distance measure for the input multiplicative case. Let $X \in \mathscr{R}$ such that $XX^* = PP^*$. Then

$$d_{\rm im}(P,P_\Delta) := \tag{20}$$
$$\begin{cases} \left\|X^{-1}(P-P_\Delta)\right\|_\infty & \text{when } \textit{Condition I/II} \text{ holds,} \\ \left\|(P^*P)^{-1}P^*(P-P_\Delta)\right\|_\infty & \text{when } \textit{Condition III} \text{ holds,} \\ \infty & \text{otherwise.} \end{cases}$$

With all the technical machinery defined, the robust stability and robust performance theorems for input multiplicative uncertainty can be stated (proofs will be given elsewhere).

Theorem 8 (Robust Stability — Input Multiplicative)**.** *Given a plant $P \in \mathscr{R}^{p\times q}$, a perturbed plant $P_\Delta \in \mathscr{R}^{p\times q}$ and a controller $C \in \mathscr{R}^{q\times p}$. Define a stability margin $b_{\rm im}(P,C)$ as in* (17) *and a distance measure $d_{\rm im}(P,P_\Delta)$ as in* (20)*. Furthermore, suppose $d_{\rm im}(P,P_\Delta) < b_{\rm im}(P,C)$. Then*

$$[P_\Delta, C] \textit{ is internally stable} \Leftrightarrow \eta(P_\Delta) = \eta(P).$$

Theorem 9 (Robust Performance — Input Multiplicative)**.** *Given the suppositions of Theorem 8 and furthermore assuming $\eta(P_\Delta) = \eta(P)$. Then*

$$\left|1 - \frac{b_{\rm im}(P,C)}{b_{\rm im}(P_\Delta,C)}\right| \le \|(I-CP_\Delta)^{-1}\|_\infty d_{\rm im}(P,P_\Delta)$$

and

$$\frac{\|\mathscr{F}_l(H_\Delta,C) - \mathscr{F}_l(H,C)\|_\infty}{\|\mathscr{F}_l(H,C)\|_\infty} \le \|(I-CP_\Delta)^{-1}\|_\infty d_{\rm im}(P,P_\Delta),$$

where $H = \left[\begin{array}{c:c} 0 & I \\ \hdashline -P & P \end{array}\right]$ *and* $H_\Delta = \left[\begin{array}{c:c} 0 & I \\ \hdashline -P_\Delta & P_\Delta \end{array}\right]$.

V. Conclusions

Specific distance measures, robust stability margins, and the associated robust stability and robust performance theorems for systems with output inverse multiplicative uncertainty have been derived, and the corresponding concepts for input multiplicative uncertainty were briefly summarised. Due to the enlarged set of allowable uncertainty ($\mathscr{RL}_\infty$ rather than $\mathscr{RH}_\infty$ as in previous results), these readily applicable theorems allow a design engineer great flexibility in modelling system uncertainty of a multiplicative nature. The results validate practical approaches for distance measures for multiplicative uncertainties, and illustrate the generic distance measure theory through an intuitive uncertainty setting.

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