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Reformulating negative imaginary frequency response systems to bounded-real systems

Alexander Lanzon Zhuoyue Song Ian R. Petersen

Abstract— **This paper provides a reformulation of closed-loop systems that have negative imaginary frequency response to closed-loop systems that have bounded gain, so that theory and results from $\mathcal{H}_\infty$ control can be borrowed to enable controller synthesis for the former class of systems. Systems with negative imaginary frequency response arise for example in structures with co-located position sensors and force actuators, and finding a systematic controller synthesis treatment for such systems has important applications in, for example, lightly damped large space structure problems. The key result in this paper assists by reformulating such systems into a bounded-real framework. An example demonstrates the feasibility of the reformulation given herein.**

Index Terms— **Positive real, bounded real, negative imaginary frequency response, lightly damped systems, $\mathcal{H}_\infty$ control, small-gain theorem, passivity.**

NOTATION AND SYMBOLS

Let $\mathcal{RH}_\infty$ denote the set of real-rational stable transfer function matrices and $\mathcal{RH}_\infty^{n\times m}$ denote the systems in $\mathcal{RH}_\infty$ that have m columns and n rows. Let $\mathbb{R}$ and $\mathbb{C}$ denote fields of real and complex numbers respectively, $\mathbb{C}_-$ and $\bar{\mathbb{C}}_-$ denote the open and closed left-half planes respectively. Let A^* denote the complex conjugate transpose of matrix A. Let $\bar{\sigma}(A)$ denote the largest singular value of matrix A. Let $\lambda_i(A)$ denote the i-th eigenvalue of a square complex matrix A. Let $\|P\|_\infty$ denote the $\mathcal{H}_\infty$-norm of $P \in \mathcal{RH}_\infty$. Let $F_\ell(G,K)$ denote the lower Linear Fractional Transformation (LFT) of transfer function matrices G and K [1]. Let $G \star K$ denote the Redheffer Star-Product of transfer function matrices G and K with respect to some appropriate partitioning [1]. Let $\mathrm{diag}(A,B)$ be shorthand for $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$. Let $\langle G, K\rangle$ denote the feedback interconnection shown in Fig.1 and correspondingly let $T(G,K)$ denote the transfer function from $\left(\begin{array}{c} \omega \\ \left(\begin{array}{c} u_1 \\ u_2 \end{array}\right) \end{array}\right)$ to $\left(\begin{array}{c} z \\ \left(\begin{array}{c} v_1 \\ v_2 \end{array}\right) \end{array}\right)$. We say $\langle G, K\rangle$ is internally stable when $T(G,K) \in \mathcal{RH}_\infty$.

Corresponding author is Zhuoyue Song. Tel: +44-161-306-2821, Fax: +44-161-306-4647. Email: Zhuoyue.Song@postgrad.manchester.ac.uk

Alexander Lanzon and Zhuoyue Song are at the Control Systems Centre, School of Electrical and Electronic Engineering, University of Manchester, Manchester M60 1QD, UK. Alexander Lanzon's Email is: a.lanzon@ieee.org

Ian R. Petersen is with the School of Information Technology and Electrical Engineering, University of New South Wales at the Australian Defence Force Academy, Canberra, ACT 2006 Australia. Email: i.r.petersen@gmail.com

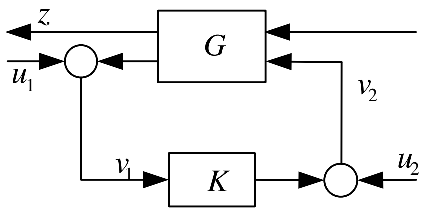

Fig. 1. LFT Interconnection

I. INTRODUCTION

It is a well-known fact that mechanical systems modelled with force actuators and co-located velocity sensors give rise to positive-real (or passive) transfer functions [2]. It is, however, a less known fact that the same systems modelled with force actuators and co-located position sensors, as opposed to velocity sensors, (i.e. when the variable to be controlled is position instead of velocity) give rise to systems with negative imaginary frequency response [3] in the LTI case (or systems with counter-clockwise input-output dynamics [4] in a nonlinear time-varying setting). This is a consequence of Newton's second law of motion that typically produces a transfer function of the form $\frac{K}{s^2+2\zeta\omega_n s+\omega_n^2}$ when co-located position measurements are used in conjunction with force actuation. Broadly speaking, systems with negative imaginary frequency response are stable systems whose Nyquist plot has a phase lag between 0 and $-\pi$ for all frequencies, and hence have a Nyquist plot below the Real axis for all frequencies. Studying these systems has important engineering applications in, for example, large space structures and multi-link robots. When these structures are lightly damped (i.e. flexible) with co-located position sensors and force actuators, they give rise to transfer functions which consist of the sum of a large number of second order terms with negative imaginary frequency response. Normally, only a small number of modes are considered for control design so that the truncated modes will give rise to spill-over dynamics that can destabilize the systems, especially at high frequencies. The SISO LTI case of such systems have been studied extensively in positive position feedback literature, where much of the analysis and synthesis is based on graphical Nyquist approaches [8] [9]. Recently, non-convex optimization methods to damp structural vibrations of lightly damped structures with co-located actuators and sensors have also been proposed [10]. In [3] [11], an *analysis* result was proposed for MIMO LTI systems with negative imaginary frequency response. It shows that a positive feed-

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back interconnection of two systems, each of which having a negative imaginary frequency response, is internally stable if and only if the DC loop gain is less than unity. Systems with negative frequency response are also closely connected to nonlinear/time-varying systems with counter-clockwise input-output dynamics [4].

This paper will take a first step towards controller synthesis for systems with negative imaginary frequency response based on the analysis result in [3] [11]. More specifically, for uncertain systems where the perturbation belongs to the class of systems with negative imaginary frequency response, it is natural to seek to design a stabilizing controller such that the closed-loop LFT satisfies negative imaginary frequency response properties. Via results in [3], it is then possible to quantity the largest family of perturbations that have negative imaginary frequency response properties in terms of the reciprocal of the DC gain of the nominal system. This paper will reformulate the problem of finding such a controller to an equivalent problem of finding an internally stabilizing controller for a transformed system such that the closed-loop is bounded-real.

II. Preliminaries

First we define some sets for compactness of notation.

Definition 1: Let the set of square stable transfer functions with negative imaginary frequency response [3] be defined by

$$\mathcal{I} := \{X \in \mathcal{RH}_\infty^{n\times n} : \\ j[X(j\omega) - X(j\omega)^*] \geq 0 \ \forall \omega \in (0,\infty)\}. \quad (1)$$

Definition 2: Let the set of square stable positive real transfer functions be defined by

$$\mathcal{P} := \{X \in \mathcal{RH}_\infty^{n\times n} : \\ [X(j\omega) + X(j\omega)^*] \geq 0 \ \forall \omega \in \mathbb{R}\}. \quad (2)$$

It is easy to see that $X \in \mathcal{I}$ implies $s[X(s) - X(\infty)] \in \mathcal{P}$ [3].

Definition 3: Let the set of square stable contractive transfer functions whose Nyquist plot does not pass through $-1 + j0$ point be defined by

$$\mathcal{B} := \{X \in \mathcal{RH}_\infty^{n\times n} : \|X\|_\infty \leq 1, \\ \det(I + X(j\omega)) \neq 0 \ \forall \omega \in \mathbb{R} \cup \{\infty\}\}. \quad (3)$$

The next three simple technical lemmas are given here to streamline presentation of the proof of the main result in the next section. The first two lemmas state that if $X \in \mathcal{P}$ or $X \in \mathcal{B}$, then $(I + X)^{-1}$ is stable. These standard properties will be used to make a connection amongst different parts of the proof of the main result.

Lemma 1: Given $X \in \mathcal{P}$, then $(I + X)^{-1} \in \mathcal{RH}_\infty$.

Proof: This lemma is trivially established via simple application of the passivity theorem [12]. ■

Lemma 2: Given $Y \in \mathcal{B}$. Then $(I + Y)^{-1} \in \mathcal{RH}_\infty$.

Proof: First note that $(I+\alpha Y)^{-1} \in \mathcal{RH}_\infty \ \forall \alpha \in (0, 1)$ and also $Y \in \mathcal{B}$ gives $\det(I + Y(j\omega)) \neq 0 \ \forall \omega \in \mathbb{R} \cup \{\infty\}$. As α increases continuously to unity, the transmission zeros of $(I + \alpha Y(s))$ vary continuously and are in $\mathbb{C}_-$ for all $\alpha \in (0, 1)$, and they do not intersect the $j\omega$-axis at $\alpha = 1$. Therefore, at $\alpha = 1$, they must remain in $\mathbb{C}_-$. Thus $(I + Y)^{-1} \in \mathcal{RH}_\infty$. ■

The third technical lemma gives a simple necessary and sufficient condition for input-output stability of a particular Redheffer Star-Product. It will be used in the next section to make a connection between systems in $\mathcal{P}$ and $\mathcal{B}$.

Lemma 3: Given $T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \in \mathcal{RH}_\infty$. Then,

$$\begin{pmatrix} I & -2I \\ I & -I \end{pmatrix} \star T \in \mathcal{RH}_\infty \Leftrightarrow \left(I + T_{11}\right)^{-1} \in \mathcal{RH}_\infty. \quad (4)$$

Proof: This equivalence can be seen directly from an expansion [1, Section 10.4] of the Redheffer Star-Product of $\begin{pmatrix} I & -2I \\ I & -I \end{pmatrix} \star T$. ■

The left side of equivalence in (4) is essentially a bilinear transformation on T which is similar to the transformation used to map systems in $\mathcal{P}$ to systems in $\mathcal{B}$ [5] [1]. Note that instead of these transformations, there are also direct solutions of systems in $\mathcal{P}$ [6] [7].

III. The Main Result

The main result of this paper is given in Theorem 4 below. The theorem broadly states that a controller internally stabilizes a generalized plant Σ and makes the input-output map satisfy a negative imaginary frequency response property if and only if the same controller internally stabilizes a different generalized plant G (constructed from Σ) and makes the input-output map contractive.

Theorem 4: Given a controller K and a generalized plant

$$\Sigma = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & 0 \\ C_2 & D_{21} & 0 \end{array}\right] \quad (5)$$

with (A, B_2) stabilizable, (C_2, A) detectable, and $D_{11} = D_{11}^*$. Then, $\langle\Sigma, K\rangle$ is internally stable and $j\left[F_\ell(\Sigma, K)(j\omega) - F_\ell(\Sigma, K)(j\omega)^*\right] \geq 0 \ \forall \omega \in (0,\infty)$ if and only if $\langle G, K\rangle$ is internally stable, $\left\|F_\ell(G, K)\right\|_\infty \leq 1$ and $\det(I + F_\ell(G, K)(j\omega)) \neq 0 \ \forall \omega \in \mathbb{R} \cup \{\infty\}$, where

$$G = \left[\begin{array}{c|cc} V^{-1}A & B_1U^{-1} & V^{-1}B_2 \\ \hline -2U^{-1}C_1A & (I - C_1B_1)U^{-1} & -2U^{-1}C_1B_2 \\ C_2 - D_{21}U^{-1}C_1A & D_{21}U^{-1} & -D_{21}U^{-1}C_1B_2 \end{array}\right],$$
$$U = I + C_1B_1 \ \ and \ \ V = I + B_1C_1. \quad (6)$$

Proof: We will prove the result via a sequence of equivalent reformulations:

(a) $\langle\Sigma, K\rangle$ is internally stable and $j\left[F_\ell(\Sigma, K)(j\omega) - F_\ell(\Sigma, K)(j\omega)^*\right] \geq 0 \ \forall \omega \in (0,\infty)$.

(b) $\langle\bar\Sigma, K\rangle$ is internally stable and $j\left[F_\ell(\bar\Sigma, K)(j\omega) - F_\ell(\bar\Sigma, K)(j\omega)^*\right] \geq 0 \ \forall \omega \in (0,\infty)$, where

$$\bar\Sigma = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & 0 \\ C_2 & D_{21} & 0 \end{array}\right]. \quad (7)$$

[The equivalence (a)⇔(b) follows on noting that $F_\ell(\Sigma, K)(\infty) = D_{11} = D_{11}^*$.]

(c) $\langle\hat{\Sigma}, K\rangle$ is internally stable and $\big[F_\ell(\hat{\Sigma}, K)(j\omega) + F_\ell(\hat{\Sigma}, K)(j\omega)^*\big] \geq 0\ \forall\omega \in \mathbb{R}$, where

$$\hat{\Sigma} = \begin{pmatrix} sI & 0 \\ 0 & I \end{pmatrix} \bar{\Sigma} = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 A & C_1 B_1 & C_1 B_2 \\ C_2 & D_{21} & 0 \end{array}\right]. \quad (8)$$

[The internal stability parts can be seen to be equivalent on noting that K is the same, $\hat{\Sigma}$ and $\bar{\Sigma}$ are both stabilizable and detectable, and that

$$\bar{\Sigma}_{22} = \hat{\Sigma}_{22} = \left[\begin{array}{c|c} A & B_2 \\ \hline C_2 & 0 \end{array}\right], \quad (9)$$

thus allowing use of [13, Lemma A.4.1]. The frequency domain inequalities are also equivalent since $F_\ell(\hat{\Sigma}, K)(j\omega) = j\omega \cdot F_\ell(\bar{\Sigma}, K)(j\omega)$.]

(d) $\langle G, K\rangle$ is internally stable, $\bar{\sigma}\big[F_\ell(G, K)(j\omega)\big] \leq 1\ \forall\omega \in \mathbb{R}$, and $\det(I + F_\ell(G, K)(j\omega)) \neq 0\ \forall\omega \in \mathbb{R} \cup \{\infty\}$, where

$$\begin{aligned} G &= \left[\begin{pmatrix} I & -2I \\ I & -I \end{pmatrix} \star \hat{\Sigma}\right] \\ &= \left[\begin{array}{c|cc} V^{-1}A & B_1 U^{-1} & V^{-1}B_2 \\ \hline -2U^{-1}C_1 A & (I - C_1 B_1)U^{-1} & -2U^{-1}C_1 B_2 \\ C_2 - D_{21}U^{-1}C_1 A & D_{21}U^{-1} & -D_{21}U^{-1}C_1 B_2 \end{array}\right], \\ U &= I + C_1 B_1 \quad \textit{and} \quad V = I + B_1 C_1. \end{aligned}$$

[(c) $\Rightarrow$ (d): Since $F_\ell(\hat{\Sigma}, K) \in \mathcal{P}$, it follows that $(I + F_\ell(\hat{\Sigma}, K))^{-1} \in \mathcal{RH}_\infty$ via Lemma 1. Then define $Y = (I - F_\ell(\hat{\Sigma}, K))(I + F_\ell(\hat{\Sigma}, K))^{-1}$ and note that $Y = F_\ell(G, K) \in \mathcal{RH}_\infty$. Also, since $(I+Y)^{-1} = \frac{1}{2}(I + F_\ell(\hat{\Sigma}, K)) \in \mathcal{RH}_\infty$, it follows that $\det(I + Y(j\omega)) \neq 0\ \forall\omega \in \mathbb{R} \cup \{\infty\}$. Also, $[F_\ell(\hat{\Sigma}, K)(j\omega) + F_\ell(\hat{\Sigma}, K)(j\omega)^*] \geq 0\ \forall\omega \in \mathbb{R}$ implies $\bar{\sigma}\,[Y(j\omega)] \leq 1\ \forall\omega \in \mathbb{R}$. Finally, since $\langle\hat{\Sigma}, K\rangle$ is internally stable, we have $T(\hat{\Sigma}, K) \in \mathcal{RH}_\infty$. Noting that $T_{11}(\hat{\Sigma}, K) = F_\ell(\hat{\Sigma}, K) \in \mathcal{P}$, we get $(I + T_{11}(\hat{\Sigma}, K))^{-1} \in \mathcal{RH}_\infty$ via Lemma 1 and this in turn gives

$$\begin{pmatrix} I & -2I \\ I & -I \end{pmatrix} \star T(\hat{\Sigma}, K) \in \mathcal{RH}_\infty \quad (10)$$

via Lemma 3, which implies $\langle G, K\rangle$ is internally stable as

$$\begin{aligned} T(G, K) &= T\left(\begin{bmatrix} I & -2I \\ I & -I \end{bmatrix} \star \hat{\Sigma}, K\right) \\ &= \begin{pmatrix} I & -2I \\ I & -I \end{pmatrix} \star T(\hat{\Sigma}, K). \quad (11) \end{aligned}$$

(d) $\Rightarrow$ (c): Since $F_\ell(G, K) \in \mathcal{B}$, $(I + F_\ell(G, K))^{-1} \in \mathcal{RH}_\infty$ via Lemma 2. Then define $X = (I - F_\ell(G, K))\,(I + F_\ell(G, K))^{-1} \in \mathcal{RH}_\infty$ and note that $X = F_\ell(\hat{\Sigma}, K)$. Then $\bar{\sigma}\,[F_\ell(G, K)(j\omega)] \leq 1\ \forall\omega \in \mathbb{R}$ implies $[X(j\omega) + X(j\omega)^*] \geq 0\ \forall\omega \in \mathbb{R}$. Finally, since $\langle G, K\rangle$ is internally stable, we have $T(G, K) \in \mathcal{RH}_\infty$. Noting that $T_{11}(G, K) = F_\ell(G, K) \in \mathcal{B}$, we get $(I + T_{11}(G, K))^{-1} \in \mathcal{RH}_\infty$ via Lemma 2 and this in turn gives

$$\begin{pmatrix} I & -2I \\ I & -I \end{pmatrix} \star T(G, K) \in \mathcal{RH}_\infty \quad (12)$$

via Lemma 3, which implies $\langle\hat{\Sigma}, K\rangle$ is internally stable as

$$\begin{aligned} &\begin{pmatrix} 0 & 2I \\ \frac{1}{2}I & 0 \end{pmatrix} \star T(\hat{\Sigma}, K) \quad (13) \\ &= \begin{pmatrix} I & -2I \\ I & -I \end{pmatrix} \star T(G, K) \text{ via (11).]} \end{aligned}$$

■

This theorem states that the original problem of synthesising an internally stabilizing controller such that a closed-loop LFT has negative imaginary frequency response can be transformed to an equivalent bounded-real problem. This is a first step towards a controller synthesis method for closed-loop systems with negative imaginary frequency response, allowing results to be borrowed from $\mathcal{H}_\infty$ control theory. The following lemma shows that the restriction of $D_{12} = 0$ and $D_{22} = 0$ in the realization of the generalized plant for Σ in (5) can be easily circumvented.

Lemma 5: Given a strictly proper controller K and a generalized plant

$$\Sigma = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right] \quad (14)$$

with (A, B_2) stabilizable, (C_2, A) detectable, and $D_{11} = D_{11}^*$. Then, $\langle\Sigma, K\rangle$ is internally stable and $j\big[F_\ell(\Sigma, K)(j\omega) - F_\ell(\Sigma, K)(j\omega)^*\big] \geq 0\ \forall\omega \in (0, \infty)$ if and only if $\langle\breve{\Sigma}, \breve{K}\rangle$ is internally stable and $j\big[F_\ell(\breve{\Sigma}, \breve{K})(j\omega) - F_\ell(\breve{\Sigma}, \breve{K})(j\omega)^*\big] \geq 0\ \forall\omega \in (0, \infty)$, where

$$\begin{aligned} \breve{\Sigma} &= \left[\begin{array}{cc|cc} A & B_2 & B_1 & 0 \\ 0 & -\tau I & 0 & \tau I \\ \hline C_1 & D_{12} & D_{11} & 0 \\ C_2 & D_{22} & D_{21} & 0 \end{array}\right], \quad (15) \\ \breve{K}(s) &= \left(\frac{s}{\tau} + 1\right) K(s), \text{ and any arbitrary } \tau > 0. \end{aligned}$$

Proof: Easily follows on noting that

$$\begin{aligned} &T(\Sigma, K) \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & \frac{1}{s/\tau+1}I \end{pmatrix} \\ &= \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & \frac{1}{s/\tau+1}I \end{pmatrix} T(\breve{\Sigma}, \breve{K}) \quad (16) \end{aligned}$$

because

$$\breve{\Sigma} = \Sigma \begin{pmatrix} I & 0 \\ 0 & \frac{1}{s/\tau+1}I \end{pmatrix}.$$

■

Two remarks are appropriate on Theorem 4 at this stage.

Remark 1: Under the suppositions (A, B_2) stabilizable and (C_2, A) detectable, the state-space realization for G given in Theorem 4 is stabilisable and detectable.

Proof: This can be easily shown via a PBH test [1]. ■

Remark 2: Whenever $\det(A) \neq 0$, we have $G_{12}(0) = 0$, and hence the Riccati method for $\mathcal{H}_\infty$ controller synthesis [1] cannot be used.

Proof: This is trivial on noting that

$$G_{12}(s) = -2U^{-1}C_1 B_2 - 2U^{-1}C_1 A(sI - V^{-1}A)^{-1}V^{-1}B_2$$

■

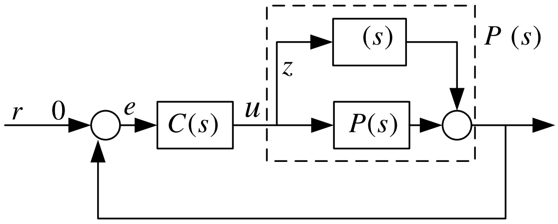

Fig. 2. Controlled closed-loop system

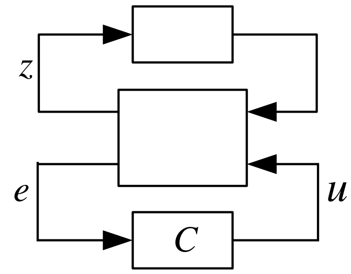

Fig. 3. Rearranged LFT interconnection

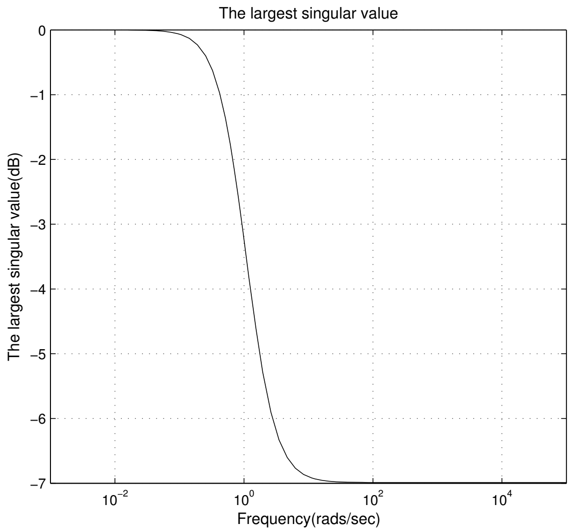

Fig. 4. The largest singular value plot of $F_\ell(G,\check{C})(j\omega)$

Remark 2 states that whenever A is nonsingular, there is a blocking zero at zero frequency for G_{12}, and thus the standard assumption of "no invariant zeros on imaginary axis" for the existence of solutions for the Riccati equations in the $DGKF$ formulae [1] is not satisfied. Thus, in this case, structured perturbation techniques [14] or techniques in [15] [16] may help address controller synthesis in this situation.

It may also not be possible to use the $DGKF$ solution to $\mathcal{H}_\infty$ control synthesis [1] because it may happen that $(-2U^{-1}C_1B_2)$ does not have full column rank or $(D_{21}U^{-1})$ does not have full row rank. Again an LMI based solution [17] may address controller synthesis in such a situation.

IV. Illustrative example

This section uses the same lightly damped mechanical plant as in [3] to illustrate the key reformulation of this paper. The uncertain plant considered is:

$$P_\triangle(s) = p(s)\delta(s) \times \begin{bmatrix} s^2+(\alpha+1)s+(k+1) & (\alpha s+k) \\ (\alpha s+k) & s^2+(\alpha+1)s+(k+1) \end{bmatrix}$$

where α and k are unknown real parameters, $p(s) = \frac{1}{s^2+s+1}$ and $\delta(s) = \frac{1}{s^2+(2\alpha+1)s+(2k+1)}$. For the purpose of robust controller synthesis, the controlled closed-loop system in Fig.2 is rearranged in a standard LTF interconnection shown in Fig.3. In these two figures, the generalized plant Σ, the nominal plant P and the uncertainty Δ are given respectively by

$$\Sigma = \begin{bmatrix} 0 & I \\ -I & -P \end{bmatrix},\ P(s) = \Psi \mathrm{diag}(\frac{1}{2}p(s),0)\Psi^*$$

$$\text{and} \quad \Delta(s) = \Psi^{-1}\mathrm{diag}(\frac{1}{2}\delta(s),0)(\Psi^{-1})^*, \qquad (17)$$

where $\Psi = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$. Since the uncertainty $\Delta(s)$ belongs to $\mathcal{I}$, a particular choice of controller $C(s)$ that internally stabilises Σ and makes $F_\ell(\Sigma,C)$ belong to $\mathcal{I}$ was chosen in [3] as $C(s) = \Psi^{-*}\mathrm{diag}(\frac{-2(s^2+s+1)}{2s^3+4s^2+4s+3}, \frac{-1}{s+1})\Psi^{-1}$. This guarantees robust stability for all perturbations in $\mathcal{I}$ as long as the DC loop gain condition is also satisfied [3, Theorem 5].

Since $C(s)$ is strictly proper and the D_{12} matrix of Σ is nonzero, we first use Lemma 5 to give $\check{C}(s) = (\frac{s}{\tau}+1)C(s)$ and $\check{\Sigma} = \begin{pmatrix} 0 & \frac{1}{s/\tau+1}I \\ -I & -\frac{1}{s/\tau+1}P \end{pmatrix}$ where we arbitrarily set $\tau = 1$. Then, using the construction in Theorem 4, we obtain the transformed generalised plant $G(s)$ as:

$$G = \left[\begin{array}{cccc|cccc} -0.9778 & -0.8526 & 0.6992 & 0.6992 & 0 & 0 & 0 & 0 \\ 1.1474 & -0.0222 & -0.1054 & -0.1054 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 2 & 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 0 & 2 & 0 & 1 & 0 & -2 \\ -0.1054 & -0.6992 & -1 & 0 & -1 & 0 & 1 & 0 \\ -0.1054 & -0.6992 & 0 & -1 & 0 & -1 & 0 & 1 \end{array}\right].$$

Since we know from [3] that the chosen $C(s)$ internally stabilises $\Sigma(s)$ and makes $F_\ell(\Sigma,C) \in \mathcal{I}$, then via Lemma 5 and Theorem 4, we should get that $\check{C}(s)$ internally stabilises $G(s)$ and $\|F_\ell(G,\check{C})\|_\infty \le 1$ and the MIMO Nyquist plot of $F_\ell(G,\check{C})$ does not intersect $-1+j0$ for all frequencies.

A simple computation gives $\|F_\ell(G,\check{C})\|_\infty = 1$ and a plot of $\bar{\sigma}\left[F_\ell(G,\check{C})(j\omega)\right]$ is given in Fig.4. Also, $\check{C}$ internally stabilises G as the poles of $T(G,\check{C})$ are at: $-0.5 \pm j\ 0.8660$, $-0.5 \pm j\ 0.8660$, -1, -0.7236, -0.2764. Finally, a Nyquist plot of $\lambda_i\left[F_\ell(G,\check{C})(j\omega)\right]$ is given in Fig.5 for $i = 1,2$. This illustrative example demonstrates that the problem of finding an internally stabilizing controller such that the input-output map has negative imaginary frequency response can

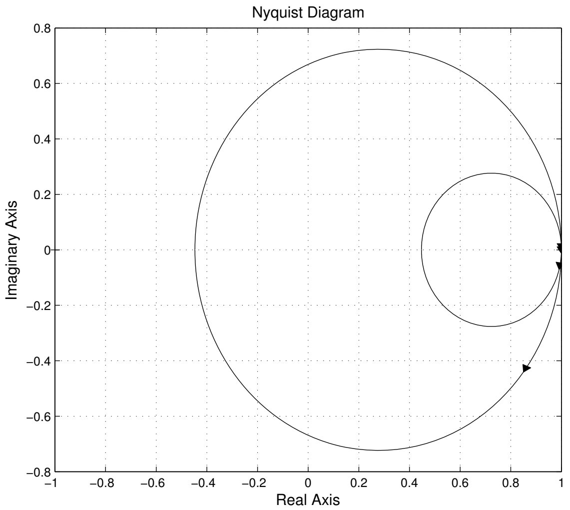

Fig. 5. Nyquist plot of $\lambda_i\left[F_\ell(G, \breve{C})(j\omega)\right]$

be reformulated to a bounded-real problem.

V. CONCLUSION

This paper is a first step towards a controller synthesis technique for systems with negative imaginary frequency response. In [3], an analysis result was proposed, similar to the small-gain or passivity theorem, for the systems with negative frequency response. This paper shows how an LFT interconnection that has negative imaginary frequency response closed-loop properties can be reformulated into a bounded-real LFT interconnection. Although this paper does not tackle the important step of explicit controller synthesis for such a class of systems, the main results in this paper could constitute a first step in allowing results from $\mathcal{H}_\infty$ control synthesis to be borrowed for controller synthesis for closed-loop systems with negative imaginary frequency response.

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