Proceedings of the 40th IEEE
Conference on Decision and Control
Orlando, Florida USA, December 2001

Simultaneous Synthesis of Weights and Controllers in $\mathscr{H}_\infty$ Loop-Shaping

Alexander Lanzon*

School of Aerospace Engineering, Georgia Institute of Technology,
270 Ferst Drive, Atlanta GA 30332-0150, USA.

Abstract

In this paper, several steps of the standard $\mathscr{H}_\infty$ loop-shaping design procedure are combined into one optimization problem that maximizes the robust stability margin over the loop-shaping weights subject to constraints which ensure that the loop-shape and the singular values/condition numbers of the weights lie in pre-specified regions. Thus, loop-shaping weights, which can be required to have either a diagonal or a non-diagonal structure, and a robustly stabilizing controller are simultaneously synthesized by one algorithm in a systematic way. This approach greatly simplifies the often long and tedious process of designing “good” loop-shaping weights directly and allows the designer to quickly get an idea of what is attainable.

Keywords: $\mathscr{H}_\infty$ loop-shaping, weight synthesis, performance optimization, robust performance, robust control.

1 Introduction

The $\mathscr{H}_\infty$ loop-shaping design procedure proposed by [7] is an effective method for designing robust controllers and has been successfully used in a variety of applications (see [10] and references therein).

Desired closed-loop performance is specified by shaping the singular values of the scaled nominal plant P using pre- and post-compensators W_1 and W_2, as shown in Figure 1, to obtain a shaped plant $P_s = W_2 P W_1$. Since the notions of classical loop-shaping carry through, W_1 and W_2 are typically chosen so that P_s has large gain at low frequency, small gain at high frequency and does not roll off at a high rate near cross-over. However, in contrast with classical loop-shaping, the designer does not need to explicitly shape the phase of P.

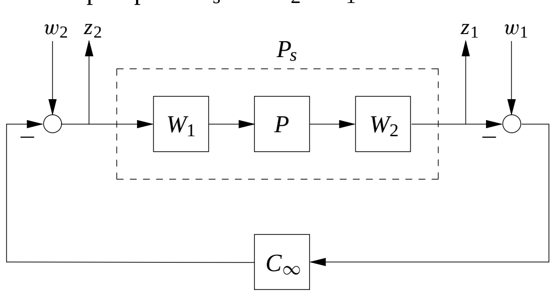

Figure 1: Typical $\mathscr{H}_\infty$ loop-shaping framework

Loop-shaping weights W_1 and W_2 are usually designed in two stages. In the first stage, the desired loop-shape is determined by translating time-response requirements and closed-loop performance specifications into the frequency domain. To do this, engineers largely rely on their intuition and their past experience with loop-shaping concepts. In the second stage, the designer selects loop-shaping weights W_1 and W_2 so that P_s has the desired loop-shape. Diagonal weights are often adequate to achieve the desired loop-shape [3]. However, some design examples have shown that diagonal weights do not work well for plants with strong cross-coupling between the channels. In such cases, non-diagonal weights are necessary which are of course more difficult to design.

Once a desired loop-shape is achieved, the optimal robust stability margin $b_{opt}(P_s)$ is computed. [2] showed that this optimal value can be explicitly calculated using a simple formula and gave a characterization of the set of all internally stabilizing controllers C_∞ that achieve some robust stability margin $b(P_s, C_\infty)$ less than this optimal value. Subsequently, [7] showed that the value $b_{opt}(P_s)$ is also a good indicator of the success of the loop-shaping stage. A large (resp. small) value of $b_{opt}(P_s)$ indicates compatibility (resp. incompatibility) between the specified loop-shape and closed-loop robust stability. A controller C for the scaled nominal plant P is finally obtained by pulling around the weights to obtain $C = W_1 C_\infty W_2$. A full tutorial on how to design robust controllers using the $\mathscr{H}_\infty$ loop-shaping design procedure can be found in [10].

2 Notation

Let the feedback interconnection of P_s and C_∞ shown in Figure 1 be denoted by $[P_s, C_\infty]$. This interconnection is said to be *internally stable* if it is well-posed and each of the four transfer functions mapping disturbances to outputs

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} P_s \\ I \end{bmatrix} (I - C_\infty P_s)^{-1} \begin{bmatrix} -C_\infty & I \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

belongs to $\mathscr{RH}_\infty$. Furthermore, given a plant P_s and a controller C_∞, the *robust stability margin* $b(P_s, C_\infty)$ is given by

$$b(P_s, C_\infty) := \left\| \begin{bmatrix} P_s \\ I \end{bmatrix} (I - C_\infty P_s)^{-1} \begin{bmatrix} -C_\infty & I \end{bmatrix} \right\|_\infty^{-1}$$

if $[P_s, C_\infty]$ is internally stable and by $b(P_s, C_\infty) := 0$ otherwise. Then, the largest value of the robust stability margin is defined by $b_{opt}(P_s) := \sup_{C_\infty} b(P_s, C_\infty)$. It is shown in [2] that $b_{opt}(P_s) \leq 1$ for any P_s.

*For correspondence email alexander.lanzon@ae.gatech.edu

3 Problem Motivation

Despite the success of the standard $\mathscr{H}_\infty$ loop-shaping design procedure, the selection of loop-shaping weights to achieve a desired loop-shape is not always straightforward, especially for plants with strong cross-coupling. This is because it is not always clear how each element in the weights affects the singular values of the scaled nominal plant and the complexity of this relationship considerably increases when non-diagonal weights are used.

In addition, lack of design experience with loop-shaping concepts may lead to a designed loop-shape that does not achieve a sufficiently large robust stability margin. In this case, the designer has to first determine the factors in the designed loop-shape that are giving rise to a small $b_{opt}(P_s)$ and then understand how to modify these factors (without compromising the specifications) in order to increase the robust stability margin. This may not be obvious and the designer may have to iterate between the selection of loop-shaping weights and the evaluation of $b_{opt}(P_s)$ several times before a sufficiently large value of $b_{opt}(P_s)$ is achieved. For instance, the designer must ensure that the loop-gain is large around the frequencies and in the directions of open-loop unstable poles, small around the frequencies and in the directions of open-loop unstable zeros and that the loop-gain does not roll-off at a high rate around cross-over. If any one of these is violated, a small $b_{opt}(P_s)$ is obtained.

All of this can be fairly time-consuming if done in an ad-hoc manner and although designers usually arrive at very good loop-shaping weights and controllers using their engineering insight and intuition, trial and error can never be guaranteed to yield the best possible results. The length of this iterative process strongly depends on the experience of the designer and on the cross-coupling present in the plant.

Consequently, it is believed that by combining several steps of the standard $\mathscr{H}_\infty$ loop-shaping design procedure into one optimization problem, the design procedure can be made even easier to use in application. The proposed optimization problem maximizes the robust stability margin over the loop-shaping weights subject to constraints which ensure that the loop-shape and the singular values/condition numbers of the weights lie in pre-specified regions. Thus, loop-shaping weights, which can be required to have a diagonal or a non-diagonal structure, and a robustly stabilizing controller are simultaneously synthesized by one algorithm in a systematic way. This algorithm enables the designer to quickly get a feel of what performance is achievable, determine whether non-diagonal weights would be beneficial and easily understand the tradeoffs involved in the particular problem at hand. A more detailed presentation of this work is given in [4].

4 A New Optimization Problem

A new optimization problem is now proposed which directly addresses the aforementioned difficulties. First, however, the following assumption is made.

Assumption: *Let the scaled nominal plant $P \in \mathscr{R}^{m\times n}$ be such that $m \geq n$.*

This assumption incurs no loss of generality but considerably simplifies notation. If the plant has strictly fewer outputs than inputs (i.e. $m < n$), then the dual problem to that shown in Figure 1 would be considered. That is, one would use the same optimization framework proposed in this paper with P^T replacing P. Then, the resulting pre-compensator for the original plant is given by W_2^T, the resulting post-compensator for the original plant is given by W_1^T and the resulting robust stabilizing controller for the shaped plant is given by C_∞^T.

Now, consider the following optimization problem:

$$\max_{\substack{W_1, W_1^{-1} \in \mathscr{RH}_\infty \\ W_2, W_2^{-1} \in \mathscr{RH}_\infty}} b_{opt}(P_s)$$

subject to

(a) $|\underline{s}(j\omega)| < \sigma_i\big(P_s(j\omega)\big) < |\overline{s}(j\omega)| \quad \forall i, \omega,$

(b) $|\underline{w}_1(j\omega)| < \sigma_i\big(W_1(j\omega)\big) < |\overline{w}_1(j\omega)| \quad \forall i, \omega,$
$|\underline{w}_2(j\omega)| < \sigma_i\big(W_2(j\omega)\big) < |\overline{w}_2(j\omega)| \quad \forall i, \omega,$

(c) $\kappa\big(W_1(j\omega)\big) < |k_1(j\omega)| \quad \forall \omega,$
$\kappa\big(W_2(j\omega)\big) < |k_2(j\omega)| \quad \forall \omega,$

where the scaled nominal plant P is given and satisfies the above assumption, and $\underline{s}$, $\overline{s}$, $\underline{w}_i$, $\overline{w}_i$, k_i ($i = 1, 2$) are SISO transfer functions specified by the designer such that:

(i) the frequency functions $|\underline{s}(j\omega)|$ and $|\overline{s}(j\omega)|$ are boundaries for an allowable loop-shape (see Figure 2),

(ii) the frequency functions $|\underline{w}_i(j\omega)|$ and $|\overline{w}_i(j\omega)|$ delimit the allowable region for the singular values of loop-shaping weight $W_i(j\omega)$ ($i = 1, 2$),

(iii) the frequency function $|k_i(j\omega)|$ bounds the condition number of loop-shaping weight $W_i(j\omega)$ ($i = 1, 2$).

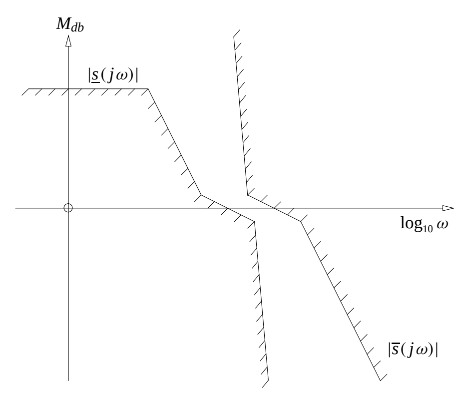

Figure 2: Typical loop-shape boundary

5 Rewriting the Optimization Problem

The optimization problem proposed above will now be rewritten into a form more suitable for a subsequent algorithm. First, however, the following set will be defined.

Definition 5.1 *Let the set of real diagonal matrices of dimension $n \times n$ be defined by:*

$$\mathbf{\Lambda}_n := \left\{ \operatorname*{diag}_{i=1}^{n} (x_i) \quad : \quad x_i \in \mathbb{R} \ \ \forall i \right\}.$$

This definition will be used at the end of this section. For the time being, note that the optimization problem posed in Section 4 can be rewritten as:

$$\text{Minimize } \gamma$$
$$\text{such that } \exists\, W_1, W_2, C_\infty \text{ satisfying}$$

(a) $W_1, W_1^{-1} \in \mathscr{RH}_\infty, \quad W_2, W_2^{-1} \in \mathscr{RH}_\infty,$
$[P_s, C_\infty]$ is internally stable,

(b) $\left\| \begin{bmatrix} P_s \\ I \end{bmatrix} (I - C_\infty P_s)^{-1} \begin{bmatrix} -C_\infty & I \end{bmatrix} \right\|_\infty < \gamma,$

(c) $|\underline{s}(j\omega)| < \sigma_i\big(P_s(j\omega)\big) < |\overline{s}(j\omega)| \quad \forall i, \omega,$

(d) $|\underline{w}_1(j\omega)| < \sigma_i\big(W_1(j\omega)\big) < |\overline{w}_1(j\omega)| \quad \forall i, \omega,$
$|\underline{w}_2(j\omega)| < \sigma_i\big(W_2(j\omega)\big) < |\overline{w}_2(j\omega)| \quad \forall i, \omega,$

(e) $\kappa\big(W_1(j\omega)\big) < |k_1(j\omega)| \quad \forall \omega,$
$\kappa\big(W_2(j\omega)\big) < |k_2(j\omega)| \quad \forall \omega.$

Then, using $P_s = W_2 P W_1$ and $C = W_1 C_\infty W_2$, it follows after some algebra that the above optimization problem can be rewritten as:

$$\text{Minimize } \gamma_\omega \text{ at each } \omega$$
$$\text{such that } \exists\, W_1, W_2, C \text{ satisfying}$$

(a) $W_1, W_1^{-1} \in \mathscr{RH}_\infty, \quad W_2, W_2^{-1} \in \mathscr{RH}_\infty,$
$[P, C]$ is internally stable,

(b) $\overline{\sigma}\left(\begin{bmatrix} W_2 & 0 \\ 0 & W_1^{-1} \end{bmatrix} \begin{bmatrix} 0 & P \\ 0 & I \end{bmatrix} \begin{bmatrix} I & P \\ C & I \end{bmatrix}^{-1} \begin{bmatrix} W_2^{-1} & 0 \\ 0 & W_1 \end{bmatrix} \right)(j\omega) < \gamma_\omega \quad \forall \omega,$

(c) $|\underline{s}(j\omega)| < \sigma_i\big(W_2(j\omega)P(j\omega)W_1(j\omega)\big) < |\overline{s}(j\omega)| \quad \forall i, \omega,$

(d) $\dfrac{1}{|\overline{w}_1(j\omega)|} < \sigma_i\big(W_1(j\omega)^{-1}\big) < \dfrac{1}{|\underline{w}_1(j\omega)|} \quad \forall i, \omega,$
$\kappa\big(W_1(j\omega)^{-1}\big) < |k_1(j\omega)| \quad \forall \omega,$

(e) $|\underline{w}_2(j\omega)| < \sigma_i\big(W_2(j\omega)\big) < |\overline{w}_2(j\omega)| \quad \forall i, \omega,$
$\kappa\big(W_2(j\omega)\big) < |k_2(j\omega)| \quad \forall \omega.$

Dropping the dependence on $(j\omega)$ for P, C, W_1 and W_2 in the interest of clarity, the above optimization problem can be restated as:

$$\text{Minimize } \gamma_\omega^2 \text{ at each } \omega$$
$$\text{such that } \exists\, W_1, W_2, C \text{ satisfying}$$

(a) $W_1, W_1^{-1} \in \mathscr{RH}_\infty, \quad W_2, W_2^{-1} \in \mathscr{RH}_\infty,$
$[P, C]$ is internally stable,

(b) $\begin{bmatrix} 0 & P \\ 0 & I \end{bmatrix}^* \begin{bmatrix} W_2^* W_2 & 0 \\ 0 & W_1^{-*} W_1^{-1} \end{bmatrix} \begin{bmatrix} 0 & P \\ 0 & I \end{bmatrix}$
$< \gamma_\omega^2 \begin{bmatrix} I & P \\ C & I \end{bmatrix}^* \begin{bmatrix} W_2^* W_2 & 0 \\ 0 & W_1^{-*} W_1^{-1} \end{bmatrix} \begin{bmatrix} I & P \\ C & I \end{bmatrix} \quad \forall \omega,$

(c) $|\underline{s}(j\omega)|^2 \left(W_1^{-*} W_1^{-1}\right) < P^*\big(W_2^* W_2\big) P \quad \forall \omega,$
$|\overline{s}(j\omega)|^2 \left(W_1^{-*} W_1^{-1}\right) > P^*\big(W_2^* W_2\big) P \quad \forall \omega,$

(d) $\forall \omega, \ \exists\, \underline{\xi}_{1\omega}, \overline{\xi}_{1\omega}$ satisfying $\underline{\xi}_{1\omega} I < W_1^{-*} W_1^{-1} < \overline{\xi}_{1\omega} I,$
$|\overline{w}_1(j\omega)|^{-2} < \underline{\xi}_{1\omega}, \ \ \overline{\xi}_{1\omega} < |\underline{w}_1(j\omega)|^{-2},$
$\overline{\xi}_{1\omega} < |k_1(j\omega)|^2 \, \underline{\xi}_{1\omega},$

(e) $\forall \omega, \ \exists\, \underline{\xi}_{2\omega}, \overline{\xi}_{2\omega}$ satisfying $\underline{\xi}_{2\omega} I < W_2^* W_2 < \overline{\xi}_{2\omega} I,$
$|\underline{w}_2(j\omega)|^2 < \underline{\xi}_{2\omega}, \ \ \overline{\xi}_{2\omega} < |\overline{w}_2(j\omega)|^2,$
$\overline{\xi}_{2\omega} < |k_2(j\omega)|^2 \, \underline{\xi}_{2\omega}.$

If the loop-shaping weights are required to have a diagonal structure, then the frequency functions $W_1^{-*} W_1^{-1}$ and $W_2^* W_2$ reduce to simple strictly positive diagonal real matrices at each frequency ω. Let these strictly positive frequency dependent diagonal matrices be denoted by $\Lambda_{1\omega}$ and $\Lambda_{2\omega}$ respectively (i.e. at each fixed ω, $0 < \Lambda_{1\omega} := W_1(j\omega)^{-*} W_1(j\omega)^{-1} \in \mathbf{\Lambda}_n$ and $0 < \Lambda_{2\omega} := W_2(j\omega)^* W_2(j\omega) \in \mathbf{\Lambda}_m$). Note that given any $\Lambda_{1\omega} \in \mathbf{\Lambda}_n$ with $\Lambda_{1\omega} > 0 \ \forall \omega$ (resp. $\Lambda_{2\omega} \in \mathbf{\Lambda}_m$ with $\Lambda_{2\omega} > 0 \ \forall \omega$), it is always possible to construct a diagonal weight W_1 (resp. W_2) that is a unit in $\mathscr{RH}_\infty$ and satisfies $W_1(j\omega)^{-*} W_1(j\omega)^{-1} = \Lambda_{1\omega} \ \forall \omega$ (resp. $W_2(j\omega)^* W_2(j\omega) = \Lambda_{2\omega} \ \forall \omega$) by fitting stable minimum phase transfer functions to each magnitude function on the diagonal of $\Lambda_{1\omega}$ (resp. $\Lambda_{2\omega}$).

If, on the other hand, the loop-shaping weights are required to have a non-diagonal structure, then the frequency functions $W_1^{-*} W_1^{-1}$ and $W_2^* W_2$ are strictly positive non-diagonal complex hermitian matrices at each frequency ω. The problem, in this case, is significantly more difficult if approached directly. However, the technique developed by [8] may be used to simplify the problem. Building on that work, let $\hat{U}(s)$ and $\hat{V}(s)$ be units in $\mathscr{RL}_\infty$ that approximately interpolate the frequency-by-frequency unitary matrices containing the left and right singular vectors of the scaled nominal plant P. Then it is possible to parameterize:

- $W_1(j\omega)^{-*} W_1(j\omega)^{-1}$ by $\hat{V}(j\omega)^{-*} \Lambda_{1\omega} \hat{V}(j\omega)^{-1}$ for some $\Lambda_{1\omega} \in \mathbf{\Lambda}_n$ with $\Lambda_{1\omega} > 0 \ \forall \omega$,
- $W_2(j\omega)^* W_2(j\omega)$ by $\hat{U}(j\omega) \Lambda_{2\omega} \hat{U}(j\omega)^*$ for some $\Lambda_{2\omega} \in \mathbf{\Lambda}_m$ with $\Lambda_{2\omega} > 0 \ \forall \omega$,

with very little restriction. This is because the parameters in $\Lambda_{1\omega}$ and $\Lambda_{2\omega}$ are able to directly influence the singular values of $P(j\omega)$. The construction of $\hat{U}(s)$ and $\hat{V}(s)$ is described in more detail in [4]. The interested reader is also referred to [8] for a full exposition of the original idea. As before, note that given any $\Lambda_{1\omega} \in \mathbf{\Lambda}_n$ with $\Lambda_{1\omega} > 0 \ \forall \omega$ (resp. $\Lambda_{2\omega} \in \mathbf{\Lambda}_m$ with $\Lambda_{2\omega} > 0 \ \forall \omega$), it is always possible to construct a diagonal weight D_1 (resp. D_2) that is a unit in $\mathscr{RH}_\infty$ and satisfies

$D_1(j\omega)^{-*}D_1(j\omega)^{-1} = \Lambda_{1\omega}\ \forall\omega$ (resp. $D_2(j\omega)^*D_2(j\omega) = \Lambda_{2\omega}\ \forall\omega$). Then, corresponding non-diagonal weights W_1 and W_2 that are units in $\mathscr{RH}_\infty$ are obtained by solving the following co-spectral and spectral factorizations:

$$W_1 W_1^{\sim} = \hat{V} D_1 D_1^{\sim} \hat{V}^{\sim},$$
$$W_2^{\sim} W_2 = \hat{U} D_2^{\sim} D_2 \hat{U}^{\sim}.$$

With the above argument in mind, it follows that the previous optimization problem can be rewritten as:

Minimize γ_ω^2 at each ω
such that
$\exists$ a C and $\forall\omega$ a $\Lambda_{1\omega} \in \mathbf{\Lambda}_n$ and a $\Lambda_{2\omega} \in \mathbf{\Lambda}_m$
satisfying

(a) $[P, C]$ is internally stable,

(b) $$\begin{bmatrix} 0 & P \\ 0 & I \end{bmatrix}^* \begin{bmatrix} \hat{U}\Lambda_{2\omega}\hat{U}^* & 0 \\ 0 & \hat{V}^{-*}\Lambda_{1\omega}\hat{V}^{-1} \end{bmatrix} \begin{bmatrix} 0 & P \\ 0 & I \end{bmatrix} < \gamma_\omega^2 \begin{bmatrix} I & P \\ C & I \end{bmatrix}^* \begin{bmatrix} \hat{U}\Lambda_{2\omega}\hat{U}^* & 0 \\ 0 & \hat{V}^{-*}\Lambda_{1\omega}\hat{V}^{-1} \end{bmatrix} \begin{bmatrix} I & P \\ C & I \end{bmatrix},$$

(c) $$|\underline{s}(j\omega)|^2 \left(\hat{V}^{-*}\Lambda_{1\omega}\hat{V}^{-1}\right) < P^*\left(\hat{U}\Lambda_{2\omega}\hat{U}^*\right)P,$$
$$|\overline{s}(j\omega)|^2 \left(\hat{V}^{-*}\Lambda_{1\omega}\hat{V}^{-1}\right) > P^*\left(\hat{U}\Lambda_{2\omega}\hat{U}^*\right)P,$$

(d) $\exists\, \underline{\xi}_{1\omega},\ \overline{\xi}_{1\omega}$ satisfying $\underline{\xi}_{1\omega} I < \left(\hat{V}^{-*}\Lambda_{1\omega}\hat{V}^{-1}\right) < \overline{\xi}_{1\omega} I,$
$$|\overline{w}_1(j\omega)|^{-2} < \underline{\xi}_{1\omega},\quad \overline{\xi}_{1\omega} < |\underline{w}_1(j\omega)|^{-2},$$
$$\overline{\xi}_{1\omega} < |k_1(j\omega)|^2\, \underline{\xi}_{1\omega},$$

(e) $\exists\, \underline{\xi}_{2\omega},\ \overline{\xi}_{2\omega}$ satisfying $\underline{\xi}_{2\omega} I < \left(\hat{U}\Lambda_{2\omega}\hat{U}^*\right) < \overline{\xi}_{2\omega} I,$
$$|\underline{w}_2(j\omega)|^2 < \underline{\xi}_{2\omega},\quad \overline{\xi}_{2\omega} < |\overline{w}_2(j\omega)|^2,$$
$$\overline{\xi}_{2\omega} < |k_2(j\omega)|^2\, \underline{\xi}_{2\omega}.$$

Note that $\Lambda_{1\omega}$ and $\Lambda_{2\omega}$ are implicitly restricted to be strictly positive matrices at each frequency ω by constraints (d) and (e) above. Also, $\hat{U}(s) = I_m$ and $\hat{V}(s) = I_n$ when diagonal weights are required, whereas $\hat{U}(s)$ and $\hat{V}(s)$ are units in $\mathscr{RL}_\infty$ that approximately interpolate the frequency-by-frequency unitary matrices containing the left and right singular vectors respectively of the scaled nominal plant P when non-diagonal weights are required. Furthermore, observe that the above problem is a quasi-convex optimization problem if the controller C is held fixed at an internally stabilizing controller for the scaled nominal plant P.

6 Solution Algorithm

This section presents a sub-optimal solution to the optimization problem proposed in Section 4. Analogous to D-K iterations and other solution methods for these types of optimizations, an iterative algorithm must be used since the posed problem is not simultaneously convex in all variables.

Inputs to the algorithm:

- Scaled nominal plant P satisfying the assumption stated in Section 4,
- Frequency functions $|\underline{s}(j\omega)|$ and $|\overline{s}(j\omega)|$ that are boundaries for an allowable loop-shape,
- Frequency functions $|\underline{w}_i(j\omega)|$ and $|\overline{w}_i(j\omega)|$ that delimit the allowable region for the singular values of loop-shaping weight $W_i(j\omega)$ $(i = 1, 2)$,
- Frequency function $|k_i(j\omega)|$ that bounds the condition number of loop-shaping weight $W_i(j\omega)$ $(i = 1, 2)$.

The solution algorithm:

1. Find a controller $C_0^\star$ such that the interconnection $[P, C_0^\star]$ is internally stable. Let $\hat{U}(s) = I_m$ (resp. $\hat{V}(s) = I_n$) if a diagonal loop-shaping weight W_2 (resp. W_1) is required, or let $\hat{U}(s)$ (resp. $\hat{V}(s)$) be constructed as described in [4] if a non-diagonal weight W_2 (resp. W_1) is required.

   Set $i = 0$ and let $\epsilon^\star_{\max,0} = -1$.

2. Increment i by 1.

3. Solve the following quasi-convex optimization problem *at each frequency* ω:

   Minimize γ_ω^2
   such that $\exists\ \Lambda_{1\omega} \in \mathbf{\Lambda}_n,\ \Lambda_{2\omega} \in \mathbf{\Lambda}_m$ satisfying

   (a) $$\begin{bmatrix} 0 & \hat{U}^*P\hat{V} \\ 0 & I \end{bmatrix}^* \begin{bmatrix} \Lambda_{2\omega} & 0 \\ 0 & \Lambda_{1\omega} \end{bmatrix} \begin{bmatrix} 0 & \hat{U}^*P\hat{V} \\ 0 & I \end{bmatrix} < \gamma_\omega^2 \begin{bmatrix} I & \hat{U}^*P\hat{V} \\ \hat{V}^{-1}C^\star_{i-1}\hat{U}^{-*} & I \end{bmatrix}^* \begin{bmatrix} \Lambda_{2\omega} & 0 \\ 0 & \Lambda_{1\omega} \end{bmatrix} \times \begin{bmatrix} I & \hat{U}^*P\hat{V} \\ \hat{V}^{-1}C^\star_{i-1}\hat{U}^{-*} & I \end{bmatrix},$$

   (b) $$|\underline{s}(j\omega)|^2\,\Lambda_{1\omega} < (\hat{U}^*P\hat{V})^*\Lambda_{2\omega}\,(\hat{U}^*P\hat{V}),$$
   $$|\overline{s}(j\omega)|^2\,\Lambda_{1\omega} > (\hat{U}^*P\hat{V})^*\Lambda_{2\omega}\,(\hat{U}^*P\hat{V}),$$

   (c) $\exists\, \underline{\xi}_{1\omega},\ \overline{\xi}_{1\omega} \in \mathbb{R}\ :\ \underline{\xi}_{1\omega}\left(\hat{V}^*\hat{V}\right) < \Lambda_{1\omega} < \overline{\xi}_{1\omega}\left(\hat{V}^*\hat{V}\right),$
   $$|\overline{w}_1(j\omega)|^{-2} < \underline{\xi}_{1\omega},\quad \overline{\xi}_{1\omega} < |\underline{w}_1(j\omega)|^{-2},$$
   $$\overline{\xi}_{1\omega} < |k_1(j\omega)|^2\, \underline{\xi}_{1\omega},$$

   (d) $\exists\, \underline{\xi}_{2\omega}, \overline{\xi}_{2\omega} \in \mathbb{R}\ :\ \underline{\xi}_{2\omega}\left(\hat{U}^*\hat{U}\right)^{-1} < \Lambda_{2\omega} < \overline{\xi}_{2\omega}\left(\hat{U}^*\hat{U}\right)^{-1},$
   $$|\underline{w}_2(j\omega)|^2 < \underline{\xi}_{2\omega},\quad \overline{\xi}_{2\omega} < |\overline{w}_2(j\omega)|^2,$$
   $$\overline{\xi}_{2\omega} < |k_2(j\omega)|^2\, \underline{\xi}_{2\omega}.$$

   Note that $C^\star_{i-1}$ is the controller synthesized in the previous iteration. The above minimization problem can thus be easily solved using LMI routines.

   Denote by $\Lambda^\star_{1\omega}$ and $\Lambda^\star_{2\omega}$ the values of $\Lambda_{1\omega}$ and $\Lambda_{2\omega}$ that achieve the minimum of the above optimization problem at each frequency ω.

4. Construct *diagonal* transfer function matrices $D_1^\star(s)$ and $D_2^\star(s)$ that are units in $\mathscr{RH}_\infty$ by fitting stable minimum phase transfer functions to each magnitude function on the main diagonal of $\left(\Lambda^\star_{1\omega}\right)^{-\frac{1}{2}}$ and $\left(\Lambda^\star_{2\omega}\right)^{\frac{1}{2}}$ respectively.

5. Solve the following co-spectral and spectral factorizations to obtain loop-shaping weights $W^\star_{1,i}(s)$ and $W^\star_{2,i}(s)$ that are units in $\mathscr{RH}_\infty$ and have the required structure.

$$\begin{aligned}(W^\star_{1,i})(W^\star_{1,i})^\sim &= (\hat{V})\,(D^\star_1)(D^\star_1)^\sim\,(\hat{V})^\sim,\\ (W^\star_{2,i})^\sim(W^\star_{2,i}) &= (\hat{U})\,(D^\star_2)^\sim(D^\star_2)\,(\hat{U})^\sim,\end{aligned}$$

6. Compute $b_{opt}\left(W^\star_{2,i} P W^\star_{1,i}\right)$ as detailed in [2] and let this value be denoted by $\epsilon^\star_{\max,i}$. Synthesize a controller $C^\star_{\infty,i}$ that achieves a robust stability margin $b\left(W^\star_{2,i} P W^\star_{1,i}, C^\star_{\infty,i}\right) = \epsilon^\star_{\max,i}$ using the state-space formula given in [1] and let $C^\star_i = W^\star_{1,i} C^\star_{\infty,i} W^\star_{2,i}$.

   These calculations are computed using well-known formulae that are coded in commercially available software.

7. Evaluate $(\epsilon^\star_{\max,i} - \epsilon^\star_{\max,i-1})$. If this difference (which is always positive) is very small and has remained very small for the last few iterations, then EXIT. Otherwise return to Step 2.

Outputs from the algorithm: (after i iterations)

- The largest value of $b_{opt}(P_s)$ obtained by the algorithm in the variable $\epsilon^\star_{\max,i}$,
- Loop-shaping weights $W^\star_{1,i}(s)$ and $W^\star_{2,i}(s)$ that achieve this maximized robust stability margin $\epsilon^\star_{\max,i}$,
- A controller $C^\star_{\infty,i}(s)$ that achieves this maximized robust stability margin $\epsilon^\star_{\max,i}$.

Note that this algorithm is an ascent algorithm. By this it is meant that the value $\epsilon^\star_{\max,i}$ is monotonically non-decreasing as i increases and that at each iteration i, the reciprocal of the square-root of the minimum cost γ^2_ω obtained in Step 3 of the algorithm is greater than or equal to $\epsilon^\star_{\max,i-1}$ for all frequency ω. Note however that iterative algorithms as the one presented above cannot be guaranteed to converge to the *global maximum*. Only monotonicity properties can be guaranteed.

7 Numerical Example

The algorithm proposed in Section 6 will now be illustrated by a numerical example. The plant used to demonstrate the applicability of the proposed algorithm is a scaled-down version of the High Incidence Research Model (HIRM) developed by the Defence Evaluation and Research Agency in Bedford, UK. A physical model of this was constructed at the Department of Engineering of the University of Cambridge in order to investigate problems associated with the control of air-vehicles at high angles of attack. Details of the identification experiments carried out on this plant may be found in [9].

The nominal open-loop plant P with the actuator model included has 8 states. Figure 3 depicts the singular values of the scaled nominal plant P and the loop-shape boundaries $|\underline{s}(j\omega)|$ and $|\overline{s}(j\omega)|$ selected so that the performance specifications are satisfied. For instance, the bandwidth determines the rise time and the low-frequency gain determines the sensitivity reduction and hence the reference tracking capabilities. The frequency functions $|\underline{w}_1(j\omega)|$, $|\overline{w}_1(j\omega)|$ and $|k_1(j\omega)|$ that confine the singular values/condition number of pre-compensator W_1 where chosen to be 10^{-10}, 10^{10} and 20 respectively. These bounds never became active in this particular design example. In other design problems (say, for plants with very large condition number and/or stringent requirements on gains from plant output disturbances to control signal or from plant input disturbances to output signal), it may be necessary to specify more complicated, perhaps frequency dependent, bounds to satisfy the problem specifications.

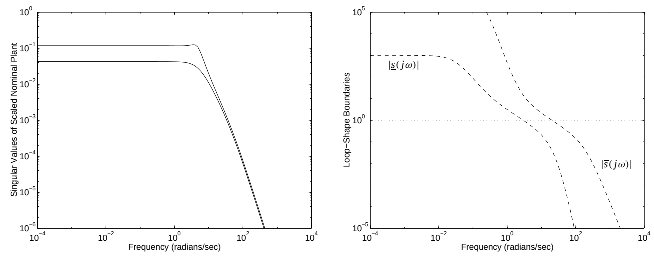

Figure 3: Singular values of P and loop-shape boundaries

Two different designs will be considered — one using a diagonal pre-compensator W_1 and the other using a non-diagonal pre-compensator W_1. The post-compensator will be held fixed in both design cases for simplicity of illustration. This will also usually be the case in practice. W_2 was in fact chosen to be a first-order low-pass filter with a corner frequency of 300 `rad/s` on each output channel for sensor noise rejection. The algorithm was coded up in MATLAB 5.3 and run on a 400 `MHz` Pentium II PC. Table 1 summarizes the results obtained for both design cases.

	Diagonal W_1	Non-diagonal W_1
No. of iterations for convergence	4 iterations	4 iterations
Time taken for convergence	$\approx$ 5 minutes	$\approx$ 6 minutes
Order of weight W_1	4 states + 4 states	38 states (model reduced to 12 states)
Condition number of weight W_1	$< 3\ \forall\omega$	$< 4\ \forall\omega$
Order of controller C_∞	17 states	46 states (model reduced to 11 states)
Order of C after model reduction	17 states	17 states
Robust stability margin	0.368	0.382

Table 1: Comparison of results for both designs

Figure 4 shows the singular values of the designed pre-compensator W_1, the correspondingly achieved loop-shape and the singular values of the simultaneously synthesized robustly stabilizing controller C_∞ for both design cases. In

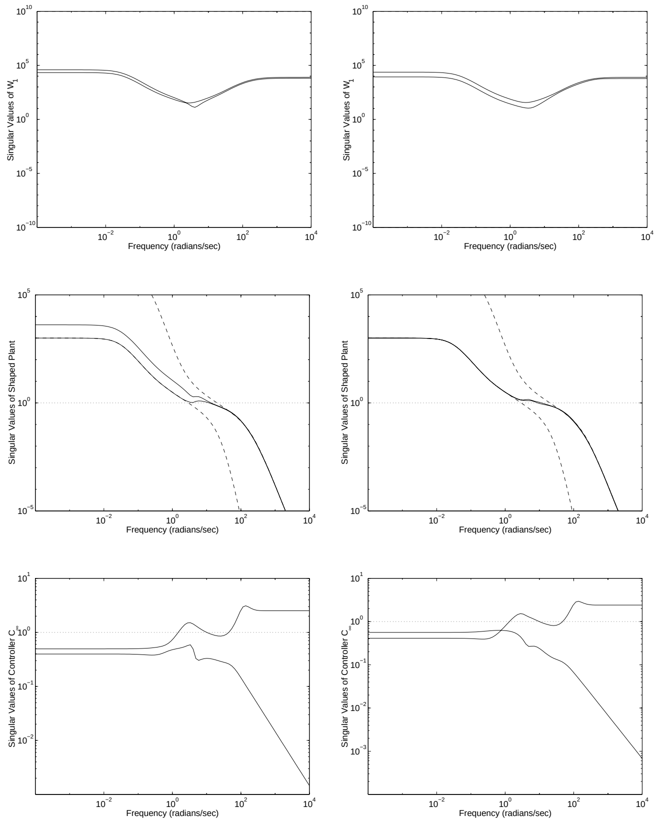

Figure 4: Left plots are for diagonal W_1 while right plots are for non-diagonal W_1

both cases, it can be easily seen that the loop-shape lies in the pre-specified region and that it rolls-off at a very small rate around cross-over. Furthermore, both the loop-shaping weight W_1 and the controller C_∞ introduce some phase lead around cross-over to improve the robust stability margin.

Although the synthesized weight and controller were of much higher order in the non-diagonal pre-compensator design case, the achieved robust stability margin was slightly higher too. It was also noted that the controller $C = W_1 C_\infty W_2$ could be model reduced to 17 states in *both* design cases with negligible deterioration of the designed properties. Thus, one should not be scared 'a priori' of a non-diagonal design simply on the grounds that it may give higher order weights, as these can often be model reduced afterwards. However, if diagonal weights achieve approximately the same robust stability margin as non-diagonal weights, then it is futile to use more complicated weights to achieve very minor improvements.

8 Conclusions

An algorithm for the simultaneous synthesis of weights and controllers in $\mathscr{H}_\infty$ loop-shaping has been presented. In this algorithm, loop-shaping weights are synthesized in a systematic way to immediately have the required structure (i.e. diagonal/non-diagonal), satisfy the designer-specified constraints on their singular values and condition numbers and achieve a loop-shape which falls in a pre-specified region and maximizes the robust stability margin $b_{opt}(P_s)$. These pre-specified regions are usually determined from the closed-loop performance specifications. A robustly stabilizing controller C_∞ is also synthesized by the algorithm to achieve the maximized robust stability margin.

Specifying acceptable regions rather than exact weights and hence an exact loop-shape makes it more difficult for inexperienced designers to obtain very bad loop-shapes. Furthermore, since the algorithm is not time-consuming, the designer can quickly determine whether a diagonal weight design is sufficiently good or if non-diagonal weights are necessary. Consequently, this algorithm allows the designer to concentrate on more fundamental design issues than simply finding weights that achieve the desired loop-shape.

Similar results have been obtained for the robust performance μ-synthesis problem by [6] and [5].

References

[1] K. Glover. All optimal Hankel-norm approximations of linear multivariable systems and their $\mathscr{L}_\infty$-error bounds. *International Journal of Control*, 39(6):1115–1193, 1984.

[2] K. Glover and D. McFarlane. Robust stabilization of normalized coprime factor plant descriptions with $\mathscr{H}_\infty$-bounded uncertainty. *IEEE Transactions on Automatic Control*, 34(8):821–830, August 1989.

[3] R. A. Hyde. $\mathscr{H}_\infty$ aerospace control design — a VSTOL flight application. In *Advances in Industrial Control Series*. Springer-Verlag, 1995.

[4] A. Lanzon. *Weight Selection in Robust Control: An Optimisation Approach*. PhD thesis, University of Cambridge, United Kingdom, October 2000. Downloadable from `http://www-control.eng.cam.ac.uk/al225`.

[5] A. Lanzon and M. Cantoni. A state-space algorithm for the simultaneous optimisation of performance weights and controllers in μ-synthesis. In *Proceeding of the 39th IEEE Conference on Decision and Control*, Sydney, Australia, December 2000.

[6] A. Lanzon and R. J. Richards. A frequency domain optimisation algorithm for simultaneous design of performance weights and controllers in μ-synthesis. In *Proceedings of the 38th IEEE Conference on Decision and Control*, Phoenix, Arizona, December 1999.

[7] D. McFarlane and K. Glover. A loop shaping design procedure using $\mathscr{H}_\infty$ synthesis. *IEEE Transactions on Automatic Control*, 37(6):759–769, June 1992.

[8] G. Papageorgiou and K. Glover. A systematic procedure for designing non-diagonal weights to facilitate $\mathscr{H}_\infty$ loop-shaping. In *Proceedings of the 36th IEEE Conference on Decision and Control*, pages 2127–2132, San Diego, CA, December 1997.

[9] G. Papageorgiou and K. Glover. Design, development and control of the HIRM wind tunnel model. In *Proceedings of the 38th IEEE Conference on Decision and Control*, Phoenix, Arizona, December 1999.

[10] G. Papageorgiou and K. Glover. $\mathscr{H}_\infty$ loop-shaping: Why is it a sensible procedure for designing robust flight controllers? In *Proceedings of the AIAA Conference on Guidance, Navigation and Control*, August 1999.