A State-Space Algorithm for the Simultaneous Optimisation of Performance Weights and Controllers in μ-Synthesis

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Abstract

A conceptually new approach to the μ-synthesis robust performance problem is proposed in this paper. Performance weights, maximised with respect to a suitable cost function that captures the desired closed-loop performance, are synthesised simultaneously with an internally stabilising controller to immediately achieve robust performance. The designer is only required to specify the plant set and an optimisation directionality. This directionality only appears in the cost function and reflects the desired closed-loop properties in particular frequency regions. Correspondingly, this approach greatly simplifies the often long and tedious process of designing "good" performance weights directly.

Keywords: optimise performance, performance weight synthesis, robust performance, μ-synthesis, D-K iterations, $\mathscr{H}_\infty$-control.

1 Introduction

It is well known that the design of performance weights for $\mathscr{H}_\infty$-control and μ-synthesis problems is non-trivial. Usually, suitable performance weights are obtained via a long and tedious trial and error process based primarily on engineering judgement and intuition. This approach becomes increasingly complicated as the number of performance channels increases, since it may not be possible to choose performance weights for each channel independently. The D-K iterative procedure [2] is probably the most popular method used in μ-synthesis to design robustly stabilising controllers. However, this procedure assumes that the performance weights have already been chosen.

In [3], a mathematical quantity (closely related to μ) was introduced to answer the question: "Determine the smallest α such that for any uncertainty bounded by unity, an $\mathscr{H}_\infty$ performance level of α is guaranteed". Although this may be considered as an initial step towards maximising robust performance (i.e. the determination of the smallest α) for a given uncertainty set, the value α is a constant bound over all frequency and spatial direction. In this paper, the following more general problem is addressed: "Determine the largest performance weights (in some sense, over frequency and spatial direction) such that for any uncertainty bounded by unity, an $\mathscr{H}_\infty$ performance level of unity is guaranteed".

This paper considers an optimisation problem that is very similar to that proposed in [8]. However, the solution algorithm derived for this optimisation problem will be based on state-space techniques that eliminate all problems of the pointwise algorithm given in that paper. The results presented here are also considerably stronger. Consequently, this paper presents a significant generalisation of that work.

2 Problem Formulation

Consider the Linear Time-Invariant (LTI) system depicted in Figure 1. Here G is the generalised plant, Δ is the uncertainty in the system and Δ_P is a fictitious uncertainty used only to transform the robust performance problem into an equivalent robust stability problem. It is desired to synthesise the largest performance weights W (in some sense) subject to the existence of an internally stabilising controller K that guarantees robust performance with respect to these maximised weights. For notational convenience all uncertainty blocks, except those for performance, are assumed to be square. This can be done without loss of generality by adding dummy inputs or outputs [6]. Before formulating the problem of interest, some sets need to be defined.

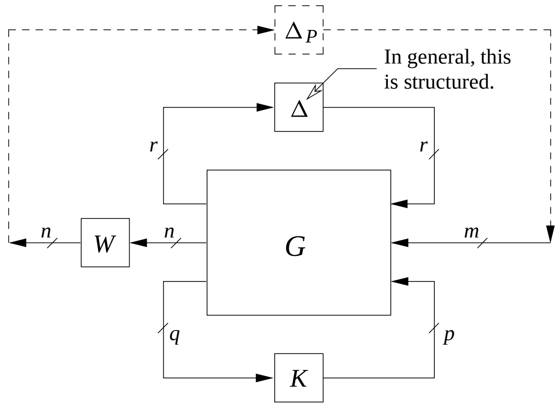

Figure 1: Typical μ-synthesis LFT framework

Definition 2.1 *Define the sets of allowable perturbations by:*

$$\begin{aligned}
\mathbf{\Delta} &:= \left\{ \operatorname*{diag}_{i=1}^{f} \left(I_{\alpha_i} \otimes \Delta_i \right) : \Delta_i \in \mathbb{C}^{\beta_i \times \beta_i}, \sum_{i=1}^{f} \alpha_i \beta_i = r \right\} \\
\mathbf{\Delta}_P &:= \left\{ \Delta_P \in \mathbb{C}^{m \times n} \right\} \\
\mathbf{\Delta}_{TOT} &:= \left\{ \operatorname{diag}(\Delta, \Delta_P) : \Delta \in \mathbf{\Delta}, \Delta_P \in \mathbf{\Delta}_P \right\} \\
\mathbf{B\Delta}^{TF} &:= \left\{ \Delta(s) \in \mathscr{RH}_\infty : \Delta(s_o) \in \mathbf{\Delta}\ \forall s_o \in \overline{\mathbb{C}}_+, \|\Delta\|_\infty \le 1 \right\} \\
\mathbf{B\Delta}_P^{TF} &:= \left\{ \Delta_P(s) \in \mathscr{RH}_\infty : \Delta_P(s_o) \in \mathbf{\Delta}_P\ \forall s_o \in \overline{\mathbb{C}}_+, \|\Delta_P\|_\infty \le 1 \right\}.
\end{aligned}$$

Definition 2.2 *Define the set of diagonal complex matrices by:*

$$\mathbf{\Lambda} := \left\{ \operatorname*{diag}_{i=1}^{n} (\ell_i) : \ell_i \in \mathbb{C} \right\}.$$

Definition 2.3 *The scaling sets $\boldsymbol{\mathcal{D}}$ and $\boldsymbol{\mathcal{D}}^{TF}$ that commute with $\mathbf{\Delta}$ and $\mathbf{B\Delta}^{TF}$ are defined by:*

$$\begin{aligned}
\boldsymbol{\mathcal{D}} &:= \left\{ D = \operatorname*{diag}_{i=1}^{f} \left(D_i \otimes I_{\beta_i} \right) : \det D \neq 0, D_i \in \mathbb{C}^{\alpha_i \times \alpha_i}, \sum_{i=1}^{f} \alpha_i \beta_i = r \right\} \\
\boldsymbol{\mathcal{D}}^{TF} &:= \left\{ D(s) \in \mathscr{RH}_\infty : D(s)^{-1} \in \mathscr{RH}_\infty, D(s_o) \in \boldsymbol{\mathcal{D}}\ \forall s_o \in \overline{\mathbb{C}}_+ \right\}.
\end{aligned}$$

Definition 2.4 *The sets of performance weights and directionality matrices are defined by:*

$$\begin{aligned}
\boldsymbol{\mathcal{W}}^{TF} &:= \left\{ W(s) \in \mathscr{RH}_\infty : W(s)^{-1} \in \mathscr{RH}_\infty, W(s_o) \in \mathbf{\Lambda}\ \forall s_o \in \overline{\mathbb{C}}_+ \right\} \\
\mathbf{\Upsilon}^{TF} &:= \left\{ \Upsilon(s) \in \mathscr{RH}_\infty : \Upsilon(\infty) = 0, \Upsilon(s_o) \in \mathbf{\Lambda}\ \forall s_o \in \overline{\mathbb{C}}_+ \right\}.
\end{aligned}$$

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Definition 2.5 *Given a generalised plant*

$$G(s) = \left[\begin{array}{c|cc:c} A & B_1 & B_2 & B_3 \\ \hline C_1 & D_{11} & D_{12} & D_{13} \\ C_2 & D_{21} & D_{22} & D_{23} \\ \hdashline C_3 & D_{31} & D_{32} & D_{33} \end{array}\right]$$

partitioned consistently with Figure 1, let the term "Standard Assumptions" refer to:

(A1) (A, B_3) *is stabilisable and* (C_3, A) *is detectable,*

(A2) $D_{33} = 0$.

Note that assumption (A1) is necessary and sufficient for the existence of an internally stabilising output-feedback controller [5, Appendix A.4], whereas assumption (A2) incurs no loss of generality but considerably simplifies calculations [4].

Definition 2.6 *Given a generalised plant G, the set of internally stabilising output-feedback controllers $K \in \mathscr{R}^{p\times q}$ for the LFT interconnection $\mathcal{F}_l\,(G, K)$ is denoted by $\boldsymbol{\mathcal{K}}_G^{TF}$.*

It follows by the Robust Performance Theorem [9, Theorem 5.4] that robust performance is achieved for the setup of Figure 1 for all $\Delta \in \mathbf{B\Delta}^{TF}$ and $\Delta_P \in \mathbf{B\Delta}_P^{TF}$ if and only if

$$\sup_{\omega} \mu_{\mathbf{\Delta}_{TOT}} \left[\begin{pmatrix} I_r & 0 \\ 0 & W(j\omega) \end{pmatrix} \mathcal{F}_l\,(G(j\omega), K(j\omega)) \right] < 1.$$

Now consider the following optimisation problem for a given generalised plant $G(s)$ satisfying the standard assumptions stated in Definition 2.5 and an 'a priori' chosen directionality transfer function matrix $\Upsilon(s) \in \mathbf{\Upsilon}^{TF}$:

$$\begin{gathered} \max_{W\in\boldsymbol{\mathcal{W}}^{TF}} \frac{1}{\|\Upsilon W^{-1}\|_2} \\ \text{such that} \\ \min_{K\in\boldsymbol{\mathcal{K}}_G^{TF}} \sup_{\omega} \mu_{\mathbf{\Delta}_{TOT}} \left[\begin{pmatrix} I_r & 0 \\ 0 & W(j\omega) \end{pmatrix} \mathcal{F}_l\,(G(j\omega), K(j\omega)) \right] < 1. \end{gathered} \tag{1}$$

Some justification will now be given to the fact that this is a sensible optimisation problem to consider. First observe that

$$\|\Upsilon W^{-1}\|_2^2 \;=\; \int_{-\infty}^{\infty} \sum_{i=1}^{n} \frac{1}{\left|\frac{w_i(j\omega)}{\upsilon_i(j\omega)}\right|^2}\, d\omega,$$

where $w_i(j\omega)$ (resp. $\upsilon_i(j\omega)$) is the i-th diagonal element of $W(j\omega)$ (resp. $\Upsilon(j\omega)$). From this decomposition, it is clear that the cost function $1\big/\|\Upsilon W^{-1}\|_2$ is a cumulative measure of the frequency-dependent size of the performance weights $w_i(j\omega)$. Each performance weight $w_i(j\omega)$ is weighted differently across frequency due to the directionality factors $\upsilon_i(j\omega)$. A gradient analysis reveals that the steepest ascent in maximising this cost function over the performance weights $w_i(j\omega)$ is always attained by maximising the smallest ratio $\left|\frac{w_i(j\omega)}{\upsilon_i(j\omega)}\right|$ for all i and ω. Thus, the directionality matrix $\Upsilon(s) \in \mathbf{\Upsilon}^{TF}$ is chosen by the designer so as to direct the maximisation as desired. In fact, $\upsilon_i(j\omega)$ will be chosen large (resp. small) where the corresponding performance weight $w_i(j\omega)$ is required to be large (resp. small).

This however does not make $\Upsilon(j\omega)$ a substitute for the performance weight $W(j\omega)$, as $\Upsilon(j\omega)$ only captures the desired directionality of the optimisation. The absolute size of each $\upsilon_i(j\omega)$ is completely irrelevant as this will only affect the value of the cost associated with the above optimisation problem. Only the shape across frequency and the relative sizes amongst the different diagonal entries of $\Upsilon(j\omega)$ are important. Furthermore, conflicting directionalities can never be specified, unlike directly specifying the performance weights. This is because the performance weights given by the above optimisation must always be feasible to its constraint and hence always satisfy $\mu < 1$. Sensible choice of $\Upsilon(j\omega)$ is of course still necessary (this is however much easier than choosing the actual performance weights) so as to obtain a controller which performs sensibly and satisfies reasonable stability/performance requirements.

The constraint in optimisation problem (1) ensures that maximisation of the performance weight $W(s)$ is limited by the fact that there must exist some internally stabilising controller $K(s)$ which guarantees robust performance for all $\Delta \in \mathbf{B\Delta}^{TF}$ and $\Delta_P \in \mathbf{B\Delta}_P^{TF}$.

3 Replacing μ with an Upper Bound

Since the μ constraint in optimisation problem (1) is not computationally tractable, it is necessary to replace $\mu_{\mathbf{\Delta}_{TOT}}[\,\cdot\,]$ with a computationally tractable upper bound. To this end, note that

$$\begin{aligned} &\sup_{\omega} \mu_{\mathbf{\Delta}_{TOT}} \left[\begin{pmatrix} I_r & 0 \\ 0 & W(j\omega) \end{pmatrix} \mathcal{F}_l\,(G(j\omega), K(j\omega)) \right] \\ &\qquad \le \inf_{D\in\boldsymbol{\mathcal{D}}^{TF}} \left\| \begin{pmatrix} D & 0 \\ 0 & W \end{pmatrix} \mathcal{F}_l\,(G, K) \begin{pmatrix} D^{-1} & 0 \\ 0 & I_m \end{pmatrix} \right\|_\infty . \end{aligned}$$

In view of this and since interest is only in the arguments of the optimisation, the following problem will be considered henceforth:

$$\begin{gathered} \min_{W\in\boldsymbol{\mathcal{W}}^{TF}} \|\Upsilon W^{-1}\|_2^2 \\ \text{such that} \\ \min_{K\in\boldsymbol{\mathcal{K}}_G^{TF}} \inf_{D\in\boldsymbol{\mathcal{D}}^{TF}} \left\| \begin{pmatrix} D & 0 \\ 0 & W \end{pmatrix} \mathcal{F}_l\,(G, K) \begin{pmatrix} D^{-1} & 0 \\ 0 & I_m \end{pmatrix} \right\|_\infty < 1. \end{gathered}$$

Furthermore, since $\|P\|_{2,(\infty)} = \|P^T\|_{2,(\infty)}$ and using the definitions $\bar{D} := D^{-T} \in \boldsymbol{\mathcal{D}}^{TF}$ and $\bar{W} := W^{-T} \in \boldsymbol{\mathcal{W}}^{TF}$, this optimisation problem may be rewritten as:

$$\begin{gathered} \min_{\bar{W}\in\boldsymbol{\mathcal{W}}^{TF}} \|\bar{W}\Upsilon\|_2^2 \\ \text{such that} \\ \min_{K\in\boldsymbol{\mathcal{K}}_G^{TF}} \inf_{\bar{D}\in\boldsymbol{\mathcal{D}}^{TF}} \left\| \begin{pmatrix} \bar{D} & 0 \\ 0 & I_m \end{pmatrix} \mathcal{F}_l\,(G, K)^T \begin{pmatrix} \bar{D}^{-1} & 0 \\ 0 & \bar{W}^{-1} \end{pmatrix} \right\|_\infty < 1. \end{gathered} \tag{2}$$

4 Commuting Properties

This section specifies the commuting properties which need to be satisfied by the state-space realisations of $\bar{D} \in \boldsymbol{\mathcal{D}}^{TF}$, $\bar{W} \in \boldsymbol{\mathcal{W}}^{TF}$ and $\Upsilon \in \mathbf{\Upsilon}^{TF}$. To this end, select *arbitrary* realisations

$$\bar{D} := \left[\begin{array}{c|c} A_{\bar{D}} & B_{\bar{D}} \\ \hline C_{\bar{D}} & D_{\bar{D}} \end{array}\right] \in \boldsymbol{\mathcal{D}}^{TF}, \qquad \bar{W} := \left[\begin{array}{c|c} A_{\bar{W}} & B_{\bar{W}} \\ \hline C_{\bar{W}} & D_{\bar{W}} \end{array}\right] \in \boldsymbol{\mathcal{W}}^{TF},$$

$$\text{and } \Upsilon := \left[\begin{array}{c|c} A_\Upsilon & B_\Upsilon \\ \hline C_\Upsilon & 0 \end{array}\right] \in \mathbf{\Upsilon}^{TF},$$

with $A_{\bar{D}}$, $A_{\bar{W}}$ and A_Υ Hurwitz. Furthermore, define

$$T_{\bar{D}}^o(s) := \begin{bmatrix} (sI_{s_D} - A_{\bar{D}})^{-1}B_{\bar{D}} \\ I_r \end{bmatrix} \text{ and } T_{\bar{W}}^o(s) := \begin{bmatrix} (sI_{s_W} - A_{\bar{W}})^{-1}B_{\bar{W}} \\ I_n \end{bmatrix}. \tag{3}$$

Now, a complete parametrisation of the frequency function:

I. $\bar{D}(j\omega)^*\bar{D}(j\omega)$ is given by $T_{\bar{D}}^o(j\omega)^*\breve{D}T_{\bar{D}}^o(j\omega)$, where $\breve{D} := \begin{bmatrix} 0 & \breve{D}_{12} \\ \breve{D}_{12}^T & \breve{D}_{22} \end{bmatrix}$ with $\breve{D}_{12} \in \mathbb{R}^{s_D\times r}$ and $\breve{D}_{22} = \breve{D}_{22}^T \in \mathbb{R}^{r\times r}$,

II. $\bar{W}(j\omega)^*\bar{W}(j\omega)$ is given by $T_{\bar{W}}^o(j\omega)^*\breve{W}T_{\bar{W}}^o(j\omega)$, where $\breve{W} := \begin{bmatrix} 0 & \breve{W}_{12} \\ \breve{W}_{12}^T & \breve{W}_{22} \end{bmatrix}$ with $\breve{W}_{12} \in \mathbb{R}^{s_W\times n}$ and $\breve{W}_{22} = \breve{W}_{22}^T \in \mathbb{R}^{n\times n}$.

Since it is required that $\bar{D} \in \boldsymbol{\mathcal{D}}^{TF}$, $\bar{W} \in \boldsymbol{\mathcal{W}}^{TF}$ and $\Upsilon \in \boldsymbol{\Upsilon}^{TF}$, it is clear that $T^o_{\bar{D}}(j\omega)^* \breve{D} T^o_{\bar{D}}(j\omega)$ should commute with $\boldsymbol{\Delta}$, $T^o_{\bar{W}}(j\omega)^* \breve{W} T^o_{\bar{W}}(j\omega)$ should commute with $\boldsymbol{\Lambda}$ and $\Upsilon(j\omega)$ should commute with $\boldsymbol{\Lambda}$. These commuting requirements determine the structure of each parameter in the above parametrisations. This structure will not be explicitly stated here due to space limitations. However, the reader is referred to [6, Section 8.4.3] for a similar approach.

Definition 4.1 *Define the structure of the parameters $\breve{D}$ and $\breve{W}$ by:*

$$\boldsymbol{\Xi}_{\breve{D}} := \left\{ \breve{D} = \begin{bmatrix} 0 & \breve{D}_{12} \\ \breve{D}_{12}^T & \breve{D}_{22} \end{bmatrix} \; : \; \breve{D}_{12} \in \mathbb{R}^{s_D \times r},\; \breve{D}_{22} = \breve{D}_{22}^T \in \mathbb{R}^{r \times r}, \right.$$
$$\left. \breve{D}_{12}, \breve{D}_{22} \textit{ have the appropriate structure} \right\},$$

$$\boldsymbol{\Xi}_{\breve{W}} := \left\{ \breve{W} = \begin{bmatrix} 0 & \breve{W}_{12} \\ \breve{W}_{12}^T & \breve{W}_{22} \end{bmatrix} \; : \; \breve{W}_{12} \in \mathbb{R}^{s_W \times n},\; \breve{W}_{22} = \breve{W}_{22}^T \in \mathbb{R}^{n \times n}, \right.$$
$$\left. \breve{W}_{12}, \breve{W}_{22} \textit{ have the appropriate structure} \right\}.$$

Definition 4.2 *Define the structure of $(A_{\bar{D}}, B_{\bar{D}})$ and $(A_{\bar{W}}, B_{\bar{W}})$ by:*

$$\boldsymbol{\Xi}_{(A_{\bar{D}}, B_{\bar{D}})} := \left\{ (A_{\bar{D}}, B_{\bar{D}}) \; : A_{\bar{D}} \in \mathbb{R}^{s_D \times s_D}, B_{\bar{D}} \in \mathbb{R}^{s_D \times r}, A_{\bar{D}} \textit{ is Hurwitz}, \right.$$
$$\left. A_{\bar{D}}, B_{\bar{D}} \textit{ have the appropriate structure} \right\},$$

$$\boldsymbol{\Xi}_{(A_{\bar{W}}, B_{\bar{W}})} := \left\{ (A_{\bar{W}}, B_{\bar{W}}) : A_{\bar{W}} \in \mathbb{R}^{s_W \times s_W}, B_{\bar{W}} \in \mathbb{R}^{s_W \times n}, A_{\bar{W}} \textit{ is Hurwitz}, \right.$$
$$\left. A_{\bar{W}}, B_{\bar{W}} \textit{ have the appropriate structure} \right\}.$$

5 Restrictions of the Optimisation Sets

As is usual with state-space methods used to address optimisation problems as the one posed here, attention has to be limited to optimisation over a subclass of performance weights and D-scales. This is necessary to obtain convex state-space conditions. Since the frequency functions $\bar{D}(j\omega)^* \bar{D}(j\omega)$ and $\bar{W}(j\omega)^* \bar{W}(j\omega)$ are completely parametrised by $T^o_{\bar{D}}(j\omega)^* \breve{D} T^o_{\bar{D}}(j\omega)$ and $T^o_{\bar{W}}(j\omega)^* \breve{W} T^o_{\bar{W}}(j\omega)$ respectively, it seems natural to restrict these parametrisations by holding the basis functions $T^o_{\bar{D}}(j\omega)$ and $T^o_{\bar{W}}(j\omega)$ fixed. This amounts to keeping $(A_{\bar{D}}, B_{\bar{D}}) \in \boldsymbol{\Xi}_{(A_{\bar{D}}, B_{\bar{D}})}$ and $(A_{\bar{W}}, B_{\bar{W}}) \in \boldsymbol{\Xi}_{(A_{\bar{W}}, B_{\bar{W}})}$ fixed.

It is desirable, however, to choose fixed values of $(A_{\bar{D}}, B_{\bar{D}}) \in \boldsymbol{\Xi}_{(A_{\bar{D}}, B_{\bar{D}})}$ and $(A_{\bar{W}}, B_{\bar{W}}) \in \boldsymbol{\Xi}_{(A_{\bar{W}}, B_{\bar{W}})}$ that are sufficiently close to the optimal values which would have been obtained if these quantities were free variables. Towards constructing such "close to optimal" values, observe that for a given G satisfying the standard assumptions stated in Definition 2.5, a fixed $K \in \boldsymbol{\mathcal{K}}_G^{TF}$ and an 'a priori' chosen $\Upsilon \in \boldsymbol{\Upsilon}^{TF}$, the following optimisation problem

$$\min_{\bar{W} \in \boldsymbol{\mathcal{W}}^{TF}} \|\bar{W}\Upsilon\|_2^2$$
$$\text{such that} \tag{4}$$
$$\inf_{\bar{D} \in \boldsymbol{\mathcal{D}}^{TF}} \left\| \begin{pmatrix} \bar{D} & 0 \\ 0 & I_m \end{pmatrix} \mathcal{F}_l(G, K)^T \begin{pmatrix} \bar{D}^{-1} & 0 \\ 0 & \bar{W}^{-1} \end{pmatrix} \right\|_\infty < 1$$

is convex if solved pointwise in frequency. To see this, define the following sets and quantities:

Definition 5.1 *Define the set of strictly-positive vector functions by:*

$$\boldsymbol{\mathcal{V}} := \left\{ f : \mathbb{R} \mapsto \mathbb{R}^n_+ \right\}$$

For ease of notation, define the following real-valued vector functions:

$$v_W(\omega) := \begin{bmatrix} \frac{1}{|w_1(j\omega)|^2} & \frac{1}{|w_2(j\omega)|^2} & \cdots & \frac{1}{|w_n(j\omega)|^2} \end{bmatrix}^T \in \boldsymbol{\mathcal{V}},$$
$$v_\Upsilon(\omega) := \begin{bmatrix} |\upsilon_1(j\omega)|^2 & |\upsilon_2(j\omega)|^2 & \cdots & |\upsilon_n(j\omega)|^2 \end{bmatrix}^T.$$

Using this notation, optimisation problem (4) can be rewritten as:

$$\min_{v_W \in \boldsymbol{\mathcal{V}}} \int_{-\infty}^{\infty} v_\Upsilon(\omega)^T v_W(\omega)\, d\omega$$
$$\text{such that}$$
$$\forall \omega \in \mathbb{R} \cup \{\infty\} \;\; \exists\, \Theta_\omega \in \boldsymbol{\mathcal{D}} \text{ with } \Theta_\omega > 0 \tag{5}$$
$$\text{satisfying}$$
$$\left[\mathcal{F}_l(G(j\omega), K(j\omega))^T \right]^* \begin{pmatrix} \Theta_\omega & 0 \\ 0 & I_m \end{pmatrix} \left[\mathcal{F}_l(G(j\omega), K(j\omega))^T \right] < \begin{pmatrix} \Theta_\omega & 0 \\ 0 & \operatorname{diag}(v_W(\omega)) \end{pmatrix}.$$

It is now easy to see that this optimisation is convex and can be solved pointwise in frequency over a finite grid using LMI routines. Once Θ_ω and $v_W(\omega)$ have been determined, $\bar{D} \in \boldsymbol{\mathcal{D}}^{TF}$ and $\bar{W} \in \boldsymbol{\mathcal{W}}^{TF}$ can be constructed as described in Section 9. Then, $(A_{\bar{D}}, B_{\bar{D}}) \in \boldsymbol{\Xi}_{(A_{\bar{D}}, B_{\bar{D}})}$ and $(A_{\bar{W}}, B_{\bar{W}}) \in \boldsymbol{\Xi}_{(A_{\bar{W}}, B_{\bar{W}})}$ are obtained from the appropriate state-space realisations of $\bar{D} \in \boldsymbol{\mathcal{D}}^{TF}$ and $\bar{W} \in \boldsymbol{\mathcal{W}}^{TF}$.

Once "close to optimal" values for $(A_{\bar{D}}, B_{\bar{D}}) \in \boldsymbol{\Xi}_{(A_{\bar{D}}, B_{\bar{D}})}$ and $(A_{\bar{W}}, B_{\bar{W}}) \in \boldsymbol{\Xi}_{(A_{\bar{W}}, B_{\bar{W}})}$ are found, optimisation problem (2) may be restricted so as to obtain convex state-space conditions.

Definition 5.2 *Given $(A_{\bar{D}}, B_{\bar{D}}) \in \boldsymbol{\Xi}_{(A_{\bar{D}}, B_{\bar{D}})}$ and $(A_{\bar{W}}, B_{\bar{W}}) \in \boldsymbol{\Xi}_{(A_{\bar{W}}, B_{\bar{W}})}$, define*

$$\boldsymbol{\mathcal{D}}^{TF}_{(A_{\bar{D}}, B_{\bar{D}})} := \left\{ \bar{D}(s) = \left[\begin{array}{c|c} A_{\bar{D}} & B_{\bar{D}} \\ \hline C_{\bar{D}} & D_{\bar{D}} \end{array} \right] \; : \; \bar{D}(s) \in \boldsymbol{\mathcal{D}}^{TF} \right\} \subset \boldsymbol{\mathcal{D}}^{TF}$$

$$\boldsymbol{\mathcal{W}}^{TF}_{(A_{\bar{W}}, B_{\bar{W}})} := \left\{ \bar{W}(s) = \left[\begin{array}{c|c} A_{\bar{W}} & B_{\bar{W}} \\ \hline C_{\bar{W}} & D_{\bar{W}} \end{array} \right] \; : \; \bar{W}(s) \in \boldsymbol{\mathcal{W}}^{TF} \right\} \subset \boldsymbol{\mathcal{W}}^{TF}.$$

Using these definitions, optimisation problem (2) can be restricted to:

$$\min_{\bar{W} \in \boldsymbol{\mathcal{W}}^{TF}_{(A_{\bar{W}}, B_{\bar{W}})}} \|\bar{W}\Upsilon\|_2^2$$
$$\text{such that} \tag{6}$$
$$\min_{K \in \boldsymbol{\mathcal{K}}_G^{TF}} \; \inf_{\bar{D} \in \boldsymbol{\mathcal{D}}^{TF}_{(A_{\bar{D}}, B_{\bar{D}})}} \left\| \begin{pmatrix} \bar{D} & 0 \\ 0 & I_m \end{pmatrix} \mathcal{F}_l(G, K)^T \begin{pmatrix} \bar{D}^{-1} & 0 \\ 0 & \bar{W}^{-1} \end{pmatrix} \right\|_\infty < 1.$$

6 The Cost Function

The following theorem states that "Minimising $\|\bar{W}\Upsilon\|_2^2$ over $\bar{W}(s) \in \boldsymbol{\mathcal{W}}^{TF}_{(A_{\bar{W}}, B_{\bar{W}})}$ subject to some constraint" is equivalent to "Minimising $c^T \operatorname{vec}(\breve{W})$ over $\breve{W} \in \boldsymbol{\Xi}_{\breve{W}}$ subject to the same constraint", provided that $T^o_{\bar{W}}(j\omega)^* \breve{W} T^o_{\bar{W}}(j\omega) > 0 \;\forall \omega \in \mathbb{R} \cup \{\infty\}$ is implicitly guaranteed by the constraint.

Theorem 6.1 *Given $\Upsilon(s) = \left[\begin{array}{c|c} A_\Upsilon & B_\Upsilon \\ \hline C_\Upsilon & 0 \end{array} \right] \in \boldsymbol{\Upsilon}^{TF}$ with A_Υ Hurwitz, $(A_{\bar{W}}, B_{\bar{W}}) \in \boldsymbol{\Xi}_{(A_{\bar{W}}, B_{\bar{W}})}$ and any $\bar{W}(s) \in \boldsymbol{\mathcal{W}}^{TF}_{(A_{\bar{W}}, B_{\bar{W}})}$, define $T^o_{\bar{W}}(s)$ as in equation (3) and parametrise $\bar{W}(j\omega)^* \bar{W}(j\omega)$ by $T^o_{\bar{W}}(j\omega)^* \breve{W} T^o_{\bar{W}}(j\omega)$ for some $\breve{W} \in \boldsymbol{\Xi}_{\breve{W}}$. Then*

$$\|\bar{W}\Upsilon\|_2^2 = c^T \operatorname{vec}(\breve{W})$$

where

$$c := -\left(\begin{bmatrix} I_{s_W} & 0 \\ 0 & C_\Upsilon \end{bmatrix} \otimes \begin{bmatrix} I_{s_W} & 0 \\ 0 & C_\Upsilon \end{bmatrix} \right)$$
$$\times \left(\begin{bmatrix} A_{\bar{W}} & B_{\bar{W}} C_\Upsilon \\ 0 & A_\Upsilon \end{bmatrix} \oplus \begin{bmatrix} A_{\bar{W}} & B_{\bar{W}} C_\Upsilon \\ 0 & A_\Upsilon \end{bmatrix} \right)^{-1}$$
$$\times \left(\begin{bmatrix} 0 \\ B_\Upsilon \end{bmatrix} \otimes \begin{bmatrix} 0 \\ B_\Upsilon \end{bmatrix} \right) \operatorname{vec}(I_n).$$

Proof Omitted for brevity, see [7]. □

7 Holding K Fixed in the Constraint

The following theorem shows that for a fixed $K \in \mathcal{K}_G^{TF}$, the constraint of optimisation problem (6) can be rewritten as a set of LMIs that are also simultaneously affine in $\breve{W}$.

Theorem 7.1 *Given $(A_{\bar{W}}, B_{\bar{W}}) \in \Xi_{(A_{\bar{W}}, B_{\bar{W}})}$ and any $\bar{W}(s) \in \mathcal{W}^{TF}_{(A_{\bar{W}}, B_{\bar{W}})}$, define $T^o_{\bar{W}}(s)$ as in equation* (3) *and parametrise $\bar{W}(j\omega)^* \bar{W}(j\omega)$ by $T^o_{\bar{W}}(j\omega)^* \breve{W} T^o_{\bar{W}}(j\omega)$ for some $\breve{W} \in \Xi_{\breve{W}}$. Then, given*

$$\mathcal{F}_l(G, K) = \left[\begin{array}{c|cc} A_{cl} & B_{1cl} & B_{2\,cl} \\ \hline C_{1cl} & D_{11cl} & D_{12\,cl} \\ C_{2\,cl} & D_{21cl} & D_{22\,cl} \end{array}\right],$$

where $A_{cl} \in \mathbb{R}^{s_{cl} \times s_{cl}}$ is Hurwitz and the partitioning is consistent with Figure 1, the following two statements are equivalent for any $(A_{\bar{D}}, B_{\bar{D}}) \in \Xi_{(A_{\bar{D}}, B_{\bar{D}})}$:

(i) $$\inf_{\bar{D} \in \mathcal{D}^{TF}_{(A_{\bar{D}}, B_{\bar{D}})}} \left\| \begin{pmatrix} \bar{D} & 0 \\ 0 & I_m \end{pmatrix} \mathcal{F}_l(G, K)^T \begin{pmatrix} \bar{D}^{-1} & 0 \\ 0 & \bar{W}^{-1} \end{pmatrix} \right\|_\infty < 1.$$

(ii) $\exists\, \breve{D} \in \Xi_{\breve{D}}$, $X = X^T \in \mathbb{R}^{s_D \times s_D}$ *and* $Y = Y^T \in \mathbb{R}^{(s_{cl}+2s_D+s_W)\times(s_{cl}+2s_D+s_W)}$ *such that*

$$\begin{bmatrix} X A_{\bar{D}} + A_{\bar{D}}^T X & X B_{\bar{D}} \\ B_{\bar{D}}^T X & 0 \end{bmatrix} + \breve{D} > 0,$$

$$\begin{bmatrix} Y \grave{A} + \grave{A}^T Y & Y \grave{B} \\ \grave{B}^T Y & 0 \end{bmatrix} + \begin{bmatrix} \grave{C}^T \\ \grave{D}^T \end{bmatrix} \grave{Q} \begin{bmatrix} \grave{C} & \grave{D} \end{bmatrix} < 0;$$

where $\grave{Q} := \mathrm{diag}\big(\breve{D},\ I_m,\ -\breve{D},\ -\breve{W}\big)$ and $\begin{bmatrix} \grave{A} & \grave{B} \\ \grave{C} & \grave{D} \end{bmatrix}$ is defined by:

$$\left[\begin{array}{cccc:cc} A_{\bar{D}} & 0 & 0 & 0 & B_{\bar{D}} & 0 \\ 0 & A_{\bar{W}} & 0 & 0 & 0 & B_{\bar{W}} \\ 0 & 0 & A_{\bar{D}} & B_{\bar{D}} B_{1cl}^T & B_{\bar{D}} D_{11cl}^T & B_{\bar{D}} D_{21cl}^T \\ 0 & 0 & 0 & A_{cl}^T & C_{1cl}^T & C_{2\,cl}^T \\ \hdashline 0 & 0 & I_{s_D} & 0 & 0 & 0 \\ 0 & 0 & 0 & B_{1cl}^T & D_{11cl}^T & D_{21cl}^T \\ 0 & 0 & 0 & B_{2\,cl}^T & D_{12\,cl}^T & D_{22\,cl}^T \\ I_{s_D} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_r & 0 \\ 0 & I_{s_W} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_n \end{array}\right].$$

Proof Omitted for brevity, see [7]. □

8 Holding $\bar{D}$ Fixed in the Constraint

The following theorem shows that for a fixed $\bar{D} \in \mathcal{D}^{TF}$, the constraint appearing in optimisation problem (6) can be rewritten as a set of LMIs that are also simultaneously affine in $\breve{W}$.

Theorem 8.1 *Given a generalised plant $G(s)$ satisfying the standard assumptions stated in Definition 2.5 and scalings $\bar{D}(s) \in \mathcal{D}^{TF}$, define the scaled generalised plant $\tilde{G}(s)$ by*

$$\tilde{G}(s) := \begin{bmatrix} \bar{D}(s)^{-T} & 0 & 0 \\ 0 & I_n & 0 \\ 0 & 0 & I_q \end{bmatrix} G(s) \begin{bmatrix} \bar{D}(s)^T & 0 & 0 \\ 0 & I_m & 0 \\ 0 & 0 & I_p \end{bmatrix}$$

and let

$$\left[\begin{array}{c|cc:c} \tilde{A} & \tilde{B}_1 & \tilde{B}_2 & \tilde{B}_3 \\ \hline \tilde{C}_1 & \tilde{D}_{11} & \tilde{D}_{12} & \tilde{D}_{13} \\ \tilde{C}_2 & \tilde{D}_{21} & \tilde{D}_{22} & \tilde{D}_{23} \\ \hdashline \tilde{C}_3 & \tilde{D}_{31} & \tilde{D}_{32} & \tilde{D}_{33} \end{array}\right]$$

be a stabilisable and detectable realisation for $\tilde{G}(s)$ with $\tilde{A} \in \mathbb{R}^{s_{\tilde{G}} \times s_{\tilde{G}}}$, $\tilde{D}_{11} \in \mathbb{R}^{r \times r}$, $\tilde{D}_{22} \in \mathbb{R}^{n \times m}$ and $\tilde{D}_{33} = 0 \in \mathbb{R}^{q \times p}$. Furthermore, given $(A_{\bar{W}}, B_{\bar{W}}) \in \Xi_{(A_{\bar{W}}, B_{\bar{W}})}$ and any $\bar{W}(s) \in \mathcal{W}^{TF}_{(A_{\bar{W}}, B_{\bar{W}})}$, define $T^o_{\bar{W}}(s)$ as in equation (3) *and parametrise $\bar{W}(j\omega)^* \bar{W}(j\omega)$ by $T^o_{\bar{W}}(j\omega)^* \breve{W} T^o_{\bar{W}}(j\omega)$ for some $\breve{W} \in \Xi_{\breve{W}}$. Then the following two statements are equivalent:*

(i) $$\min_{K \in \mathcal{K}_G^{TF}} \left\| \begin{pmatrix} \bar{D} & 0 \\ 0 & I_m \end{pmatrix} \mathcal{F}_l(G, K)^T \begin{pmatrix} \bar{D}^{-1} & 0 \\ 0 & \bar{W}^{-1} \end{pmatrix} \right\|_\infty < 1.$$

(ii) $\exists\, P = P^T \in \mathbb{R}^{(s_W+s_{\tilde{G}})\times(s_W+s_{\tilde{G}})}$, $R = R^T \in \mathbb{R}^{s_W \times s_W}$, $S \in \mathbb{R}^{s_W \times s_{\tilde{G}}}$ *and* $T = T^T \in \mathbb{R}^{s_{\tilde{G}} \times s_{\tilde{G}}}$ *such that*

$$P > 0, \quad R > 0, \quad T > 0,$$

$$\begin{pmatrix} P & \begin{bmatrix} R & -S \\ 0 & I_{s_{\tilde{G}}} \end{bmatrix} \\ \begin{bmatrix} R & 0 \\ -S^T & I_{s_{\tilde{G}}} \end{bmatrix} & \begin{bmatrix} R & 0 \\ 0 & T \end{bmatrix} \end{pmatrix} \geq 0,$$

$$\Psi_P^T \begin{pmatrix} P \begin{bmatrix} A_{\bar{W}} & 0 \\ 0 & \tilde{A}^T \end{bmatrix} + \{\cdot\}^T & P \begin{bmatrix} 0 & B_{\bar{W}} \\ \tilde{C}_1^T & \tilde{C}_2^T \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ \tilde{B}_1 & \tilde{B}_2 \end{bmatrix} \\ * & \begin{bmatrix} -I_r & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} \tilde{D}_{11} & \tilde{D}_{12} \\ \tilde{D}_{21} & \tilde{D}_{22} \end{bmatrix} \\ * & * & \begin{bmatrix} -I_r & 0 \\ 0 & -I_m \end{bmatrix} \end{pmatrix} \Psi_P$$

$$< \begin{pmatrix} \begin{bmatrix} I_{s_W} & 0 \\ 0 & \psi_3^T \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{pmatrix} \breve{W} \left(\begin{bmatrix} I_{s_W} & 0 \\ 0 & \psi_3 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right),$$

and

$$\Psi_Q^T \begin{pmatrix} \begin{bmatrix} R & S \\ -S^T & T \end{bmatrix} \begin{bmatrix} A_{\bar{W}} & 0 \\ 0 & \tilde{A} \end{bmatrix} + \{\cdot\}^T & * & * \\ \begin{bmatrix} 0 & \tilde{B}_1^T \\ 0 & \tilde{B}_2^T \end{bmatrix} \begin{bmatrix} I_{s_W} & 0 \\ S^T & T \end{bmatrix} & \begin{bmatrix} -I_r & 0 \\ 0 & -I_m \end{bmatrix} & * \\ \begin{bmatrix} 0 & \tilde{C}_1 \\ B_{\bar{W}}^T & \tilde{C}_2 \end{bmatrix} \begin{bmatrix} R & -S \\ 0 & I_{s_{\tilde{G}}} \end{bmatrix} & \begin{bmatrix} \tilde{D}_{11} & \tilde{D}_{12} \\ \tilde{D}_{21} & \tilde{D}_{22} \end{bmatrix} & \begin{bmatrix} -I_r & 0 \\ 0 & 0 \end{bmatrix} \end{pmatrix} \Psi_Q$$

$$< \begin{pmatrix} \begin{bmatrix} I_{s_W} & 0 \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \\ 0 & I_n \end{bmatrix} \end{pmatrix} \breve{W} \left(\begin{bmatrix} I_{s_W} & 0 \\ 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 \\ 0 & I_n \end{bmatrix} \right),$$

where

$$\Psi_P := \begin{pmatrix} \begin{bmatrix} I_{s_W} & 0 \\ 0 & \psi_1 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & \psi_2 \\ 0 & \psi_3 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} I_r & 0 \\ 0 & I_m \end{bmatrix} \end{pmatrix}, \quad \Psi_Q := \begin{pmatrix} \begin{bmatrix} I_{s_W} & 0 \\ 0 & \psi_4 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & \psi_5 \\ 0 & \psi_6 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} I_r & 0 \\ 0 & I_n \end{bmatrix} \end{pmatrix}$$

and the columns of $\begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{bmatrix}$ (resp. $\begin{bmatrix} \psi_4 \\ \psi_5 \\ \psi_6 \end{bmatrix}$) form bases for the null space of $\begin{bmatrix} \tilde{B}_3^T & \tilde{D}_{13}^T & \tilde{D}_{23}^T \end{bmatrix}$ (resp. $[\tilde{C}_3 \quad \tilde{D}_{31} \quad \tilde{D}_{32}]$).

Proof Omitted for brevity, see [7]. □

The following corollary gives a necessary and sufficient condition for the existence of controllers in $\mathcal{K}_G^{TF}$ of order s_K together with a parametrisation of all such controllers.

Corollary 8.2 *If the conditions stated in Part* (ii) *of Theorem 8.1 hold, then there exist controllers $K \in \mathcal{K}_G^{TF}$ of order s_K satisfying*

$$\left\| \begin{pmatrix} \bar{D} & 0 \\ 0 & I_m \end{pmatrix} \mathcal{F}_l(G, K)^T \begin{pmatrix} \bar{D}^{-1} & 0 \\ 0 & \bar{W}^{-1} \end{pmatrix} \right\|_\infty < 1$$

if and only if

$$\mathrm{rank}\left(\begin{bmatrix} I_{s_W} & S \\ 0 & I_{s_{\tilde{G}}} \end{bmatrix} P \begin{bmatrix} I_{s_W} & 0 \\ S^T & I_{s_{\tilde{G}}} \end{bmatrix} - \begin{bmatrix} R & 0 \\ 0 & T^{-1} \end{bmatrix} \right) \leq s_K.$$

In fact, $K(s) = \left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array}\right]$ *is such a controller if and only if* $\Phi_K := \begin{bmatrix} A_K & B_K \\ C_K & D_K \end{bmatrix}$ *satisfies*

$$F + U^T \Phi_K V + V^T \Phi_K^T U < 0,$$

where F, U and V are defined by

$$F := \begin{pmatrix} X \begin{bmatrix} A_{\bar{W}} & 0 & 0 \\ 0 & \tilde{A}^T & 0 \\ 0 & 0 & 0 \end{bmatrix} + \{\cdot\}^T & X \begin{bmatrix} 0 & B_{\bar{W}} \\ \tilde{C}_1^T & \tilde{C}_2^T \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ \tilde{B}_1 & \tilde{B}_2 \\ 0 & 0 \end{bmatrix} \\ * & \begin{bmatrix} -I_r & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} \tilde{D}_{11} & \tilde{D}_{12} \\ \tilde{D}_{21} & \tilde{D}_{22} \end{bmatrix} \\ * & * & \begin{bmatrix} -I_r & 0 \\ 0 & -I_m \end{bmatrix} \end{pmatrix}$$

$$- \begin{pmatrix} \begin{bmatrix} I_{s_W} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \\ 0 & I_n \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{pmatrix} \breve{W} \left(\begin{bmatrix} I_{s_W} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 \\ 0 & I_n \end{bmatrix} \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right),$$

$$U := \left(\begin{bmatrix} 0 & 0 & I_{s_K} \\ 0 & \tilde{B}_3^T & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 \\ \tilde{D}_{13}^T & \tilde{D}_{23}^T \end{bmatrix} \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right),$$

$$V := \left(\begin{bmatrix} 0 & 0 & I_{s_K} \\ 0 & \tilde{C}_3 & 0 \end{bmatrix} X \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 \\ \tilde{D}_{31} & \tilde{D}_{32} \end{bmatrix} \right),$$

and X is constructed as follows:

- *Define* $Q := \begin{bmatrix} I_{s_W} & 0 \\ S^T & I_{s_{\tilde{G}}} \end{bmatrix} \begin{bmatrix} R^{-1} & 0 \\ 0 & T \end{bmatrix} \begin{bmatrix} I_{s_W} & S \\ 0 & I_{s_{\tilde{G}}} \end{bmatrix}$.
- *Factorise* $P - Q^{-1} = HH^T$ *with* $H \in \mathbb{R}^{(s_W + s_{\tilde{G}}) \times s_K}$.
- *Define* $X := \begin{bmatrix} P & H \\ H^T & I_{s_K} \end{bmatrix}$.

Proof Omitted for brevity, see [7]. □

9 Solution Algorithm

This section summarises a sub-optimal iterative algorithm for solving optimisation problem (2).

Inputs to the algorithm:

- Generalised plant $G(s)$ satisfying the standard assumptions,
- Directionality transfer function matrix $\Upsilon(s) \in \boldsymbol{\Upsilon}^{TF}$.

The solution algorithm:

1. First find a controller $K_0^\star$ which robustly stabilises the interconnection $\mathcal{F}_u\left(\mathcal{F}_l(\hat{G}, K_0^\star), \Delta\right)$ for all $\Delta \in \mathbf{B}\boldsymbol{\Delta}^{TF}$, where

$$\hat{G} := \left[\begin{array}{c|cc} A & B_1 & B_3 \\ \hline C_1 & D_{11} & D_{13} \\ C_3 & D_{31} & 0 \end{array}\right].$$

Set $i = 0$, where i denotes the iteration number, and $\eta_0^\star = \infty$.

2. Increment i by 1.
3. • *During the first few iterations:*

(a) Solve the following convex optimisation problem

$$\min_{\bar{W} \in \boldsymbol{\mathcal{W}}^{TF}} \|\bar{W}\Upsilon\|_2^2$$

such that

$$\inf_{\bar{D} \in \boldsymbol{\mathcal{D}}^{TF}} \left\| \begin{pmatrix} \bar{D} & 0 \\ 0 & I_m \end{pmatrix} \mathcal{F}_l\left(G, K_{i-1}^\star\right)^T \begin{pmatrix} \bar{D}^{-1} & 0 \\ 0 & \bar{W}^{-1} \end{pmatrix} \right\|_\infty < 1$$

pointwise in frequency on a sufficiently dense but finite grid using the reformulation given in optimisation problem (5). Let $v_{W,\omega_k}^\star$ and $\Theta_{\omega_k}^\star$ be the respective values of v_{W,ω_k} and Θ_{ω_k} at every ω_k (these are vector/matrix decision variables in optimisation problem (5)) that achieve the above minimum.

(b) Construct a $\bar{W}^\star \in \boldsymbol{\mathcal{W}}^{TF}$ by fitting a stable minimum-phase transfer function to each magnitude function in $v_{W,\omega_k}^\star$.

(c) Construct a self-adjoint real-rational unit in $\mathscr{RL}_\infty$ which is positive at infinity by fitting real-rational functions to each element in $\Theta_{\omega_k}^\star$. Denote this unit by $\Theta^\star(s)$. Then compute a spectral factor $\bar{D}_i^\star \in \boldsymbol{\mathcal{D}}^{TF}$ for $\Theta^\star(s)$ and model reduce if necessary.

(d) Let $(A_{\bar{W}}, B_{\bar{W}}) \in \boldsymbol{\Xi}_{(A_{\bar{W}}, B_{\bar{W}})}$ and $(A_{\bar{D}}, B_{\bar{D}}) \in \boldsymbol{\Xi}_{(A_{\bar{D}}, B_{\bar{D}})}$ be the A and B matrices of the appropriate state-space realisations of $\bar{W}^\star(s)$ and $\bar{D}_i^\star(s)$ respectively.

• *During the last few iterations:*

(a) Solve the following convex optimisation problem

$$\min_{\bar{W} \in \boldsymbol{\mathcal{W}}^{TF}_{(A_{\bar{W}}, B_{\bar{W}})}} \|\bar{W}\Upsilon\|_2^2$$

such that

$$\inf_{\bar{D} \in \boldsymbol{\mathcal{D}}^{TF}_{(A_{\bar{D}}, B_{\bar{D}})}} \left\| \begin{pmatrix} \bar{D} & 0 \\ 0 & I_m \end{pmatrix} \mathcal{F}_l\left(G, K_{i-1}^\star\right)^T \begin{pmatrix} \bar{D}^{-1} & 0 \\ 0 & \bar{W}^{-1} \end{pmatrix} \right\|_\infty < 1$$

by making use of Theorem 6.1 and Theorem 7.1. Let the value of $\breve{D}$ (a matrix decision variable in the LMI constraints of Theorem 7.1) that achieves the minimum be denoted by $\breve{D}_i^\star$.

(b) Using the previously fixed values of $(A_{\bar{D}}, B_{\bar{D}}) \in \boldsymbol{\Xi}_{(A_{\bar{D}}, B_{\bar{D}})}$ and the value of $\breve{D}_i^\star \in \boldsymbol{\Xi}_{\breve{D}}$ just obtained, define

$$\Theta_i^\star(s) := \begin{bmatrix} B_{\bar{D}}^T(-sI_{s_D} - A_{\bar{D}}^T)^{-1} & I_r \end{bmatrix} \breve{D}_i^\star \begin{bmatrix} (sI_{s_D} - A_{\bar{D}})^{-1} B_{\bar{D}} \\ I_r \end{bmatrix}$$

and compute a spectral factor $\bar{D}_i^\star \in \boldsymbol{\mathcal{D}}^{TF}_{(A_{\bar{D}}, B_{\bar{D}})}$ for $\Theta_i^\star(s)$.

4. *(a)* Solve the following convex optimisation problem

$$\min_{\bar{W} \in \boldsymbol{\mathcal{W}}^{TF}_{(A_{\bar{W}}, B_{\bar{W}})}} \|\bar{W}\Upsilon\|_2^2$$

such that

$$\min_{K \in \boldsymbol{\mathcal{K}}_G^{TF}} \left\| \begin{pmatrix} \bar{D}_i^\star & 0 \\ 0 & I_m \end{pmatrix} \mathcal{F}_l\left(G, K\right)^T \begin{pmatrix} \bar{D}_i^{\star-1} & 0 \\ 0 & \bar{W}^{-1} \end{pmatrix} \right\|_\infty < 1$$

by making use of Theorem 6.1 and Theorem 8.1. Let the value of this minimum cost be denoted by $\eta_i^\star$ and let the value of $\breve{W}$ (a matrix decision variable in the LMI constraints of Theorem 8.1) that achieves this minimum be denoted by $\breve{W}_i^\star$.

(b) Using the previously fixed values of $(A_{\bar{W}}, B_{\bar{W}}) \in \boldsymbol{\Xi}_{(A_{\bar{W}}, B_{\bar{W}})}$ and the value of $\breve{W}_i^\star \in \boldsymbol{\Xi}_{\breve{W}}$ just obtained, define

$$\Pi_i^\star(s) := \begin{bmatrix} B_{\bar{W}}^T(-sI_{s_W} - A_{\bar{W}}^T)^{-1} & I_n \end{bmatrix} \breve{W}_i^\star \begin{bmatrix} (sI_{s_W} - A_{\bar{W}})^{-1} B_{\bar{W}} \\ I_n \end{bmatrix}$$

and compute a spectral factor $\bar{W}_i^\star \in \boldsymbol{\mathcal{W}}^{TF}_{(A_{\bar{W}}, B_{\bar{W}})}$ for $\Pi_i^\star(s)$.

(c) Using Corollary 8.2, find a controller of smallest order that achieves the minimum cost $\eta_i^\star$ obtained in the above optimisation problem. Denote this controller by $K_i^\star$.

5. Evaluate $(\eta_{i-1}^\star - \eta_i^\star)$. If this difference (which is always positive) is very small and has remained very small for the last few iterations, then EXIT. Otherwise return to Step 2.

Outputs from the algorithm: (after i iterations)

- The inverse of the largest performance weights obtained by the algorithm in $\bar{W}_i^\star \in \mathcal{W}^{TF}$,
- The controller $K_i^\star \in \mathcal{K}_G^{TF}$ that achieves robust performance with respect to these weights,
- The final scalings $\bar{D}_i^\star \in \mathcal{D}^{TF}$,
- The value of the minimum cost $\eta_i^\star$ obtained.

10 Numerical Example

The algorithm proposed in the previous section will now be illustrated by a numerical example. The same example used in [1] to illustrate the standard D-K iterative procedure will be used here for ease of comparison. This example considers the design of a pitch axis controller for an experimental highly maneuverable aeroplane, the HIMAT. A block diagram for the closed-loop system is shown in Figure 2. The

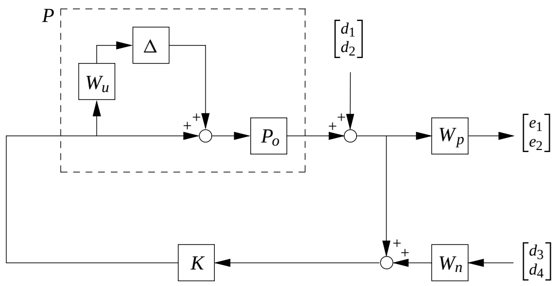

Figure 2: Block diagram of HIMAT and required feedback structure

state-space realisations of P_o, W_u and W_n are given in [1].

It is required to maximise the performance weight W_p subject to the existence of an internally stabilising controller K that guarantees robust performance with respect to this maximised weight. For a sensible control problem, W_p should be maximised in the low-frequency region, thereby achieving disturbance rejection at the plant output. Consequently, the directionality function used in this design example was simply $\frac{5}{(s+0.005)} I_2$.

The results obtained from using this algorithm are depicted in Figure 3 together with the results obtained from the standard D-K iterative procedure, so that comparison can be made. Observe that the final μ-curve obtained from using the proposed algorithm is flat across frequency and very close to unity. This reflects that robust performance has been maximised. In fact, the inverse performance weights synthesised by the proposed algorithm are everywhere less than those used in [1] to explain the standard D-K iterative procedure. That is, a higher level of robust performance is synthesised by the proposed algorithm. Of course, different performance weights then lead to different D-scales and a different internally stabilising controller.

11 Conclusions

The problem of maximising performance weights subject to the existence of an internally stabilising controller that guarantees robust performance with respect to these maximised weights was posed as an optimisation problem in Section 2. This optimisation problem is very similar to the one posed in [8], the important differences being such that the optimisation problem proposed here admits a solution algorithm based on state-space techniques.

The algorithm presented here eliminates all of the disadvantages of the pointwise approach in [8] and considerably enhances the benefits of using such a method. More specifically, (a) the controller is no longer parametrised by a basis function, (b) the order of the controller

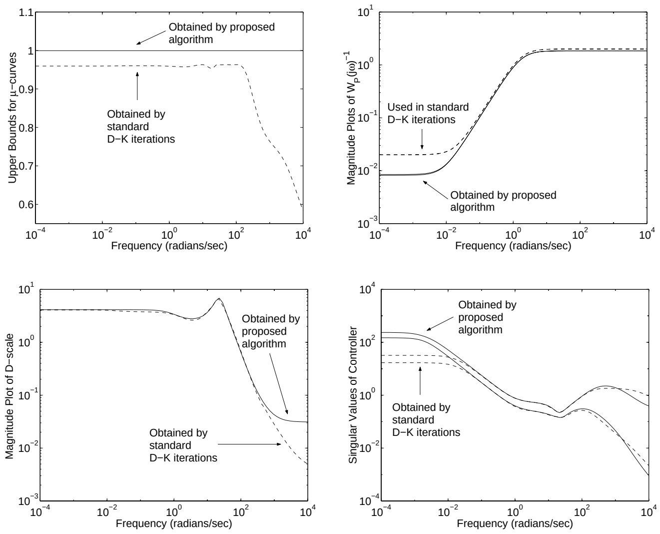

Figure 3: Clockwise from top left: Upper bounds for μ-curves, Magnitude plots of $|w_{11}(j\omega)|^{-1}$ and $|w_{22}(j\omega)|^{-1}$, Magnitude plot of $|d(j\omega)|$, Singular values of controller K

is guaranteed to be less than or equal to that of the scaled generalised plant, (c) frequency gridding is no longer necessary and hence the conditions given here guarantee that $\mu < 1$ rather than simply give confidence, (d) maximisation of performance weights occurs over the entire frequency range from $-\infty$ to ∞, (e) the additional freedom provided by the parametrisation of *all* controllers that achieve the designed robust performance may be exploited by adding additional LMI constraints to simultaneously achieve other closed-loop objectives such as regional pole placement, $\mathscr{H}_2$-norm minimisation, etc.

In summary, this paper presents a conceptually new method for performing μ-synthesis robust performance based designs and is hence a valuable alternative to the standard D-K iterative procedure. The approach presented here greatly simplifies the often long and tedious trial and error process of designing "good" performance weights directly.

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