A Stability Result on the Feedback Interconnection of Negative Imaginary Systems with Poles at the Origin

Mohamed A. Mabrok, Abhijit G. Kallapur, Ian R. Petersen, and Alexander Lanzon

Abstract— **This paper presents a new definition of negative imaginary (NI) systems to include poles at the origin. Also, a necessary and sufficient stability conditions of positive feedback interconnection of NI systems that have poles at the origin with a strictly NI system is provided. As an application of this class of systems, an example of a train system is presented.**

Index Terms— **Negative imaginary systems, flexible systems.**

I. Introduction

Highly resonant structural modes in machines and robots, ground and aerospace vehicles, and precision instrumentation, such as atomic force microscopes and optical systems, can limit the ability of control systems in achieving a desired level of performance [1]. This problem is simplified to some extent by using force actuators combined with collocated measurements of velocity, position, or acceleration.

The use of force actuators combined with velocity measurements has been studied using positive real (PR) theory for linear time invariant (LTI) systems; e.g., see [2], [3]. PR systems, in the single-input single-output (SISO) case, can be defined as systems where the real part of the transfer function is nonnegative. Many systems that dissipate energy fall under the category of PR systems. For instance, they can arise in electric circuits with linear passive components and magnetic couplings. In spite of its success, a drawback of the PR theory is the requirement for the relative degree of the underlying system transfer function to be either zero or one [3]. Hence, the control of flexible structures with force actuators combined with position measurements, cannot use the theory of PR systems.

Lanzon and Petersen introduce a new class of systems in [4] called negative imaginary (NI) systems, which has fewer restrictions on the relative degree of the system transfer function than in the PR case. In the SISO case, such systems are defined by considering the properties of the imaginary part of the frequency response $G(j\omega) = D + C(j\omega I - A)^{-1}B$, and requiring the condition $j\left(G(j\omega) - G(j\omega)^*\right) \geq 0$ for all $\omega \in (0, \infty)$.

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In general, NI systems are stable systems having a phase lag between 0 and $-\pi$ for all $\omega > 0$. That is, their Nyquist plot lies below the real axis when the frequency varies in the open interval $(0, \infty)$ (for strictly negative-imaginary systems, the Nyquist plot should not touch the real axis except at zero frequency and/or at infinity). This is similar to PR systems where the Nyquist plot is constrained to lie in the right half of the complex plane [2], [3]. However, in contrast to PR systems, transfer functions for NI systems can have relative degree greater than unity.

NI systems can be transformed into PR systems and vice versa under some technical assumptions. However, this equivalence is not complete. For instance, such a transformation applied to a strictly negative imaginary (SNI) system always leads to a non-strict PR system. Hence, the passivity theorem [2], [3] cannot capture the stability of the closed-loop interconnection of an NI and an SNI system.

Many practical systems can be considered as NI systems. For example, such systems arise when considering the transfer function from a force actuator to a corresponding collocated position sensor (for instance, a piezoelectric sensor) in a lightly damped structure [1], [4]–[7]. Also, stability results for interconnected systems with an NI frequency response have been applied to the decentralized control of large vehicle platoons in [8]. In [8], the authors discuss the stability of various designs to enhance the robust stability of the system with respect to small variations in coupling gains.

NI systems theory has been extended by Xiong et. al. in [1], [9] by allowing for simple poles on the imaginary axis of the complex plane except at the origin. In addition, it has been shown in [4] that a necessary and sufficient condition for the internal stability of a positive-feedback interconnection of an NI system with transfer function matrix $G_1(s)$ and an SNI system with transfer function matrix $G_2(s)$ is given by the DC gain condition $\lambda_{max}(G_1(0)G_2(0)) < 1$. Here, the notation $\lambda_{max}(\cdot)$ denotes the maximum eigenvalue of a matrix with only real eigenvalues.

A generalization of the NI lemma in [9] to allow for systems with a simple pole at the origin was presented in [10], [11]. In [10], stability analysis for a special class of generalized NI systems with the inclusion of an integrator connected in parallel with an NI system was discussed. The assumption in [10] restricts the application of the proposed stability result to NI systems which can be decomposed into a parallel connection of an NI system (with no pole at the origin) and an integrator. A sufficient condition for the internal stability of a feedback interconnection for NI systems including a sample pole at the origin given in [11].

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In this paper, we extend the results in [1], [4], [9]–[11] for NI systems to allow for the existence of one or two poles at the origin, with a more general structure than allowed in the result of [10], [11]. Also, a necessary and sufficient condition for the internal stability of a feedback interconnection for NI systems including a simple or a double pole at the origin is presented. This extension allows us to stabilize any NI system with a pole at the origin without any parallel decomposition assumption. Such NI systems commonly arise in the case of force actuators and position sensors where the system to be controlled is allowed free body motion.

This paper is further organized as follows: Section II introduces the concept of PR and NI systems and presents a relationship between them. The main results of this paper are presented in Section III. Section IV provides an example to support the main result and the paper is concluded with a summary and remarks on future work in Section V.

II. Preliminaries

In this section, we introduce the definitions of PR and NI systems. We also present a lemma describing the connection between PR and NI systems and some technical results which will be used in deriving the main results of the paper.

The definition of PR systems can be motivated by the study of linear electric circuits composed of resistors, capacitors, and inductors. For a detailed discussion of PR systems, see [2], [3] and references therein.

Definition 1: A square transfer function matrix $F(s)$ is positive real if:

1) $F(s)$ has no pole in $Re[s] > 0$.
2) $F(j\omega) + F(j\omega)^* \geq 0$ for all positive real ω such that $j\omega$ is not a pole of $F(j\omega)$.
3) If $j\omega_0$, is a finite or infinite pole of $F(j\omega)$, it is a simple pole and the corresponding residual matrix $K_0 = \lim_{s \longrightarrow j\omega_0} (s - j\omega_0)F(s)$ is positive semidefinite Hermitian.

To establish the main results of this paper, we consider a new generalized definition for NI systems which allows for poles at the origin as follows:

Definition 2: A square transfer function matrix $G(s)$ is NI if the following conditions are satisfied:

1) $G(s)$ has no pole in $Re[s] > 0$.
2) For all $\omega \geq 0$ such that $j\omega$ is not a pole of $G(s)$, $j\left(G(j\omega) - G(j\omega)^*\right) \geq 0$.
3) If $s = j\omega_0$, $\omega_0 > 0$ is a pole of $G(s)$ then it is a simple pole. Furthermore, if $s = j\omega_0$, $\omega_0 > 0$ is a pole of $G(s)$, the residual matrix $K = \lim_{s \longrightarrow j\omega_0} (s - j\omega_0)jG(s)$ is positive semidefinite Hermitian. If $s = 0$ is a pole of $G(s)$, then it is a simple pole or a double pole. If it is double pole then $\lim_{s \longrightarrow 0} s^2 G(s) \geq 0$.

Definition 3: A square transfer function matrix $G(s)$ is SNI if the following conditions are satisfied:

1) $G(s)$ has no pole in $Re[s] \geq 0$.
2) For all $\omega > 0$, $j\left(G(j\omega) - G(j\omega)^*\right) > 0$.

In order to use advances in the theory of PR systems and the complementary definitions of PR and NI systems to further develop the theory of NI systems, it is useful to establish a lemma which shows the relationship between these notions. In order to do so, we consider the possibility of having a simple pole or double pole at the origin and relax the determinant condition required in a corresponding result given in [9]. This leads to a modification of the relationship between PR and NI systems as follows:

Lemma 1: Consider a square real rational proper transfer function matrix $G(s)$ with state space realization $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ such that $D = D^T$ and the transfer function matrix $\tilde{G}(s) = G(s) - D$. Then the transfer function matrix $G(s)$ is NI if and only if the transfer function matrix $F(s) = s\tilde{G}(s)$ is PR. Here, we assume that any pole-zero cancellation which occurs in $s\tilde{G}(s)$ has been carried out to obtain $F(s)$.

Proof: (Necessity) It is straightforward to show that if $\tilde{G}(s)$ is NI then $G(s)$ is NI and vice-versa; e.g., see [9]. Now suppose that $j\left(\tilde{G}(j\omega) - \tilde{G}(j\omega)^*\right) \geq 0$, for all $\omega > 0$ such that $j\omega$ is not a pole of $G(s)$. Then given any such $\omega > 0$, $F(j\omega) + F(j\omega)^* = j\omega\left(\tilde{G}(j\omega) - \tilde{G}(j\omega)^*\right) \geq 0$, and $\overline{\left(F(j\omega) + F(j\omega)^*\right)} \geq 0$. This means that $F(-j\omega) + F(-j\omega)^* \geq 0$ for all $\omega > 0$ which implies that $F(j\omega) + F(j\omega)^* \geq 0$ for all $\omega < 0$ such that $j\omega$ is not a pole of $G(s)$. Hence, $\left(F(j\omega) + F(j\omega)^*\right) \geq 0$ for all $\omega \in (-\infty, \infty)$ such that $j\omega$ is not a pole of $\tilde{G}(j\omega)$.

Now, consider the case where $j\omega_0$ is a pole of $\tilde{G}(s)$ and $\omega_0 = 0$. In the case that $\tilde{G}(s)$ has only a simple pole at the origin, $F(s) = s\tilde{G}(s)$ will have no pole at the origin because of the pole zero cancellation. This implies that $F(0)$ is finite. Since $F(j\omega) + F(j\omega)^* \geq 0$ for all $\omega > 0$ such that $j\omega$ is not a pole of $G(s)$ and $F(j\omega)$ is continuous, this implies that $F(0) + F(0)^* \geq 0$. In the case where $\tilde{G}(s)$ has a double pole at the origin, $F(s) = s\tilde{G}(s)$ will have a simple pole at the origin because of the pole zero cancellation. Since $\tilde{G}(s)$ is NI, then $\lim_{s \longrightarrow 0} s^2 G(s) \geq 0$ which implies that $\lim_{s \longrightarrow 0} sF(s) \geq 0$.

Also, if $j\omega_0$ is a pole of $\tilde{G}(s)$ and $\omega_0 > 0$, then $\tilde{G}(s)$ can be factored as $\frac{1}{s^2+\omega_0^2}R(s)$, which according to the definition for NI systems implies that the residual matrix $K_0 = \frac{1}{2\omega_0}R(j\omega_0)$ is positive semidefinite Hermitian. Hence, $R(j\omega_0) = R(j\omega_0)^* \geq 0$. Now, the residual matrix of $F(s)$ at $j\omega_0$ with $\omega_0 > 0$ is given by,

$$\begin{aligned} \lim_{s \longrightarrow j\omega_0} (s - j\omega_0)F(s) &= \lim_{s \longrightarrow j\omega_0} (s - j\omega_0)s\tilde{G}(s), \\ &= \lim_{s \longrightarrow j\omega_0} (s - j\omega_0)s\frac{1}{s^2 + \omega_0^2}R(s), \\ &= \frac{1}{2}R(j\omega_0) \end{aligned}$$

which is positive semidefinite Hermitian. Hence, $F(s)$ is positive real.

(Sufficiency) Suppose that $F(s)$ is positive real. Then, $F(j\omega) + F(j\omega)^* \geq 0$ for all $\omega \in (-\infty, \infty)$ such that $j\omega$ is not a pole of $F(s)$. This implies $j\omega\left(\tilde{G}(j\omega) - \tilde{G}(j\omega)^*\right) \geq 0$ for all $\omega \geq 0$ such that $j\omega$ is not a pole of $G(s)$. Then

$\tilde{G}(j\omega) - \tilde{G}(j\omega)^* \geq 0$ for all such $\omega \in [0, \infty)$. In addition, if $j\omega_0$ is a pole of $F(s)$, then it follows from the definition of PR systems that the residual matrix $\lim_{s \longrightarrow j\omega_0} (s - j\omega_0)F(s)$ is positive semidefinite Hermitian. Also,

$$\begin{aligned}\lim_{s \longrightarrow j\omega_0} (s - j\omega_0)F(s) &= \lim_{s \longrightarrow j\omega_0} (s - j\omega_0)s\tilde{G}(s), \\ &= \omega_0 \lim_{s \longrightarrow j\omega_0} (s - j\omega_0)j\tilde{G}(s).\end{aligned}$$

Then using Definition 2, we can conclude that $\tilde{G}(s)$ is NI and hence $G(s)$ is NI. ∎

Now, we present a generalized NI lemma, which allows for a simple pole or a double pole at the origin.

Consider the following LTI system,

$$\dot{x}(t) = Ax(t) + Bu(t), \quad (1)$$
$$y(t) = Cx(t) + Du(t), \quad (2)$$

where, $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}, C \in \mathbb{R}^{m\times n}$, and $D \in \mathbb{R}^{m\times m}$.

Lemma 2: Let $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ be a minimal realization of the transfer function matrix $G(s)$ for the system in (1)-(2). Then, $G(s)$ is NI if and only if $D = D^T$ and there exists a matrix $P = P^T \geq 0$ such that the following LMI is satisfied:

$$\begin{bmatrix} PA + A^T P & PB - A^T C^T \\ B^T P - CA & -(CB + B^T C^T) \end{bmatrix} \leq 0. \quad (3)$$

Furthermore, if $G(s)$ is SNI, then $\det(A) \neq 0$ and there exists a matrix $P > 0$ such that (3) holds.

Proof: Suppose that $G(s)$ is NI, which implies from Lemma 1 that $F(s) = s\tilde{G}(s)$ with state space realization $\left[\begin{array}{c|c} A & B \\ \hline CA & CB \end{array}\right]$ is PR. It follows from Corollary 2 and Corollary 3 in [12] that there exists a matrix $P = P^T \geq 0$, such that the LMI (3) is satisfied.

Conversely , suppose that the LMI (3) is satisfied. Then $F(s)$ is PR via Corollary 1 and Corollary 3 in [12], which implies from Lemma 1 that $G(s)$ is NI. The proof of the last statement follows from Lemma 7 [9]. ∎

III. Main results

The main result of this paper is to generalize the stability results in [10] and combine these results with the main results in [4] to give a complete set of stability conditions. This generalization is stated in Theorem 1. which is the main result of this paper:

Theorem 1: Suppose that the transfer function matrix $G_1(s)$ is strictly proper and NI and the transfer function matrix $G_2(s)$ is SNI. Then, we consider the following two cases:

Case 1) $G_1(s)$ has no pole at the origin and $G_2(\infty) \geq 0$. In this case, the closed-loop positive-feedback interconnection between $G_1(s)$ and $G_2(s)$ is internally stable if and only if $\lambda_{max}(G_1(0)G_2(0)) < 1$.

Case 2) $G_1(s)$ has either a simple pole or a double at the origin such that $\lim_{s\to 0} sG_1(s) > 0$ in case of simple pole or $\lim_{s\to 0} s^2 G_1(s) > 0$ in case of a double pole. In this case, the closed-loop positive-feedback interconnection between $G_1(s)$ and $G_2(s)$ is internally stable if and only if $G_2(0) < 0$ and the matrix $A_1 + B_1 G_2(0) C_1$ is nonsingular.

Proof: The proof of necessity and sufficiency of Case 1) in the theorem has been established in [4], [9]. To prove sufficiency in Case 2) of the theorem, suppose the transfer function matrix $G_1(s)$ with a minimal realization $\left[\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & D_1 \end{array}\right]$ is NI, and $G_2(s)$ with a minimal realization $\left[\begin{array}{c|c} A_2 & B_2 \\ \hline C_2 & D_2 \end{array}\right]$ is SNI. Using Lemma 2, it follows that there exist $P_1 \geq 0$, $P_2 > 0$, $W_i \in \mathbb{R}^{m\times m}$, and $L_i \in \mathbb{R}^{m\times n}$ $(i = 1, 2)$ such that can be written as,

$$\begin{aligned} P_1 A_1 + A_1^T P_1 = -L_1^T L_1, &\quad P_2 A_2 + A_2^T P_2 = -L_2^T L_2, \\ P_1 B_1 - A_1^T C_1^T = -L_1^T W_1, &\quad P_2 B_2 - A_2^T C_2^T = -L_2^T W_2, \\ C_1 B_1 + B_1^T C_1^T = W_1^T W_1, &\quad C_2 B_2 + B_2^T C_2^T = W_2^T W_2, \end{aligned} \quad (4)$$

The internal stability of the closed-loop positive-feedback interconnection of $G_1(s)$ and $G_2(s)$ can be guaranteed by considering the transfer function matrix,

$$T(s) = G_1(s)(I - G_2(s)G_1(s))^{-1}$$

with a corresponding system matrix $\breve{A}$, where,

$$\breve{A} = \begin{bmatrix} A_1 & B_1 C_2 \\ 0 & A_2 \end{bmatrix} + \begin{bmatrix} B_1 D_2 \\ B_2 \end{bmatrix} (I - D_1 D_2)^{-1} \begin{bmatrix} C_1 & D_1 C_2 \end{bmatrix}. \quad (5)$$

Now, we show that the matrix $\breve{A}$ in (5) is Hurwitz; i.e., all the poles of $\breve{A}$ lie in the left-half of the complex plane. Let $T = \begin{bmatrix} P_1 - C_1^T D_2 C_1 & -C_1^T C_2 \\ -C_2^T C_1 & P_2 \end{bmatrix}$ be a candidate Lyapunov matrix. Since $G_2(0) < 0$ and $P_1 \geq 0$, we have

$$P_1 - C_1^T G_2(0) C_1 \geq 0. \quad (6)$$

We claim that

$$P_1 - C_1^T G_2(0) C_1 > 0. \quad (7)$$

In order to prove this claim, consider $M = P_1 - C_1^T G_2(0) C_1 \geq 0$ and $\mathcal{N}(M) = \{x : Mx = 0\}$, where $\mathcal{N}(\cdot)$ denotes the null space. Also, given any $x \in \mathcal{N}(M)$ we have $P_1 x = 0$ and $C_1 x = 0$. Now, consider the equations

$$P_1 A_1 + A_1^T P_1 = -L_1^T L_1, \quad (8)$$
$$B_1^T P_1 - C_1 A_1 = -W_1^T L_1 \quad (9)$$

given in (4). Pre-multiplying and post-multiplying (8) by x^T and x respectively, we get,

$$L_1 x = 0. \quad (10)$$

Also, post-multiplying (8) by x results in

$$P_1 A_1 x = 0. \quad (11)$$

Subsequently, post-multiplying (9) by x, gives

$$C_1 A_1 x = 0. \tag{12}$$

Now, let $y = A_1 x$, which from (11) and (12) gives

$$P_1 y = 0, \qquad C_1 y = 0 \tag{13}$$

which implies $y \in \mathcal{N}(M)$. Thus, we have established that

$$A_1\ \mathcal{N}(M) \subset \mathcal{N}(M) \text{ and } \mathcal{N}(M) \subset \mathcal{N}(C_1). \tag{14}$$

This leads to the fact that $\mathcal{N}(M)$ is a subset of the unobservable subspace of (A_1, C_1); e.g., see Chapter 18 of [13]. It now follows from the minimality of (A_1, B_1, C_1, D_1) that $\mathcal{N}(M) = \{0\}$. Hence, $M = P_1 - C_1^T G_2(0) C_1 > 0$. This completes the proof of the claim.

Now, using this claim, we have

$$\begin{aligned}
&P_2 > 0 \text{ and } P_1 - C_1^T(D_2 + G_2(0) - D_2)C_1 > 0,\\
\Rightarrow &P_2 > 0 \text{ and } P_1 - C_1^T(D_2 + C_2(-A_2)^{-1}B_2)C_1 > 0,\\
\Rightarrow &P_2 > 0 \text{ and } P_1 - C_1^T D_2 C_1 - C_1^T C_2 P_2^{-1} C_2^T C_1 > 0,
\end{aligned}$$

where $B_2 = -A_2^{-1} P_2^{-1} C_2^T$ via Lemma 7 in [9]. It follows that

$$\begin{bmatrix} P_1 - C_1^T D_2 C_1 & -C_1^T C_2 \\ -C_2^T C_1 & P_2 \end{bmatrix} > 0.$$

That is, $T > 0$.

Now, the corresponding Lyapunov inequality is given by,

$$\begin{aligned}
T\breve{A} + \breve{A}^T T = &\begin{bmatrix} P_1 - C_1^T D_2 C_1 & -C_1^T C_2 \\ -C_2^T C_1 & P_2 \end{bmatrix}\\
&\times \begin{bmatrix} A_1 + B_1 D_2 C_1 & B_1 C_2 \\ B_2 C_1 & A_2 \end{bmatrix}\\
&+ \begin{bmatrix} A_1 + B_1 D_2 C_1 & B_1 C_2 \\ B_2 C_1 & A_2 \end{bmatrix}^T\\
&\times \begin{bmatrix} P_1 - C_1^T D_2 C_1 & -C_1^T C_2 \\ -C_2^T C_1 & P_2 \end{bmatrix},\\
= &- \begin{bmatrix} (C_1^T D_2 W_1^T + L_1^T) & C_1^T W_2^T \\ C_2^T W_1^T & (L_2^T) \end{bmatrix}\\
&\times \begin{bmatrix} (W_1 D_2 C_1 + L_1) & W_1 C_2 \\ W_2 C_1 & (L_2) \end{bmatrix}\\
\le &0.
\end{aligned}$$

This implies that $\breve{A}$ has all its poles in the closed left-half of the complex plane [13]. We now show that $\det(\breve{A}) \neq 0$. Indeed, using the assumption that $(A_1 + B_1 G_2(0) C_1)$ is nonsingular, we obtain

$$\begin{aligned}
\det(\breve{A}) &= \det(A_2)\det((A_1 + B_1 D_2 C_1 - B_1 C_2 (A_2)^{-1} B_2 C_1)\\
&= \det(A_2)\det(A_1 + B_1 G_2(0) C_1)\\
&= \det(A_2)\det(A_1 + B_1 G_2(0) C_1)\\
&\neq 0
\end{aligned} \tag{15}$$

since $\det(A_2) \neq 0$. Also, using Lemma 4 in [4] and the fact that $G_1(s)$ is NI and $G_2(s)$ is SNI, we conclude that $\det(I - G_1(j\omega)G_2(j\omega)) \neq 0$ for all $\omega > 0$. This implies that $\breve{A}$ has no eigenvalues on the imaginary axis for $\omega > 0$. Hence, the matrix $\breve{A}$ is Hurwitz. This completes the proof of sufficiency for Case 2).

To prove necessity for Case 2), suppose that the matrix $\breve{A}$ is Hurwitz. It follows that $\det(\breve{A}) \neq 0$ which implies that $\det(A_1 + B_1 G_2(0) C_1) \neq 0$ as in (15). Also, the fact that $\breve{A}$ is Hurwitz implies that $[G_1(s), G_2(s)]$ is internally stable, which leads to the fact that $T(s) = G_1(s)(I - G_2(s)G_1(s))^{-1}$ is SNI via Theorem 2 in [1]. Now, since $T(\infty) = 0$, this implies that $T(0) > 0$ via Lemma 2 in [4].

Now if the pole at the origin is a simple pole, we write

$$\begin{aligned}
&G_1(s) = \frac{1}{s}\bar{G}_1(s),\\
&\Rightarrow T(s) = \bar{G}_1(s)(sI - G_2(s)\bar{G}_1(s))^{-1},\\
&\Rightarrow T(0) = \bar{G}_1(0)(-G_2(0)\bar{G}_1(0))^{-1},\\
&\Rightarrow G_2(0) < 0,
\end{aligned}$$

where $\bar{G}_1(0) > 0$, since $\lim_{s\to 0} sG_1(s) > 0$.

If the pole at the origin is a double pole, we have

$$\begin{aligned}
&G_1(s) = \frac{1}{s^2}\bar{G}_1(s),\\
&\Rightarrow T(s) = \bar{G}_1(s)(s^2 I - G_2(s)\bar{G}_1(s))^{-1},\\
&\Rightarrow T(0) = \bar{G}_1(0)(-G_2(0)\bar{G}_1(0))^{-1},\\
&\Rightarrow G_2(0) < 0,
\end{aligned}$$

where $\bar{G}_1(0) > 0$, since $\lim_{s\to 0} s^2 G_1(s) > 0$. This complete the proof of necessity for Case 2).

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Remark 1: In the case where $G_1(s)$ is the controller which is required to include an integrator and the plant transfer function $G_2(s)$ does not satisfy the condition $G_2(0) < 0$, a negative direct feed-through can be added to the plant to enforce the condition $G_2(0) < 0$; e.g., see [14], [15] for more details including a discussion onthe modelling of flexible structures.

Also, the condition $G_2(0) < 0$ can be satisfied in spring-mass-damper systems with combined acceleration, position, and velocity sensors. For instance, consider the following system

$$\begin{aligned}
\ddot{x} + \alpha\dot{x} + \beta x &= u,\\
y &= \gamma_1 \ddot{x} + \gamma_2 \dot{x} + \gamma_3 x,
\end{aligned}$$

where x is the position state variable and y is the output. Also, $\alpha, \beta, \gamma_1, \gamma_2$, and γ_3 are given constants. The corresponding transfer function for this system is

$$G_2(s) = \frac{\gamma_1 s^2 + \gamma_2 s + \gamma_3}{s^2 + \alpha s + \beta}. \tag{16}$$

For $\alpha > 0, \beta > 0, \gamma_3 < 0$ and any γ_1, γ_2 satisfying $\beta\gamma_2 - \alpha\gamma_3 < 0, \alpha\gamma_1 - \gamma_2 < 0$, the transfer function (16) will be

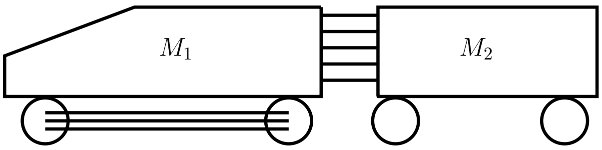

Fig. 1. A train system consisting of an engine with mass M_1 and a car with mass M_2 .

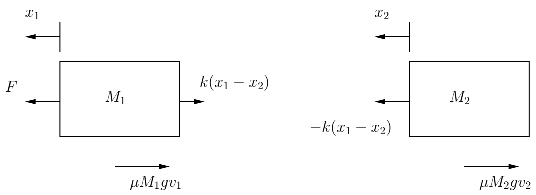

Fig. 2. Free body diagram of the train system depicted in Fig. 1. Here F is the force acting on the engine, k is the spring constant of the spring holding the two cars together, M_1 is the engine mass, M_2 is the mass of the car, and μ is the coefficient of rolling friction.

SNI with $G_2(0) < 0$. For example, if we choose $\alpha = 3, \beta = 5, \gamma_1 = -3, \gamma_2 = -5, \gamma_3 = -5$, the transfer function (16) will be SNI with $G_2(0) = -1$.

IV. ILLUSTRATIVE EXAMPLE

In this section, we will consider an example of a physical system involving a train consisting of an engine and one car as shown in Fig.1. Here, the transfer function from the applied force to the measured position of the train satisfies the NI property and has a simple pole at the origin. For this system the stability results in [4] and [10] are not applicable.

A. Free body diagram

A free body diagram for the train system depicted in Fig. 1 is given in Fig. 2. As shown in Fig. 2, the forces acting on the engine with mass M_1 consist of the forces due to the spring with spring constant k, the friction force with fraction coefficient μ, and the force generated by the engine F. The forces acting on car with mass M_2 are those due to the spring and friction. In the vertical direction, the force due to gravity is canceled by the normal force of the ground acting on the train.

B. Equations of motion

From Newton's second law of motion, the force, mass, and acceleration of the bodies are related according to

$$\begin{aligned} M_1 \ddot{x_1} &= F - k(x_1 - x_2) - \mu M_1 g \dot{x_1}, \\ M_2 \ddot{x_2} &= k(x_1 - x_2) - \mu M_2 g \dot{x_2}. \end{aligned} \tag{17}$$

The equations of motion (17) can be rewritten in state-space form (18) by considering $x_1, x_2, \dot{x_1}$ and $\dot{x_2}$ as state variables. Here, x_1 and x_2 represent the positions of the engine and the car respectively:

$$\begin{bmatrix} \dot{x_1} \\ \ddot{x_1} \\ \dot{x_2} \\ \ddot{x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{K}{M_1} & -\mu g & \frac{K}{M_1} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{K}{M_2} & 0 & -\frac{K}{M_2} & -\mu g \end{bmatrix} \begin{bmatrix} x_1 \\ \dot{x_1} \\ x_2 \\ \dot{x_2} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{M_1} \\ 0 \\ 0 \end{bmatrix} F. \tag{18}$$

The output equation which gives the position of the engine is given by

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ \dot{x_1} \\ x_2 \\ \dot{x_2} \end{bmatrix}. \tag{19}$$

By choosing the following system constants $M_1 = 2\ kg; M_2 = 1\ kg; k = 1\ Nm^{-1}; F = 5N; \mu = .001; g = 9.8\ m/s^2$, the corresponding transfer function from the force input to the position output is given by

$$G_1(s) = \frac{0.5s^2 + 0.0049s + 0.5}{s^4 + 0.0196s^3 + 1.5s^2 + 0.0147s}. \tag{20}$$

This transfer function satisfies the NI property and has a pole at the origin. This fact is illustrated in the plot of the imaginary part of $G_1(j\omega)$ given in Fig. 3 and in the Bode plot given in Fig. 4.

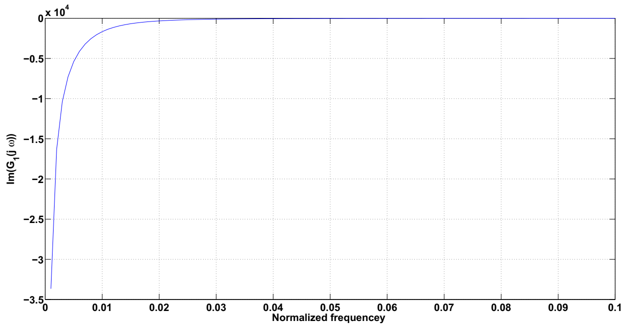

Fig. 3. Imaginary part of $G_1(j\omega)$.

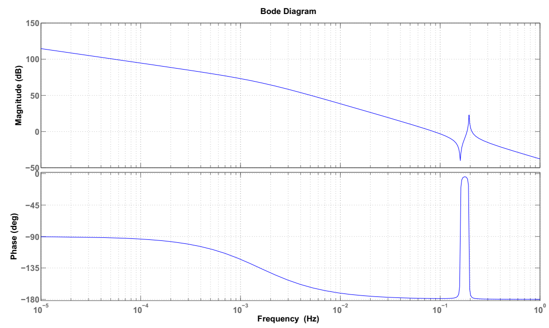

Fig. 4. Bode plot of $G_1(j\omega)$, which shows that the phase lies between 0 and $-\pi$ for all $\omega > 0$.

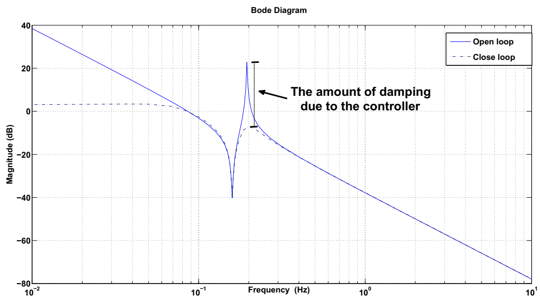

Fig. 5. Open- and closed-loop frequency responses for the train system and an integral resonant controller with a transfer function $G_2(s) = \frac{\Gamma}{s+\Phi\Gamma} - D$, where $\Phi = 5 \times 10^{-3}, D = 200.7$ and $\Gamma = 23 \times 10^3$. These parameters are chosen to provide adequate damping of the resonant mode.

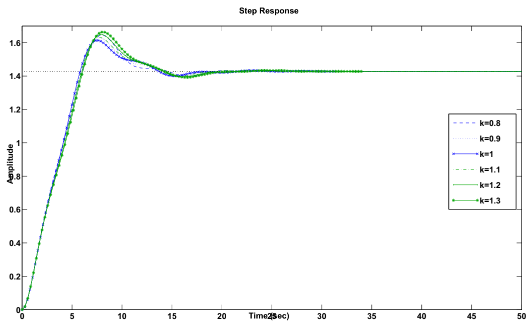

Fig. 6. The step response of the closed-loop system corresponding to an integral resonant controller with a transfer function $G_2(s) = \frac{\Gamma}{s+\Phi\Gamma} - D$, with $k \in [0.8, 1.3]$ where $\Phi = 5 \times 10^{-3}, D = 200.7$ and $\Gamma = 23 \times 10^3$.

C. Controller design

By using the results in Theorem 1, we can stabilize the train system by designing an SNI controller $G_2(s)$ satisfying $G_2(0) < 0$. A modified integral resonant controller is given by

$$G_2(s) = \frac{\Gamma}{s + \Phi\Gamma} - D. \tag{21}$$

This controller will stabilize the system for any $\Phi > 0$ provided $G_2(0) = \frac{1}{\Phi} - D < 0$. Also, a desired level of performance can be obtained by varying the parameter Γ. By choosing $\Phi = 5 \times 10^{-3}, D = 200.7$, and $\Gamma = 23 \times 10^3$, we can achieve adequate damping as depicted in Fig. 5. Also, this controller is robust against plant uncertainty which my may arise due to an uncertain spring constant k. To illustrate this robustness, the step response of the closed-loop system with different values of $k \in [0.8, 1.3]$ is depicted in Fig. 6.

V. Conclusion

In this paper, stability results for a positive-feedback interconnection of negative imaginary (NI) systems have been derived. A generalization of the NI lemma, allowing for a simple pole or double pole at the origin has been used in deriving these results. This work can be used in controller design to allow for a broader class of NI systems than considered in previous work. Also, the stability result for an NI system with poles at the origin connected with an SNI system with positive feedback can be used for controller design including integral action. The validity of the main results in this paper have been illustrated via a physical example corresponding to a train system.

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