A Negative Imaginary Lemma for Descriptor Systems

Mohamed A. Mabrok, Abhijit G. Kallapur, Ian R. Petersen, and Alexander Lanzon

***Abstract*— Flexible structure dynamics with collocated force actuators and position sensors lead to negative imaginary (NI) systems. In this paper, we study the extension of NI theory to descriptor systems. We derive an NI lemma for descriptor systems. An illustrative example is presented to support the result.**

***Index Terms*— Negative imaginary systems, positive real systems, descriptor systems, Negative imaginary lemma.**

I. Introduction

The Kalman Yakubovich Popov lemma and associated results are used to characterize positive real (PR) systems in terms of state-space realizations [1], [2]. PR systems, in the single-input single-output (SISO) case, can be defined as systems where the real part of the transfer function is nonnegative. In general, most systems that dissipate energy fall under the category of PR systems. For instance, they can be realized by electric circuits with passive components and magnetic couplings. However, in spite of their success, a drawback of PR theory is the requirement for the relative degree of the underlying system transfer function to be either zero or one [2]. Due to this restriction on the relative degree of the transfer function, PR systems theory cannot be applied to systems such as flexible structures with collocated force actuators and position sensors [3].

Lanzon and Petersen introduce a new class of systems in [4], [5] called negative imaginary (NI) systems, which have less restriction on the relative degree of the transfer function. In the SISO case, such systems are defined by considering the properties of the imaginary part of the frequency response $G(j\omega) = D + C(j\omega I - A)^{-1}B$, satisfying the condition $j\left(G(j\omega) - G(j\omega)^*\right) \geq 0$ for all $\omega \in (0, \infty)$.

In general, NI systems are stable systems with their frequency response having a phase lag between 0 and $-\pi$ for all $\omega > 0$. That is, the Nyquist plot lies below the real axis when the frequency varies in the open interval $(0, \infty)$ (for strictly negative-imaginary systems, the Nyquist plot should not touch the real axis except at zero frequency and at infinity). This is similar to PR systems where the frequency response is constrained to lie in the right half of the complex plane [1], [2]. However, in constrast to PR systems, transfer functions for NI systems can have relative degree more than unity.

NI systems can be transformed into PR systems and vice versa under certain technical assumptions. However, this equivalence is not complete. For instance, such a transformation applied to a strictly negative imaginary (SNI) system always leads to a non-strict PR system. Hence, the passivity theorem [1], [2] cannot capture the stability of the closed-loop interconnection of an NI and an SNI system. Also, any approach based on strict PR synthesis cannot be used for the control of an NI system, irrespective of whether it is strict or non-strict. Also, transformations of NI systems to bounded-real systems for application of the small-gain theorem also suffer from the exact same difficulty of giving a non-strict bounded real system despite the original system being SNI; see [6] for details.

Many practical systems can be considered as NI systems. For example, when considering the transfer function from a force actuator to a corresponding collocated position sensor (for instance, a piezoelectric sensor) in a lightly damped structure [3]–[5], [7]–[9] and in the case of large vehicle platoons [10]. In [10], the authors apply stability results for interconnecting negative imaginary systems to the decentralized control of large vehicle platoons and demonstrate that various designs can be used to enhance robust stability with respect to small variations of coupling gains.

NI systems theory has been extended by Xiong et. al. in [11]–[13] by allowing for simple poles on the imaginary axis of the complex plane except at the origin. Furthermore, NI controller synthesis has also been discussed in [4], [5]. In addition, it has been shown in [4], [5] that a necessary and sufficient condition for the internal stability of a positive-feedback interconnection of an NI system with transfer function matrix $M(s)$ and an SNI system with transfer function matrix $N(s)$ is given by the DC gain condition $\lambda_{max}(M(0)N(0)) < 1$. Here, the notation $\lambda_{max}(\cdot)$ denotes the maximum eigenvalue of a matrix with only real eigenvalues. In addition, [5] presents an NI lemma which gives a state space characterization of the NI property. This NI lemma is used in the proof of the stability result of [5] and the controller synthesis result of [3]. The more general version of the NI lemma are contained in the papers [11]–[14].

The existing NI lemma is restricted to regular time-invariant linear systems. However, in many applications, the underlying linear systems are in fact singular *descriptor systems* and the existing NI theory cannot be applied to such systems. Another motivation to consider the descriptor form is that it often provides a more natural and general system

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representation than regular state-space systems see; e.g., [15]. The descriptor form is useful in representing systems such as mechanical systems, electric circuits, interconnected systems, and parameter-varying systems, to name a few.

In this paper, we extend the NI lemma in [3]–[5], [11]–[14] to derive an NI lemma for descriptor systems.

This paper is further organized as follows: Section II introduces the concept of PR and NI systems. The main results of this paper are presented in Section III. A numerical example is provided in Section IV. The paper is concluded with a summary and remarks on future work in Section V.

II. Preliminaries

In this section, we introduce the concept of PR and NI systems in terms of previously established definitions and lemmas.

A. Positive Real Systems

The definition of PR systems is motivated by the study of linear electric circuits composed of resistors, capacitors, and inductors. The same definition applies for analogous mechanical and hydraulic systems. This idea can be extended to study electric circuits with nonlinear passive components and magnetic couplings. For a detailed discussion on PR systems, see [1], [2] and references therein.

Definition 1: A transfer function $f(s)$ is said to be positive real if:

1. $f(s)$ is analytic in $Re[s] > 0$.
2. $Re(f(s)) \geq 0$ for all $Re[s] > 0$.
3. $f(s)$ is real for positive real s.

Definition 2: A square transfer function matrix $F(s)$ is positive real if:

1. $F(s)$ has no pole in $Re[s] > 0$.
2. $F(s)$ is real for all positive real s.
3. $F(s) + F(s)^* \geq 0$ for all $Re\ [s] > 0$.

Here $F(s)^*$ denotes the complex conjugate transpose of $F(s)$.

B. Negative Imaginary Systems

Definition 3: [14] A square transfer function matrix $G(s)$ is said to be NI if the following conditions are satisfied:

1) $G(s)$ has no pole in $Re[s] > 0$.
2) For all $\omega \geq 0$, such that $j\omega$ is not a pole of $G(s)$, $j\left(G(j\omega) - G(j\omega)^*\right) \geq 0$.
3) if $s = j\omega_0$ is a pole of $G(s)$ then it is a simple pole. Furthermore, if $\omega_0 > 0$ the residual matrix $K_0 = \lim_{s \longrightarrow j\omega_0} (s - j\omega_0) jG(s)$ is positive semidefinite Hermitian.
4) $s = \infty$ is not a pole of $G(s)$.

The following lemma is the existing NI lemma that we will extend to descriptor systems. Consider the following LTI system,

$$\dot{x}(t) = Ax(t) + Bu(t), \quad (1)$$

$$y(t) = Cx(t) + Du(t), \quad (2)$$

where, $A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}, C \in \mathbb{R}^{m \times n}$, and $D \in \mathbb{R}^{m \times m}$.

Lemma 1: [14] Let (A, B, C, D) be a minimal realization of the transfer function matrix $G(s)$ for the system in (1)-(2). Then, $G(s)$ is NI if and only if there exist matrices $P = P^T \geq 0$, $W \in \mathbb{R}^{m \times m}$, and $L \in \mathbb{R}^{m \times n}$ such that the following LMI is satisfied:

$$\begin{bmatrix} PA + A^T P & PB - A^T C^T \\ B^T P - CA & -(CB + B^T C^T) \end{bmatrix} = \begin{bmatrix} -L^T L & -L^T W \\ -W^T L & -W^T W \end{bmatrix} \leq 0. \quad (3)$$

C. Descriptor Systems

Here, we consider some basic concepts concerning descriptor systems, which can be found in [15], [16]. Consider the following LTI descriptor system.

$$E\dot{x}(t) = Ax(t) + Bu(t), \quad (4)$$

$$y(t) = Cx(t) + Du(t), \quad (5)$$

where, $E, A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}, C \in \mathbb{R}^{m \times n}$, and $D \in \mathbb{R}^{m \times m}$.

A pair (E, A) is said to be regular if $\det(sE - A)$ is not identically zero, and is said to be impulsive-free if $deg \det(sE - A)$ is equal to rank(E). The zeros of $\det(sE - A)$ are called finite poles of (E, A). A pair (E, A) is called stable if and only if all the finite poles of (E, A) lie in $Re[s] < 0$, and (E, A) is called admissible if it is regular, impulsive-free and stable.

D. Reduction to Weierstrass Form

To derive the main results in the paper, we introduce the Weierstrass Form transformation [17]. The Weierstrass transformation shows that the matrices A and E can be assumed to be in a special form. The matrices A and E are real $n \times n$ matrices and $A - sE$ is assumed to be a regular pencil. For any such pair of matrices A and E there exist real nonsingular matrices T and Q such that [17]

$$QAT = \begin{bmatrix} A_1 & 0 \\ 0 & I_{n-q} \end{bmatrix}, QET = \begin{bmatrix} I_q & 0 \\ 0 & N \end{bmatrix}, \quad (6)$$

where $N \in R^{(n-q) \times (n-q)}$, $A_1 \in R^{q \times q}$ and $N^l = 0$ for some integer $l \geq 1$.

III. Main results

The main result of this paper is to extend Lemma 1 to include descriptor systems.

Theorem 1: Suppose the descriptor system (4)-(5) such that $A - sE$ is a regular pencil, $rank[A - sE \;\; B] = n$ and $rank[A^T - sE^T \;\; C^T] = n$. Then the corresponding transfer function matrix $G(s)$ is NI if:

1) The matrix N in (6) satisfies $N = 0$,
2) There exists a matrix $P = P^T \geq 0$ such that the following LMIs are satisfied:

$$\begin{bmatrix} PA + A^T P & PB - A^T E^T C^T \\ B^T P - CEA & -(CEB + B^T E^T C^T) \end{bmatrix} \leq 0,$$
$$PE^T = EP. \quad (7)$$

Proof: Without loss of generality, we can assume that the matrices A and E are in Weierstrass form:

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & I_{n-q} \end{bmatrix}, E = \begin{bmatrix} I_q & 0 \\ 0 & N \end{bmatrix} \tag{8}$$

and the positive semidefinite matrix P takes the form

$$P = \begin{bmatrix} P_1 & P_3 \\ P_3^T & P_2 \end{bmatrix} \geq 0. \tag{9}$$

It follows that

$$(sE - A)^{-1} = \begin{bmatrix} (sE - A_1)^{-1} & 0 \\ 0 & (sN - I)^{-1} \end{bmatrix}. \tag{10}$$

Also, since $N^l = 0$ for some $l \geq 1$, we have

$$(sN - I)^{-1} = -\sum_{i=0}^{\infty} N^i s^i = -\sum_{i=0}^{l-1} N^i s^i. \tag{11}$$

Then, the transfer function of the descriptor system (4)-(5) takes the form

$$G(s) = D + C_1(sI - A_1)^{-1}B_1 - C_2(\sum_{i=0}^{l-1} N^i s^i)B_2 \tag{12}$$

where

$$B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}; C = \begin{bmatrix} C_1 & C_2 \end{bmatrix}. \tag{13}$$

Now, since $N = 0$ and the matrix $A - sE$ is a regular pencil, it follows from (12) that the transfer function $G(s)$ takes the form

$$G(s) = D + C_1(sI - A_1)^{-1}B_1 - C_2B_2 \tag{14}$$

$$= D_1 + C_1(sI - A_1)^{-1}B_1, \tag{15}$$

where $D_1 = D - C_2B_2$.

Substituting (8) and (9) with $N = 0$ in the equation $PE^T = EP$, it follows that

$$P = \begin{bmatrix} P_1 & 0 \\ 0 & 0 \end{bmatrix} \geq 0. \tag{16}$$

Substituting (8) and (16) in the LMI

$$\begin{bmatrix} PA + A^TP & PB - A^TE^TC^T \\ B^TP - CEA & -(CEB + B^TE^TC^T) \end{bmatrix} \leq 0,$$

we get:

$$\begin{bmatrix} P_1A_1 + A_1^TP_1 & 0 & P_1B_1 - A_1^TC_1^T \\ 0 & 0 & 0 \\ B_1^TP_1 - C_1A_1 & 0 & -(C_1B_1 + B_1^TC_1^T) \end{bmatrix} \leq 0. \tag{17}$$

This implies that

$$\begin{bmatrix} P_1A_1 + A_1^TP_1 & PB_1 - A_1^TC_1^T \\ B_1^TP_1 - C_1A_1 & -(C_1B_1 + B_1^TC_1^T) \end{bmatrix} \leq 0. \tag{18}$$

Finally, Substituting (8) into $[A - sE \;\; B]$ and $[A^T - sE^T \;\; C^T]$ we get

$$[A - sE \;\; B] = \begin{bmatrix} A_1 - sI & 0 & B_1 \\ 0 & I_{n-q} & B_2 \end{bmatrix}, \tag{19}$$

$$[A^T - sE^T \;\; C^T] = \begin{bmatrix} A_1^T - sI & 0 & C_1^T \\ 0 & I_{n-q} & C_2^T \end{bmatrix}. \tag{20}$$

Since, $rank[A - sE \;\; B] = n$ and $rank[A^T - sE^T \;\; C^T] = n$ this implies that $rank[A_1 - sI \;\; B_1] = q$ and $rank[A_1^T - sI \;\; C_1^T] = q$ which implies that the state space realization $\left[\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & D_1 \end{array}\right]$ is minimal.

It now follows from the LMI (18) and Lemma 1 that the transfer function matrix $G(s)$ is NI. ■

Remark 1: Using the Weierstrass form in (8) and (13), a necessary and sufficient condition for the descriptor system to be NI can be given as follows:

Consider the transfer function matrix $G(s)$ for the system (4), (5), (8) and (13) with $A - sE$ a regular pencil, (A_1, B_1) controllable and (A_1, C_1) observable. Then $G(s)$ is NI if and only if:

1) The matrix $C_2N^lB_2 = 0$ for all $l \geq 1$.
2) There exist matrices $P_1 = P_1^T \geq 0$, $W \in \mathbb{R}^{m \times m}$, and $L \in \mathbb{R}^{m \times n}$ such that the following LMI is satisfied:

$$\begin{bmatrix} P_1A_1 + A_1^TP_1 & P_1B_1 - A_1^TC_1^T \\ B_1^TP_1 - C_1A_1 & -(C_1B_1 + B_1^TC_1^T) \end{bmatrix} = \begin{bmatrix} -L^TL & -L^TW \\ -W^TL & -W^TW \end{bmatrix}. \tag{21}$$

This fact follows directly from (12) and Lemma 1.

IV. Illustrative Example

To illustrate the results in this paper, consider a descriptor system with the following state space representation:

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}; E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}; C = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}; D = 0.$$

Note that this descriptor system is already in Weierstrass form.

For this descriptor system, the transfer function is:

$$G(s) = \left(\frac{2s + 3}{s^2 + 3s + 2} \right). \tag{22}$$

By solving the LMI in (7), we get the following positive semidefinite solution:

$$P = \begin{bmatrix} 2.484 & -2.969 & 0 \\ -2.969 & 7.939 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

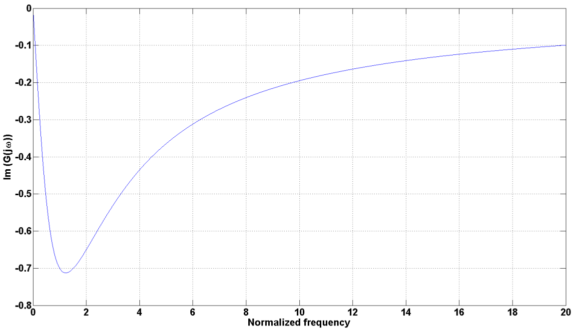

Fig. 1. The imaginary part of the frequency response of $G(j\omega)$.

which implies that the system is NI using Theorem 1.

Also, a plot of the imaginary part of the transfer function (22) is depicted in Fig 1, which verifies that the system is NI.

V. Conclusion

In this paper, we presented an extension of the NI lemma for regular time-invariant linear systems to the singular (descriptor) case. The NI lemma for descriptor systems is given by formulating LMI conditions in a similar fashion to the corresponding regular case. The results in this article can be employed to check for the negative imaginary property of descriptor systems via solution to certain LMIs.

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