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Stability Analysis for a Class of Negative Imaginary Feedback Systems Including an Integrator

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***Abstract*—In this paper, we study a class of negative imaginary linear time invariant multiple-input multiple-output systems . We generalize an existing negative imaginary lemma to include systems containing a simple pole at the origin. Also, a stability analysis result is presented for generalized negative imaginary systems including an integrator and an illustrative example is presented to support the results.**

***Index Terms*—Negative imaginary systems, Positive real systems, Internal stability, Systems with an Integrator.**

I. INTRODUCTION

Passivity theory and positive real (PR) linear time invariant (LTI) systems have been well researched in both theory and practice; e.g., see [1], [2]. PR systems in the single-input single-output (SISO) case can be defined as systems where the real part of the transfer function is non-negative. In general, most systems that dissipate energy fall under the category of PR systems. For instance, they can be realized by electric circuits with linear and nonlinear passive components and magnetic couplings. In spite of its success, a drawback of PR theory is the requirement for the relative degree of the underlying system transfer function to be either zero or one [2], and they cannot be applied to systems involving flexible structures with collocated force actuators and position sensors; e.g., see [3].

Negative imaginary (NI) systems were introduced by Lanzon and Petersen in [4], [5]. In the SISO case, such systems are defined so that the imaginary part of the transfer function $G(j\omega) = D + C(j\omega I - A)^{-1}B$, satisfies the condition $j\left(G(j\omega) - G(j\omega)^*\right) \geq 0$ for all $\omega \in (0, \infty)$.

In general, NI systems are stable systems with their Nyquist plots having a phase lag between 0 and $-\pi$ for all $\omega > 0$. That is, they lie below the real axis when the frequency varies in the open interval $(0, \infty)$ (for strictly negative-imaginary systems, the Nyquist plot should not touch the real axis except at zero frequency and at infinity). This is similar to PR systems where the frequency response is constrained to lie in the right half of the complex plane [1], [2]. However, in constrast to PR systems, transfer functions for NI systems can have relative degrees more than unity.

NI systems can be transformed into PR systems and vice versa under some technical assumptions. However, this equivalence is not complete. For instance, such a transformation applied to a strictly negative imaginary (SNI) system always leads to a non-strict PR system. Hence, the passivity theorem [1], [2] cannot capture the stability of the closed-loop interconnection of an NI and an SNI system. Also, any approach based on strictly PR synthesis cannot be used for the control of an NI system irrespective of whether it is strict or non-strict. Also, transformations of NI systems to bounded-real systems for application of the small-gain theorem also suffer from the exact same difficult of giving a non-strict bounded real system despite the original system being SNI, see [6] for details.

Xiong et. al. extended the NI systems theory in [7]–[10] by allowing for simple poles on the imaginary axis of the complex plane except at the origin. Furthermore, NI controller synthesis has also been discussed in [4], [5]. In addition, it has been shown in [4], [5] that a necessary and sufficient condition for the internal stability of a positive-feedback interconnection of an NI system with transfer function matrix $M(s)$ and an SNI system with transfer function matrix $N(s)$ is given by the DC gain condition $\lambda_{max}(M(0)N(0)) < 1$. Here, the notation $\lambda_{max}(\cdot)$ denotes the maximum eigenvalue of a matrix with only real eigenvalues. The the stability results in [3]–[5], [7]–[10] almost depend on the DC gain condition which is not helpful when we include the integrator.

Many practical systems can be considered as NI systems. For example, when considering the transfer function from a force actuator to a corresponding collocated position sensor (for instance, a piezoelectric sensor) in a lightly damped structure [3]–[5], [11]–[13] and in the case of large vehicle platoons [14], the NI property will often arise.

In this paper, we extend the results in [3]–[5], [7]–[10] for NI systems to allow for the existence of a pole at the origin. This extension of NI systems with a pole at the origin can be applied to controller design with integral action in a similar fashion as in linear quadratic Gaussian integral control [15].

This paper is organized as follows: Section II introduces the concept of PR and NI systems and presents a relationship between them. The main results of this paper are presented in Section III. Section IV provides a numerical example and the paper is concluded with a summary and remarks on future work in Section V.

II. Preliminaries

In this section, we introduce the definitions of PR and NI systems. We also present a lemma describing the transformation between PR and NI systems, and some technical results which will be used in deriving the main results of the paper.

The definition of PR systems has been motivated by the study of linear electric circuits composed of resistors, capacitors, and inductors. For a detailed discussion on PR systems see [1], [2] and references therein.

Definition 1: A square transfer function matrix $F(s)$ is positive real if:

1) $F(s)$ has no pole in $Re[s] > 0$.
2) $F(j\omega) + F(j\omega)^* \geq 0$ for all positive real $j\omega$ such that $j\omega$ is not a pole of $F(j\omega)$.
3) If $j\omega_0$, finite or infinite, is a pole of $F(j\omega)$, it is a simple pole and the corresponding residual $K_0 = \lim_{s \longrightarrow j\omega_0} (s - j\omega_0)F(s)$ is a positive semidefinite Hermitian matrix.

To establish the main results of this paper, we consider a generalized definition for NI systems which allows for a simple pole at the origin as follows:

Definition 2: A square transfer function matrix $G(s)$ is NI if the following conditions are satisfied:

1) $G(s)$ has no pole in $Re[s] > 0$.
2) For all $\omega \geq 0$, such that $j\omega$ is not a pole of $G(s)$, $j\left(G(j\omega) - G(j\omega)^*\right) \geq 0$.
3) if $s = j\omega_0$ is a pole of $G(s)$ then it is a simple pole. Furthermore, if $\omega_0 > 0$ the residual matrix $K_0 = \lim_{s \longrightarrow j\omega_0} (s - j\omega_0)jG(s)$ is positive semidefinite Hermitian.

Definition 3: A square transfer function matrix $G(s)$ is SNI if the following conditions are satisfied:

1) $G(s)$ has no pole in $Re[s] \geq 0$.
2) For all $\omega \geq 0$, $j\left(G(j\omega) - G(j\omega)^*\right) > 0$.

Due to the advances in the theory of PR systems and the complementary definitions of PR and NI systems, it is useful to establish a lemma which considers the relationship between these systems to further develop the theory of NI systems. In order to do so, we consider the possibility of having a simple pole at the origin, and relax the condition $\det(A) \neq 0$ in [5], [7], [10]. This leads to a modification of the relationship between PR and NI systems as follows:

Lemma 1: Given a real rational proper transfer function matrix $G(s)$ with state space realization $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ and the transfer function matrix $\tilde{G}(s) = G(s) - D$, the transfer function matrix $G(s)$ is NI if and only if the transfer function matrix $F(s) = s\tilde{G}(s)$ is PR. Here, we assume that any pole zero cancellation which occurs in $s\tilde{G}(s)$ have been carried out to obtain $F(s)$.

Proof: (Necessity) It is straightforward to show that if $\tilde{G}(s)$ is NI then $G(s)$ is NI and vice-versa. Suppose that $j\left(\tilde{G}(j\omega) - \tilde{G}(j\omega)^*\right) \geq 0$, for all $\omega > 0$ such that $j\omega$ is not a pole of $G(s)$. Then given any such $\omega > 0$, $F(j\omega) + F(j\omega)^* = j\omega\left(\tilde{G}(j\omega) - \tilde{G}(j\omega)^*\right) \geq 0$, and $\overline{(F(j\omega) + F(j\omega)^*)} \geq 0$. This means that $F(-j\omega) + F(-j\omega)^* \geq 0$ for all and $\omega > 0$ which implies that $F(j\omega) + F(j\omega)^* \geq 0$ for all $\omega < 0$ such that $j\omega$ is not a pole of $G(s)$. Hence, $(F(j\omega) + F(j\omega)^*) \geq 0$ for all $\omega \in (-\infty, \infty)$ such that $j\omega$ is not a pole of $\tilde{G}(j\omega)$.

Now, consider the case where $j\omega_0$ is a pole of $\tilde{G}(s)$ and $\omega_0 = 0$. Since $\tilde{G}(s)$ has only a simple pole at the origin, $F(s) = s\tilde{G}(s)$ will have no pole at the origin because of the pole zero cancellation. This implies that $F(0)$ is finite. Since $F(j\omega) + F(j\omega)^* \geq 0$ for all $\omega > 0$ and $F(j\omega)$ is continuous, this implies that $F(0) + F(0)^* \geq 0$. Also, if $j\omega_0$ is a pole of $\tilde{G}(s)$ and $\omega_0 > 0$, then $\tilde{G}(s)$ can be factored as $\frac{1}{s^2+\omega_0^2}R(s)$, which according to the definition for NI systems implies that the residual matrix $K_0 = \frac{1}{2\omega_0}R(j\omega_0)$ is positive semidefinite Hermitian. This implies that $R(j\omega_0) = R(j\omega_0)^* \geq 0$. Now, the residual matrix of $F(s)$ at $j\omega_0$ with $\omega_0 > 0$ is given by,

$$\begin{aligned}\lim_{s \longrightarrow j\omega_0} (s - j\omega_0)F(s) &= \lim_{s \longrightarrow j\omega_0} (s - j\omega_0)s\tilde{G}(s), \\ &= \lim_{s \longrightarrow j\omega_0} (s - j\omega_0)s\frac{1}{s^2 + \omega_0^2}R(s), \\ &= \frac{1}{2}R(j\omega_0)\end{aligned}$$

which is positive semidefinite Hermitian. Hence, $F(s)$ is positive real.

(Sufficiency) Suppose that $F(s)$ is positive real. Then, $F(j\omega) + F(j\omega)^* \geq 0$ for all $\omega \in (-\infty, \infty)$ such that $j\omega$ is not a pole of $F(s)$. This implies $j\omega\left(\tilde{G}(j\omega) - \tilde{G}(j\omega)^*\right) \geq 0$ for all $\omega \geq 0$ such that $j\omega$ is not a pole of $G(s)$. Then $\tilde{G}(j\omega) - \tilde{G}(j\omega)^* \geq 0$ for all such $\omega \in [0, \infty)$. In addition, if $j\omega_0$ is a pole of $F(s)$, then it follows from the definition of PR systems that the residual matrix $\lim_{s \longrightarrow j\omega_0} (s - j\omega_0)F(s)$ is positive semidefinite Hermitian. Also,

$$\begin{aligned}\lim_{s \longrightarrow j\omega_0} (s - j\omega_0)F(s) &= \lim_{s \longrightarrow j\omega_0} (s - j\omega_0)s\tilde{G}(s), \\ &= \omega_0 \lim_{s \longrightarrow j\omega_0} (s - j\omega_0)j\tilde{G}(s).\end{aligned}$$

Then using Definition 2, we can conclude that $\tilde{G}(s)$ is NI. ■

Remark 1: Note that the pole zero cancellation at the origin $F(s) = s\tilde{G}(s)$ will not affect the use of the PR lemma since the minimality condition is relaxed in the generalized version of the PR lemma [16], [17].

In studying the internal stability of an interconnection of NI and SNI systems, we shall use the following SNI lemma:

Lemma 2: [5], [7], [10] Suppose that the proper transfer function matrix $G(s) = C(sI - A)^{-1}B + D$ with a minimal realization (A, B, C, D) is SNI, then the following conditions are satisfied:

1) $\det(A) \neq 0$, $D = D^T$.
2) There exists a square matrix $P = P^T > 0$, $W \in \mathbb{R}^{m\times m}$ and $L \in \mathbb{R}^{m\times n}$ such that the following LMI is satisfied:

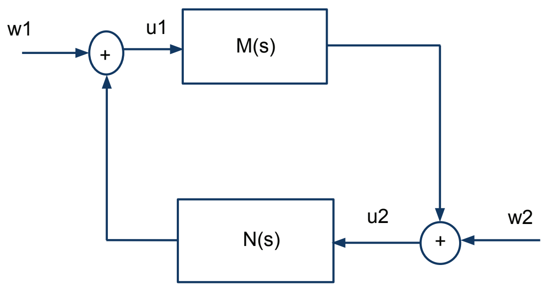

Fig. 1. Transfer functions M(s) and N(s) interconnected via positive feedback.

$$\begin{bmatrix} PA + A^TP & PB - A^TC^T \\ B^TP - CA & -(CB + B^TC^T) \end{bmatrix} = \begin{bmatrix} -L^TL & -L^TW \\ -W^TL & -W^TW \end{bmatrix}. \quad (1)$$

The following lemma shows that the transfer function of a closed-loop system corresponding to the positive feedback interconnection of two NI systems is NI. Furthermore, if, in addition, either one of the systems is SNI, then the transfer function of the closed-loop system is SNI.

Lemma 3: (See Theorem 2 of [3]) Consider NI transfer function matrices $M(s)$ and $N(s)$ and suppose that the positive-feedback interconnection shown in Fig. 1 is internally stable. Then the corresponding closed-loop transfer function matrix,

$$T(s) = \begin{bmatrix} M(s)\Xi(s) & M(s)\Xi(s)N(s) \\ N(s)\Xi(s)M(s) & N(s)\Xi(s) \end{bmatrix} \quad (2)$$

is NI. Here, $\Xi(s) = (I - N(s)M(s))^{-1}$. Furthermore, if, in addition, either $M(s)$ or $N(s)$ is SNI, then $T(s)$ in (2) is SNI.

The following lemma is a modification of Theorem 5 in [5], which outlines conditions for a positive feedback interconnection of an NI and an SNI system to be internally stable.

Lemma 4: Consider an NI transfer function matrix $M(s)$ with minimal state space realization $(A, B, C, 0)$ and an SNI transfer function matrix $N(s)$ with minimal state space realization $(\bar{A}, \bar{B}, \bar{C}, \bar{D})$, and suppose that $\det(A) \neq 0$. Then the positive-feedback interconnection shown in Fig. 1 is internally stable if and only if $N(0) \leq 0$.

Proof: Since $M(s)$ is NI with $\det(A) \neq 0$ and $N(s)$ is SNI, it follows from [7], [10] that there exist real matrices $Y > 0$ and $\bar{Y} > 0$ such that,

$$AY + YA^* \leq 0 \text{ and } B = -AYC^*, \quad (3)$$

$$\bar{A}\bar{Y} + \bar{Y}\bar{A}^* \leq 0 \text{ and } \bar{B} = -\bar{A}\bar{Y}\bar{C}^*. \quad (4)$$

According to [5], $[M(s), N(s)]$ is internally stable if and only if the matrix

$$T = \begin{bmatrix} Y^{-1} - C^*\bar{D}C & -C^*\bar{C} \\ -\bar{C}^*C & \bar{Y}^{-1} \end{bmatrix} \quad (5)$$

is positive definite. Now, $T > 0$ if and only if,

$$\begin{aligned} & Y^{-1} - C^*\bar{D}C - C^*\bar{C}\bar{Y}\bar{C}^*C > 0, \\ \Leftrightarrow\ & Y^{-1} - C^*(\bar{D} + \bar{C}\bar{Y}\bar{C}^*)C > 0, \\ \Leftrightarrow\ & Y^{-1} - C^*N(0)C > 0. \end{aligned}$$

This is because $\bar{C}\bar{Y}\bar{C}^* = N(0) - \bar{D}$ via [5]. This shows that $T > 0$ if $N(0) \leq 0$. ∎

Also, consider the following lemma, which will be used to derive the main results of this paper in Section III,

Lemma 5: [5] Given $A \in \mathbb{C}^{n\times n}$ with $j(A - A^*) \geq 0$ and $B \in \mathbb{C}^{n\times n}$ with $j(B - B^*) > 0$, then $\det(I - AB) \neq 0$.

III. Main Results

The key result of this paper is to derive the stability conditions for an interconnection between an NI system with a simple pole at the origin and an SNI system. This requires the generalization of the NI lemma in [7] to include systems with a simple pole at the origin.

Consider the following LTI system,

$$\dot{x}(t) = Ax(t) + Bu(t), \quad (6)$$

$$y(t) = Cx(t) + Du(t), \quad (7)$$

where, $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}, C \in \mathbb{R}^{m\times n}$, and $D \in \mathbb{R}^{m\times m}$.

Theorem 1: Let (A, B, C, D) be a minimal realization of the transfer function matrix $G(s) \in \mathbb{R}^{m\times m}$ for the system in (6)-(7). Then, $G(s)$ is NI if and only if there exist matrices $P = P^T \geq 0$, $W \in \mathbb{R}^{m\times m}$, and $L \in \mathbb{R}^{m\times n}$ such that the following LMI is satisfied:

$$\begin{bmatrix} PA + A^TP & PB - A^TC^T \\ B^TP - CA & -(CB + B^TC^T) \end{bmatrix} = \begin{bmatrix} -L^TL & -L^TW \\ -W^TL & -W^TW \end{bmatrix} \leq 0. \quad (8)$$

Proof: Suppose that $G(s)$ is NI, which implies from Lemma 1 that $F(s) = s\tilde{G}(s)$ with state space realization $\left[\begin{array}{c|c} A & B \\ \hline CA & CB \end{array}\right]$ is PR. It follows from Corollary 2 and Corollary 3 in [17] that there exists a matrix $P = P^T \geq 0$, such that the LMI (8) is satisfied.

On the other hand, suppose that LMI (8) is satisfied, then $F(s)$ is PR via Corollary 1 and Corollary 3 in [17], which implies from Lemma 1 that $G(s)$ is NI. ∎

To derive the conditions for internal stability, suppose the transfer function matrix $G_1(s)$ with a minimal realization (A_1, B_1, C_1, D_1) is NI, and $G_2(s)$ with a minimal realization (A_2, B_2, C_2, D_2) is SNI. According to Theorem 1 and Lemma 2, we have,

$$\begin{aligned} P_1A_1 + A_1^TP_1 &= -L_1^TL_1, & P_2A_2 + A_2^TP_2 &= -L_2^TL_2, \\ P_1B_1 - A_1^TC_1^T &= -L_1^TW_1, & P_2B_2 - A_2^TC_2^T &= -L_2^TW_2, \\ C_1B_1 + B_1^TC_1^T &= W_1^TW_1, & C_2B_2 + B_2^TC_2^T &= W_2^TW_2. \end{aligned}$$

The internal stability of the closed-loop positive-feedback interconnection of $G_1(s)$ and $G_2(s)$ can be achieved by considering the stability of the transfer function matrix,

$$(I - G_1(s)G_2(s))^{-1} = \breve{D} + \breve{C}(sI - \breve{A})^{-1}\breve{B},$$

where,

$$\breve{A} = \begin{bmatrix} A_1 & B_1 C_2 \\ 0 & A_2 \end{bmatrix} + \begin{bmatrix} B_1 D_2 \\ B_2 \end{bmatrix} (I - D_1 D_2)^{-1} \begin{bmatrix} C_1 & D_1 C_2 \end{bmatrix}, \tag{9}$$

$$\breve{B} = \begin{bmatrix} B_1 D_2 \\ B_2 \end{bmatrix} (I - D_1 D_2)^{-1}, \tag{10}$$

$$\breve{C} = (I - D_1 D_2)^{-1} \begin{bmatrix} C_1 & D_1 C_2 \end{bmatrix}, \tag{11}$$

$$\breve{D} = (I - D_1 D_2)^{-1},$$

and $\breve{A}$ is Hurwitz.

Theorem 2: Suppose that $G_1(s)$ Suppose that $G_1(s)$ is NI and a pure integrator; i.e, $A_1 = 0$ and $G_2(s)$ is SNI. Then, the positive-feedback interconnection of $G_1(s)$ and $G_2(s)$ is internally stable if $G_2(0) < 0$ and $\det(B_1 G_2(0) C_1) \neq 0$.

Proof: $[G_1(s), G_2(s)]$ is internally stable if and only if the matrix $\breve{A}$ defined in (9), which becomes

$$\breve{A} = \begin{bmatrix} B_1 D_2 C_1 & B_1 C_2 \\ B_2 C_1 & A_2 \end{bmatrix}$$

is Hurwitz. Suppose that $T = \begin{bmatrix} -C_1^T D_2 C_1 & -C_1^T C_2 \\ -C_2^T C_1 & P_2 \end{bmatrix}$. Then $T > 0$

$$\begin{aligned} &\Rightarrow \; -C_1^T D_2 C_1 > C_1^T C_2 P_2^{-1} C_2^T C_1, \\ &\Rightarrow \; C_1^T D_2 C_1 + C_1^T C_2 P_2^{-1} C_2^T C_1 < 0, \\ &\Rightarrow \; D_2 + C_2 P_2^{-1} C_2^T < 0, \\ &\Rightarrow \; D_2 + G_2(0) - D_2 < 0, \\ &\Rightarrow \; G_2(0) < 0. \end{aligned}$$

Now, consider the quantity,

$$\begin{aligned} &T\breve{A} + \breve{A}^T T \\ &= \begin{bmatrix} -C_1^T D_2 C_1 & -C_1^T C_2 \\ -C_2^T C_1 & P_2 \end{bmatrix} \begin{bmatrix} B_1 D_2 C_1 & B_1 C_2 \\ B_2 C_1 & A_2 \end{bmatrix} \\ &\quad + \begin{bmatrix} B_1 D_2 C_1 & B_1 C_2 \\ B_2 C_1 & A_2 \end{bmatrix}^T \begin{bmatrix} -C_1^T D_2 C_1 & -C_1^T C_2 \\ -C_2^T C_1 & P_2 \end{bmatrix}, \\ &= - \begin{bmatrix} C_1^T W_2^T W_2 C_1 & C_1^T W_2^T L_2 \\ L_2^T W_2 C_1 & L_2^T L_2 \end{bmatrix} \\ &\quad - \begin{bmatrix} C_1^T D_2 W_1^T W_1 D_2 C_1 & C_1^T D_2 W_1^T W_1 C_2 \\ C_2^T W_1^T W_1 D_2 C_1 & C_2^T W_1^T W_1 C_2 \end{bmatrix}, \\ &= - \begin{bmatrix} C_1^T W_2^T \\ L_2^T \end{bmatrix} \begin{bmatrix} W_2 C_1 & L_2 \end{bmatrix} \\ &\quad - \begin{bmatrix} C_1^T D_2 W_1^T \\ C_2^T W_1^T \end{bmatrix} \begin{bmatrix} W_1 D_2 C_1 & W_1 C_2 \end{bmatrix} \\ &\leq 0. \end{aligned}$$

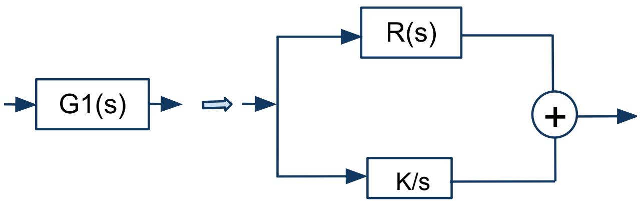

Fig. 2. The transfer function $G_1(s)$ can be decomposed into a sum of transfer function $R(s)$ and an integrator $\frac{K}{s}$.

Also, for the matrix $\breve{A}$ to be Hurwitz, the condition $\det(\breve{A}) \neq 0$ must hold, which can be guaranteed as follows:

$$\begin{aligned} \det(\breve{A}) &= \det(A_2) \det(B_1 (D_2 - C_2 A_2^{-1} B_2) C_1), \\ &= \det(A_2) \det(B_1 G_2(0) C_1). \end{aligned}$$

Hence, $\det(\breve{A}) \neq 0 \Leftrightarrow \det(B_1 G_2(0) C_1) \neq 0$, since, $\det(A_2) \neq 0$.

Finally, using Lemma 5 and the fact that $G_1(s)$ is NI and $G_2(s)$ is SNI, we conclude that $\det(I - G_1(j\omega) G_2(j\omega)) \neq 0$. This is equivalent to $\breve{A}$ having no eigenvalues on the imaginary axis. These conditions imply that the matrix $\breve{A} = \begin{bmatrix} B_1 D_2 C_1 & B_1 C_1 \\ B_2 C_1 & A_2 \end{bmatrix}$ is Hurwitz. ■

Now consider a more general case where $G_1(s)$ has other poles as well as a simple pole at the origin.

Assumption 1: The following conditions are assumed to be satisfied.

1) $G_1(s) = C_1(sI - A_1)^{-1} B_1$ is NI and can be decomposed into $R(s) + \frac{K}{s}$ as follows (see Fig. 2):

$$\begin{aligned} G_1(s) &= \begin{bmatrix} C_{11} & C_{12} \end{bmatrix} \begin{bmatrix} sI - A_{11} & 0 \\ 0 & sI \end{bmatrix}^{-1} \begin{bmatrix} B_{11} \\ B_{12} \end{bmatrix}, \\ &= C_{11}(sI - A_{11})^{-1} B_{11} + \frac{K}{s}, \\ &= R(s) + \frac{K}{s}, \end{aligned}$$

where,

$$A_1 = \begin{bmatrix} A_{11} & 0 \\ 0 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} B_{11} \\ B_{12} \end{bmatrix},$$
$$C_1 = \begin{bmatrix} C_{11} & C_{12} \end{bmatrix}, \text{ and } K = B_{12} C_{12}.$$

2) $R(s) = C_{11}(sI - A_{11})^{-1} B_{11}$ is NI.
3) $G_2(s) = C_2(sI - A_2)^{-1} B_2 + D_2$ is SNI.

Now, we can obtain the following result:

Theorem 3: Suppose that $G_1(s)$ and $G_2(s)$ satisfy Assumption 1. Then the closed-loop system corresponding to the positive feedback interconnection between $G_1(s)$ and $G_2(s)$ is internally stable if $G_2(0) < 0$.

Proof: It follows from Lemma 3 that the closed-loop transfer function $V(s) = G_2(s)(I - G_2(s)R(s))^{-1}$ is SNI.

Also, it follows from Lemma 2 of [5] that $R(0) \geq 0$. Let $U = -G_2(0) > 0$, then the matrix

$$\begin{aligned}[I - G_2(0)R(0)] &= [I + UR(0)], \\ &= U^{\frac{1}{2}}\left[U^{\frac{-1}{2}} + U^{\frac{1}{2}}R(0)\right], \\ &= U^{\frac{1}{2}}\left[I + U^{\frac{1}{2}}R(0)U^{\frac{1}{2}}\right]U^{\frac{-1}{2}} \end{aligned} \quad (12)$$

which means that the matrix $[I - G_2(0)R(0)]$ is similar to $\left[I + U^{\frac{1}{2}}R(0)U^{\frac{1}{2}}\right] > 0$. This implies that $\det(I - R(0)G_2(0)) \neq 0$.

Now, the gain of $V(s)$ at zero frequency is given by,

$$\begin{aligned}&-[I + UR(0)]^{-1}U \\ &= -[I + UR(0)]^{-1}U^{\frac{1}{2}}U^{\frac{1}{2}}, \\ &= -\left[U^{\frac{-1}{2}}[I + UR(0)]\right]^{-1}U^{\frac{1}{2}}, \\ &= -U^{\frac{1}{2}}U^{\frac{-1}{2}}\left[U^{\frac{-1}{2}} + U^{\frac{1}{2}}R(0)\right]^{-1}U^{\frac{1}{2}}, \\ &= -U^{\frac{1}{2}}\left[\left(U^{\frac{-1}{2}} + U^{\frac{1}{2}}R(0)\right)U^{\frac{1}{2}}\right]^{-1}U^{\frac{1}{2}}, \\ &= -U^{\frac{1}{2}}\left[I + U^{\frac{1}{2}}R(0)U^{\frac{1}{2}}\right]^{-1}U^{\frac{1}{2}} \\ &< 0 \end{aligned} \quad (13)$$

which implies that $V(0) < 0$. As in Fig. 3, the positive feedback interconnection between $G_1(s)$ and $G_2(s)$ is the same as the positive-feedback interconnection between $V(s)$ and $\frac{K}{s}$. Finally, a state space realization of $V(s)$ is $\left[\begin{array}{c|c}\bar{H} & \bar{G} \\ \hline \bar{F} & \bar{D}\end{array}\right]$, where,

$$\bar{H} = \begin{bmatrix} A_2 & B_2C_{11} \\ B_{11}C_2 & A_{11} + B_{11}D_2C_{11} \end{bmatrix},$$

$$\bar{G} = \begin{bmatrix} B_2 \\ B_{11}D_2 \end{bmatrix}, \qquad \bar{F} = \begin{bmatrix} C_2 & D_2C_{11} \end{bmatrix}.$$

Also,

$$\begin{aligned}&\det(\bar{H}) \\ &= \det(A_2)\det(A_{11} + B_{11}D_2C_{11} - B_{11}C_2A_2^{-1}B_2C_{11}), \\ &= \det(A_2)\det(A_{11} + B_{11}(D_2 - C_2A_2^{-1}B_2)C_{11}), \\ &= \det(A_2)\det(A_{11} + B_{11}G_2(0)C_{11}), \\ &= \det(A_2)\det(A_{11})\det(I + C_{11}A_{11}^{-1}B_{11}G_2(0)), \\ &= \det(A_2)\det(A_{11})\det(I - R(0)G_2(0)) \\ &\neq 0 \end{aligned}$$

since $\det(A_2) \neq 0, \det(A_{11}) \neq 0$, and from (12) $\det(I - R(0)G_2(0)) \neq 0$. Using Theorem 2 and the fact that $\det(I - R(0)G_2(0)) \neq 0$, implies that $[G_1(s), G_2(s)]$ is internally stable.

■

Remark 2: In a more general setting, we can consider the case where the SNI system has two inputs and two outputs as in Fig. 4. In this case, the proof of stability for such an interconnection is similar to that in Theorem 3 in this paper, but uses Theorem 3 of [3] rather than Theorem 2 of [3].

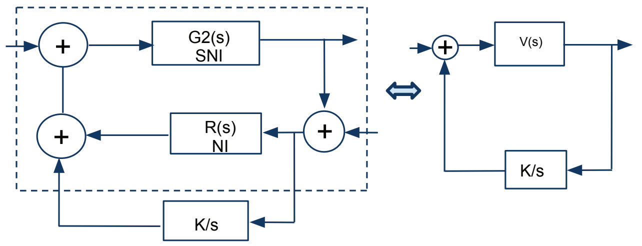

Fig. 3. Positive-feedback interconnection between $G_1(s) = (R(s) + \frac{K}{s})$, and $G_2(s)$ can be assumed as positive-feedback interconnection between $V(s)$ and $\frac{K}{s}$.

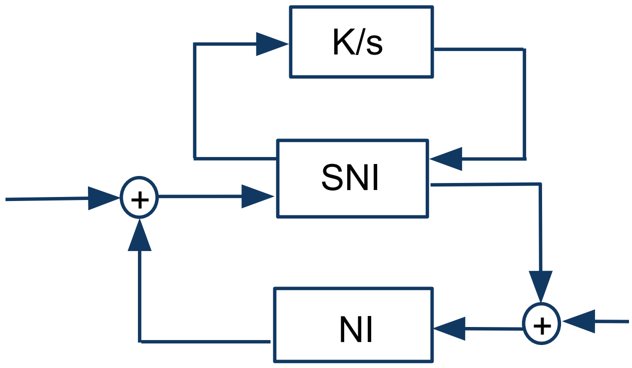

Fig. 4. An SNI system doubled in the input and the output.

IV. Illustrative Example

To illustrate the main results of this paper, consider the SNI transfer function $G_2(s) = \frac{1}{s+3} - 1$ and the NI transfer function $G_1(s) = \frac{2s+1}{s(s+1)}$. This can be decomposed into the sum of an integrator $\frac{1}{s}$ and a NI transfer function $R(s) = \frac{1}{s+1}$. The closed-loop transfer function corresponding to $G_2(s)$ and $R(s)$ is $\frac{-(s+1)}{s+3}$. As shown in Figure 5, $R(s)$ is SNI. Now, the overall closed-loop transfer function corresponding to $G_1(s)$ and $G_2(s)$ which is the same as the closed-loop transfer function corresponding to the integrator and the transfer function $\frac{-(s+1)}{s+3}$, is $\frac{-(s+1)}{(s+3)(s^2+4s+1)}$. This transfer function is asymptotically stable.

V. Conclusion

In this paper, stability results for a positive-feedback interconnection for NI systems including an integrator have been derived. A generalization of the existing NI lemma, allowing for a simple pole at the origin was also presented. This work can be used in NI controller design to allow for integral action, which will be useful in a wide range of NI control systems.

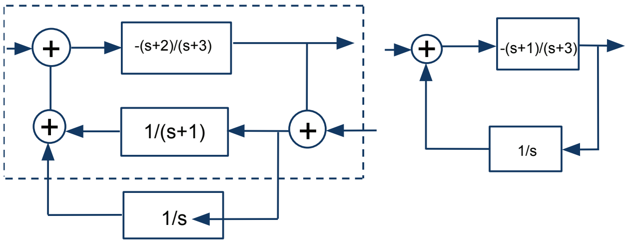

Fig. 5. Illustrative example

References

[1] B. D. O. Anderson and S. Vongpanitlerd, *Network Analysis and Synthesis: A Modern Systems Approach*. Englewood Cliffs, N.J., USA: Prentice-Hall, 1973.

[2] B. Brogliato, R. Lozano, B. Maschke, and O. Egeland, *Dissipative Systems Analysis and Control*, 2nd ed., ser. Communications and Control Engineering. London, UK: Springer, 2007.

[3] I. R. Petersen and A. Lanzon, "Feedback control of negative imaginary systems: Flexible structures with colocated actuators and sensors," *IEEE Control System Magazine*, vol. 30, no. 5, pp. 54–72, 2010.

[4] A. Lanzon and I. R. Petersen, "A modified positive-real type stability condition," in *Proceedings of the European Control Conference*, Kos, Greece, Jul. 2007, pp. 3912–3918.

[5] ——, "Stability robustness of a feedback interconnection of systems with negative imaginary frequency response," *IEEE Transactions on Automatic Control*, vol. 53, no. 4, pp. 1042–1046, 2008.

[6] Z. Song, A. Lanzon, and S. P. . I. Petersen, "Towards controller synthesis for systems with negative imaginary frequency response," *IEEE Transactions on automatic control*, vol. 55, pp. 1506–1511, June 2010.

[7] J. Xiong, I. R. Petersen, and A. Lanzon, "Stability analysis of positive feedback interconnections of linear negative imaginary systems," in *Proceedings of the American Control Conference*, St. Louis, Missouri, USA, Jun. 2009, pp. 1855–1860.

[8] ——, "On lossless negative imaginary systems," in *Proceedings of the 7th Asian Control Conference*, Hong Kong, Aug. 2009, pp. 824–829.

[9] ——, "Finite frequency negative imaginary systems," in *Proceedings of the American Control Conference*, Baltimore, MD, USA, 2010, pp. 323–328.

[10] ——, "A negative imaginary lemma and the stability of interconnections of linear negative imaginary systems," *IEEE Transactions on Automatic Control*, vol. 55, pp. 2342–2347, 2010.

[11] J. L. Fanson and T. K. Caughley, "Positive position feedback control for large space structures," *AIAA Journal*, vol. 28, no. 4, pp. 717–724, Apr. 1990.

[12] B. Bhikkaji and S. Moheimani, "Fast scanning using piezoelectric tube nanopositioners: A negative imaginary approach," in *Proc. IEEE/ASME Int. Conf. Advanced Intelligent Mechatronics AIM 2009*, Singapore, Jul. 2009, pp. 274–279.

[13] Y. K. Yong, B. Ahmed, and S. O. R. Moheimani, "Atomic force microscopy with a 12-electrode piezoelectric tube scanner," *Review of Scientific Instruments*, vol. 81, no. 3, March 2010.

[14] C. Cai and G. Hagen, "Stability analysis for a string of coupled stable subsystems with negative imaginary frequency response," *IEEE Transactions on Automatic Control*, vol. 55, pp. 1958–1963, Aug. 2010.

[15] M. J. Grimble, "Design of optimal stochastic regulating systems including integral action," in *Proc. IEE Control & Science*, vol. 126, no. 9, Sept 1979, pp. 841–848.

[16] A. Lanzon, Z. Song, S. Patra, and I. R. Petersen, "A strongly strict negative-imaginary lemma for non-minimal linear systems," *Accepted for publication in Communications in Information and Systems*, to be published in January 2011.

[17] R. Scherer and W. Wendler, "A generalization of the positive real lemma," *IEEE Transactions on Automatic Control*, vol. 39, no. 4, pp. 882–886, 1994.