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On Lossless Negative Imaginary Systems

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***Abstract*— The paper is concerned with the notion of lossless negative imaginary systems and their stabilization using a strictly negative imaginary controller through positive feedback. Firstly, some properties of lossless negative imaginary transfer functions are studied. Secondly, a Lossless Negative Imaginary Lemma is given which establishes conditions on matrices appearing in a minimal state-space realization that are necessary and sufficient for a transfer function to be lossless negative imaginary. Thirdly, a necessary and sufficient condition is developed for the stabilization of a lossless negative imaginary system by a strictly negative imaginary controller. Finally, a numerical example is presented to illustrate the theory.**

I. Introduction

Positive real systems theory has been widely applied to problems of control system design [2], [3]. Systems which dissipate energy often lead to positive real transfer functions. The positive realness of a square transfer function matrix may be seen as a generalization of the positive definiteness of a matrix to the case of a dynamic system [3], where only the real part of the transfer function is considered. Positive real systems have many uses in practice. For instance, they can be realized with an electrical circuit using only resistors, inductors and capacitors [2]. For mechanical positive real systems, the use of velocity sensors and force actuators can be used to implement a control system with a guarantee of closed loop stability.

However, one major limitation of positive real systems is that their relative degree must be zero or one [3]. This limits the application of positive real theory. For example, a lightly damped flexible structure with collocated velocity sensors and force actuators can typically be modeled by a sum of second-order transfer functions as $G(s) = \sum_{i=0}^{\infty} \frac{\psi_i^2 s}{s^2+2\zeta_i\omega_i s+\omega_i^2}$, where ω_i is the mode frequency, and $\zeta_i > 0$ is the damping coefficient associated with the i-th mode, and ψ_i is determined by the boundary conditions on the partial differential equation. However, in some cases (for example, when using piezoelectric sensors), the sensor output is proportional to position rather than velocity. So the transfer function $G(s)$ given above is the transfer function from the actuator input to the derivative of the sensor output. In the case of a lightly damped flexible structure with collocated position sensors and force actuators, the transfer function will be of the form $G(s) = \sum_{i=0}^{\infty} \frac{\psi_i^2}{s^2+2\zeta_i\omega_i s+\omega_i^2}$. It can be seen that the relative degree of the system is more than unity. Hence, the standard positive real theory will not be helpful in establishing closed loop stability. However, such a transfer function would satisfy the following negative imaginary condition: $j[G(j\omega) - G^*(j\omega)] \geq 0$ for all $\omega \in (0, \infty)$. Such systems are called *negative imaginary systems* in [4], [5].

The negative imaginary property of a square transfer function matrix may be seen as a generalization of the negative definiteness of a matrix to the case of a dynamic system, where only the imaginary part of the transfer function matrix is considered. A complete state-space characterization of negative imaginary transfer functions has been established in [4]. A necessary and sufficient condition has also been derived to guarantee the internal stability of a positive feedback interconnection of linear time-invariant multiple-input multiple-output negative imaginary systems. The results in [4] have been recently extended to the case where the system poles may be on the imaginary axis [5].

There exists a special class of negative imaginary systems whose transfer functions satisfy the lossless negative imaginary condition: $j[G(j\omega) - G^*(j\omega)] = 0$ for all $\omega \in (0, \infty)$ except values of ω where $j\omega$ is a pole. Such systems will be referred to as *lossless negative imaginary systems*. This notion is analogous to the notion of lossless positive real systems which occurs widely in circuit theory [2, page 56]. The study of lossless negative imaginary systems is significant since many applications can be found. For example, an m-port electrical network consisting of lossless circuit elements such as capacitors, inductors, and transformers is lossless [2]. The transfer function of a lossless m-port network can be negative imaginary, and hence such a lossless m-port network is actually a lossless negative imaginary system. An undamped flexible structure with collocated position sensors and force actuators is also a lossless negative imaginary system since its transfer function $G(s) = \sum_{i=0}^{\infty} \frac{\psi_i^2}{s^2+\omega_i^2}$ is a lossless negative imaginary transfer function. This paper is concerned with the characterization of lossless negative imaginary systems and with their stabilization via positive feedback control. The main results of this paper are a specialisation of the results in [4], [5] to the case where transfer functions meet the lossless negative imaginary condition.

The organization of the paper is as follows. Section II recalls some basic concepts and results on positive real transfer functions [2] and negative imaginary transfer functions [4], [5]. The notion of lossless negative imaginary transfer functions is introduced in Section III. The relation-

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ship between lossless negative imaginary transfer function matrices and lossless positive real transfer function matrices is also established in this section. In Section IV, a Lossless Negative Imaginary Lemma is established when the transfer functions satisfy the lossless negative imaginary condition. This lemma can be considered as a specialisation of the Negative Imaginary Lemma in [5] to the lossless negative imaginary case. Section V studies the stabilization of a lossless negative imaginary system by a strictly negative imaginary controller through a positive feedback interconnection. A necessary and sufficient condition for stabilization is proposed in terms of the DC loop gain (i.e., the loop gain at zero frequency) of the systems. This result is a special case of the main result in [5] with the system being lossless negative imaginary. A numerical example is presented in Section VI to illustrate the theory and Section VII concludes the paper.

Notation: $\mathbb{R}$ and $\mathbb{C}$ denote the fields of real and complex numbers, respectively. Let $\mathcal{R}^{m\times n}$ denote the set of $m \times n$ real-rational proper transfer function matrices. Also, $\mathbb{R}^{m\times n}$ and $\mathbb{C}^{m\times n}$ denote the sets of $m \times n$ real and complex matrices, respectively. $\Re[s]$ denotes the real part of a complex number $s \in \mathbb{C}$. $\lambda_i(A)$ is the ith eigenvalue of a square complex matrix A, while $\lambda_{\max}(A)$ denotes the maximum eigenvalue for a square complex matrix A that has only real eigenvalues. $\bar{A}$, A^T and A^* denotes the complex conjugate, the transpose and the complex conjugate transpose of a complex matrix A. $R^{\sim}(s)$ presents the adjoint of transfer function matrix $R(s)$ given by $R^T(-s)$. When $s = j\omega$, we have $R^{\sim}(j\omega) = R^T(-j\omega) = R^*(j\omega)$.

II. Preliminary Results

Before developing our lossless negative imaginary theory, we will recall some concepts on positive real and negative imaginary transfer functions. These concepts are closely related to each other. Also some preliminary results for negative imaginary systems are stated, which will be used to develop the main results of this paper.

A. Positive Real Transfer Functions

This subsection gives the definitions of positive real and lossless positive real transfer functions. The readers are referred to [2, Chapter 2] for more details.

Definition 1: [2] A real-rational proper transfer function matrix $F(s) \in \mathcal{R}^{m\times m}$ is positive real if

1) No element of $F(s)$ has a pole in $\Re[s] > 0$;
2) $F(s) + F^*(s) \geq 0$ in $\Re[s] > 0$.

Definition 2: [2] A real-rational proper transfer function matrix $F(s) \in \mathcal{R}^{m\times m}$ is lossless positive real if

1) $F(s)$ is positive real;
2) $F(j\omega) + F^*(j\omega) = 0$ for all real ω except values of ω where $j\omega$ is a pole of $F(s)$.

Lemma 1: [2] Let $F(s)$ be a real-rational matrix of functions of s. Then $F(s)$ is positive real if and only if

1) No element of $F(s)$ has a pole in $\Re[s] > 0$;
2) $F(j\omega) + F^*(j\omega) \geq 0$ for all real ω except values of ω where $j\omega$ is a pole of $F(s)$;
3) If $j\omega_0$ is a pole of any element of $F(s)$, it is at most a simple pole, and the residue matrix, $K_0 = \lim_{s\to j\omega_0}(s - j\omega_0)F(s)$ in case ω_0 is finite, and $K_\infty = \lim_{\omega\to\infty} \frac{F(j\omega)}{j\omega}$ in case ω_0 is infinite, is positive semidefinite Hermitian.

B. Negative Imaginary Transfer Functions

The notion of negative imaginary transfer functions and several related results developed in [4], [5] are recalled. They provide the basis to develop our lossless negative imaginary theory.

Definition 3: [5] The real-rational proper transfer function matrix $R(s) \in \mathcal{R}^{m\times m}$ is negative imaginary if

1) $R(\infty) = R^T(\infty)$;
2) $\hat{R}(s) \triangleq R(s) - R(\infty)$ satisfies:
   a) $\hat{R}(s)$ has no poles at the origin and in $\Re[s] > 0$;
   b) $j[\hat{R}(j\omega) - \hat{R}^*(j\omega)] \geq 0$ for all $\omega \in (0, \infty)$ except values of ω where $j\omega$ is a pole of $\hat{R}(s)$;
   c) If $j\omega_0$ is a pole of $\hat{R}(s)$, it is at most a simple pole, and the residue matrix of $s\hat{R}(s)$, $K_0 \triangleq \lim_{s\to j\omega_0}(s - j\omega_0)s\hat{R}(s)$, is positive semidefinite Hermitian.

Definition 4: [5] The real-rational proper transfer function matrix $R(s) \in \mathcal{R}^{m\times m}$ is strictly negative imaginary if

1) $R(\infty) = R^T(\infty)$;
2) $R(s)$ has no poles in $\Re[s] \geq 0$;
3) $j[R(j\omega) - R^*(j\omega)] > 0$ for $\omega \in (0, \infty)$.

As stated above, the concept of negative imaginary transfer functions is closely related to corresponding positive real counterparts. The following lemma describes a formal relationship between these concepts.

Lemma 2: [5] Given a real-rational strictly proper transfer function matrix $\hat{R}(s) \in \mathcal{R}^{m\times m}$. Then $\hat{R}(s)$ is negative imaginary if and only if

1) $\hat{R}(s)$ has no poles at the origin;
2) $F(s) \triangleq s\hat{R}(s)$ is positive real.

The following lemma is analogous to the Positive Real Lemma (e.g., see Lemma 3.1 of [3] or Theorem 3 of [1]), and describes a necessary and sufficient condition for $R(s)$ to be negative imaginary in terms of matrices appearing a minimal state-space realization of $R(s)$.

Lemma 3 (Negative Imaginary Lemma): [4], [5] Let (A, B, C, D) be a minimal state-space realization of a real-rational proper transfer function $R(s) \in \mathcal{R}^{m\times m}$, where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{m\times n}$, $D \in \mathbb{R}^{m\times m}$, $m \leq n$. Then $R(s)$ is negative imaginary if and only if

1) $\det(A) \neq 0$, $D = D^T$, and
2) there exists a matrix $Y = Y^T > 0$, $Y \in \mathbb{R}^{n\times n}$, such that

$$AY + YA^T \leq 0, \quad \text{and} \quad B + AYC^T = 0.$$

A useful property of negative imaginary systems is the spectral-factor property, which is stated in the following lemma.

Lemma 4: [5] If $R(s)$ is negative imaginary and has the minimal state-space realization (A, B, C, D), then there exists a real-rational strictly proper transfer function matrix $M(s) \sim \left[\begin{array}{c|c} A & B \\ \hline LY^{-1}A^{-1} & 0 \end{array}\right]$ such that

$$j[R(j\omega) - R^*(j\omega)] = \omega M^*(j\omega)M(j\omega)$$

whenever $j\omega$ is not a pole of $R(s)$. Here, $Y = Y^T > 0$ and L are the solutions of $L^T L = -AY - YA^T$ and $B + AYC^T = 0$.

Remark 1: According to [5], we also have $R(s) - R^\sim(s) = -sM^\sim(s)M(s)$ for all $s \in \mathbb{C}$ with s not a pole of $R(s)$.

III. Lossless Negative Imaginary Transfer Functions

With the preliminary concepts and results provided in previous section, we are now ready to introduce the concept of lossless negative imaginary transfer functions. The definition is inspired by the analogous concept in Definition 2.

Definition 5: A real-rational proper transfer function matrix $R(s) \in \mathcal{R}^{m\times m}$ is *lossless negative imaginary* if

1) $R(s)$ is negative imaginary;
2) $j[R(j\omega) - R^*(j\omega)] = 0$ for all $\omega \in (0, \infty)$ except values of ω where $j\omega$ is a pole of $R(s)$.

Remark 2: It can be seen from Definition 3 that the lossless negative imaginary property of a transfer function is simply defined by replacing the "$\geq$" sign with the "$=$" sign in the condition b) of Definition 3.

As expected, the concept of lossless negative imaginary transfer functions is closely related to the concept of lossless positive real transfer functions. The following lemma provides such a relationship.

Lemma 5: Given a real-rational strictly proper transfer function matrix $\hat{R}(s) \in \mathcal{R}^{m\times m}$. Then $\hat{R}(s)$ is lossless negative imaginary if and only if

1) $\hat{R}(s)$ has no poles at the origin;
2) $F(s) \triangleq s\hat{R}(s)$ is lossless positive real.

Proof: (Necessity) Suppose $\hat{R}(s)$ is lossless negative imaginary. The condition 1 of Definition 5 implies that $\hat{R}(s)$ is also a negative imaginary transfer function. In view of Lemma 2, we have that $\hat{R}(s)$ has no poles at the origin and $F(s)$ is positive real.

Now we prove that $F(s)$ is lossless positive real. Note that $F(s)$ and $\hat{R}(s)$ have the same set of poles. For any $\omega \geq 0$ where $j\omega$ is not a pole of $\hat{R}(s)$, we have $F(j\omega) + F^*(j\omega) = j\omega[\hat{R}(j\omega) - \hat{R}^*(j\omega)] = 0$. Also, by taking complex conjugate we have $\overline{F(j\omega) + F^*(j\omega)} = 0$ for $\omega \geq 0$, which is equivalent to $F(-j\omega) + F^*(-j\omega) = 0$ for $\omega \geq 0$. That is, $F(j\omega) + F^*(j\omega) = 0$ for $\omega \leq 0$. So $F(j\omega) + F^*(j\omega) = 0$ for all $\omega \in \mathbb{R}$ with $j\omega$ not a pole. According to Definition 2, $F(s)$ is a lossless positive real transfer function.

(Sufficiency) Suppose $F(s)$ is lossless positive real, and $R(s)$ has no poles at the origin. Then $F(s)$ is also positive real according to Definition 2. In view of Lemma 2, we have that $\hat{R}(s)$ is negative imaginary.

Moreover, for any $\omega > 0$ where $j\omega$ is not a pole of $\hat{R}(s)$, the condition 2 of Definition 2 implies the condition 2 of Definition 5 in view of $F(s) = s\hat{R}(s)$. Therefore, $\hat{R}(s)$ is lossless negative imaginary. ■

Lemma 6: A real-rational strictly proper transfer function matrix $\hat{R}(s)$ is lossless negative imaginary if and only if

1) $\hat{R}(s)$ has no poles at the origin;
2) All poles of elements of $\hat{R}(s)$ are simple poles and purely imaginary, and the residue matrix of $s\hat{R}(s)$ at any pole, $K_0 \triangleq \lim_{s\to j\omega_0}(s - j\omega_0)s\hat{R}(s)$, is positive semidefinite Hermitian;
3) $\hat{R}(s) = \hat{R}^\sim(s)$ for all s such that s is not a pole of $\hat{R}(s)$.

Proof: (Necessity) Suppose $\hat{R}(s)$ is lossless negative imaginary. Then Lemma 5 implies $F(s) = s\hat{R}(s)$ is lossless positive real and the condition 1 holds. In view of [2, page 57], we have $F(s) + F^\sim(s) = 0$ for all s such that s is not a pole of $F(s)$. So $s\hat{R}(s) - s\hat{R}^\sim(s) = 0$ for all s such that s is not a pole of $\hat{R}(s)$. Hence, we have $\hat{R}(s) = \hat{R}^\sim(s)$ for all s such that s is not a pole of $\hat{R}(s)$. Here $\hat{R}(0) = \hat{R}^\sim(0)$ is due to the continuity of $\hat{R}(s)$. Thus the condition 3 holds.

Suppose that s_0 is a pole of $\hat{R}(s)$. Then it follows from $\hat{R}(s) = \hat{R}^\sim(s) = \hat{R}^T(-s)$ and continuity that $-s_0$ is also a pole of $\hat{R}(s)$. On the other hand, $\hat{R}(s)$ has no pole in $\Re[s] > 0$ according to the definition. Therefore, all poles of elements of $\hat{R}(s)$ must be purely imaginary. Moreover, the condition 3 of Definition 3 implies that the poles are simple poles and the residue matrix of $s\hat{R}(s)$ at any pole is positive semidefinite Hermitian. Thus the condition 2 holds.

(Sufficiency) Suppose the conditions 1–3 hold. First, the condition 1 together with the condition 2 implies the condition a) and the condition c) of Definition 3. Secondly, from the condition 3, we have $\hat{R}(j\omega) = \hat{R}^\sim(j\omega) = \hat{R}^*(j\omega)$, so the condition 2 of Definition 5 is also true. Therefore, $\hat{R}(s)$ is lossless negative imaginary. ■

Example 1: As an application of Lemma 5, we can say that $\hat{R}(s) = \frac{1}{s^2+1}$ is lossless negative imaginary if and only if $F(s) = \frac{s}{s^2+1}$ is lossless positive real. This can be actually verified by directly using Definition 5, Definition 2 and Lemma 1. In view of Lemma 6, we also have $\hat{R}^\sim(s) = \frac{1}{(-s)^2+1} = \hat{R}(s)$.

The following lemma characterizes the properties of a sum of negative imaginary transfer functions.

Lemma 7: Given two lossless negative imaginary transfer functions $R_1(s)$ and $R_2(s)$, and a negative imaginary transfer function $R(s)$. Then

1) $R_1(s) + R_2(s)$ is lossless negative imaginary;
2) $R_1(s) + R(s)$ is negative imaginary.

Proof: The proof follows along the same lines as that of Lemma 5 in [5], and is hence omitted. ■

IV. Lossless Negative Imaginary Lemma

The Lossless Negative Imaginary Lemma proposed in this section is a modification of the Negative Imaginary Lemma in [5] (i.e., Lemma 3) applied to the case where the transfer functions being considered are lossless negative imaginary. This Lossless Negative Imaginary Lemma is analogous to the Lossless Positive Real Lemma [2, pages 221–222].

Lemma 8 (Lossless Negative Imaginary Lemma): Let (A, B, C, D) be a minimal state-space realization of a real-rational proper transfer function $R(s) \in \mathcal{R}^{m\times m}$, where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{m\times n}$, $D \in \mathbb{R}^{m\times m}$, $m \le n$. Then $R(s)$ is lossless negative imaginary if and only if

1) $\det(A) \neq 0$, $D = D^T$;
2) there exists a matrix $Y = Y^T > 0$, $Y \in \mathbb{R}^{n\times n}$, such that

$$AY + YA^T = 0, \quad \text{and} \quad B + AYC^T = 0.$$

Proof: (Necessity) Note that a lossless negative imaginary transfer function $R(s)$ is also negative imaginary. Hence, according to Lemma 3, the condition 1 holds. Furthermore, it follows from Lemma 4 that there exists a transfer function matrix $M(s) \sim \left[\begin{array}{c|c} A & B \\ \hline LY^{-1}A^{-1} & 0 \end{array}\right]$, where $Y = Y^T > 0$ and L are the solutions of $L^T L = -AY - YA^T$ and $B + AYC^T = 0$. Moreover, $M(s)$ satisfies the spectral-factor property

$$j[R(j\omega) - R^*(j\omega)] = \omega M^*(j\omega)M(j\omega)$$

whenever $j\omega$ is not a pole of $R(s)$.

Since $R(s)$ is lossless, according to Lemma 6, we have $R(j\omega) = R^\sim(j\omega) = R^*(j\omega)$ for any real ω with $j\omega$ not a pole of $R(s)$. That is, $j[R(j\omega) - R^*(j\omega)] = 0$. Thus $M^*(j\omega)M(j\omega) = 0$, and hence $M(j\omega) = 0$ for any $\omega \in \mathbb{R}$ with $j\omega$ not a pole of $M(s)$. Moreover, $M(0) = 0$ due to the continuity of $M(s)$. Therefore $M(s) = LY^{-1}A^{-1}(sI - A)^{-1}B = 0$. By the controllability of the pair (A, B) (see Theorem 3.1 of [6]), we conclude that $LY^{-1}A^{-1} = 0$. That is, $L = 0$. Therefore, the condition 2 holds. This completes the necessity part of the proof.

(Sufficiency) Suppose the conditions 1–2 hold. We know that $R(s)$ is negative imaginary in view of Lemma 3, and $M(s) = 0$ in view of Lemma 4. Therefore, $j[R(j\omega) - R^*(j\omega)] = \omega M^*(j\omega)M(j\omega) = 0$, whenever $j\omega$ is not a pole of $R(s)$. It follows from the definition that $R(s)$ is lossless negative imaginary. This completes the proof. ■

V. Stabilization of Lossless Negative Imaginary Systems

In this section, we consider the stabilization of lossless negative imaginary systems via positive feedback as shown in Fig. 1. The positive feedback interconnection is denoted by $[R(s), R_s(s)]$.

A necessary and sufficient condition is provided for the stability of the interconnected system given in Fig. 1 in terms of the DC loop gain (i.e., the loop gain at zero frequency).

Theorem 1: Given a lossless negative imaginary transfer function $R(s)$ and a strictly negative imaginary transfer function $R_s(s)$ that satisfy $R(\infty)R_s(\infty) = 0$ and $R_s(\infty) \ge 0$. Then the positive feedback interconnection $[R(s), R_s(s)]$ is internally stable if and only if $\lambda_{\max}(R(0)R_s(0)) < 1$.

Proof: The proof follows along the same lines as the proof of Theorem 5 of [4] except that Theorem 5.7 of [6] is used (instead of Corollary 5.6 of [6]). Also, this result is actually a special case of Theorem 1 of [5]. ■

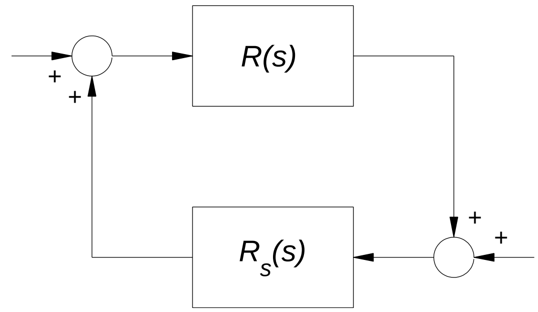

Fig. 1. Positive feedback interconnection

The following corollaries are a restatement of the above theorem, written in the same form as the small gain theorem (see Theorem 9.1 of [6]).

Corollary 1: Given $\gamma > 0$ and a strictly negative imaginary transfer function $R(s)$ with $R(\infty) \ge 0$. Then, the positive feedback interconnection $[\Delta(s), R(s)]$ is internally stable for all lossless negative imaginary transfer functions $\Delta(s)$ satisfying $\Delta(\infty)R(\infty) = 0$ and $\lambda_{\max}(\Delta(0)) < \gamma$ (respectively $\le \gamma$) if and only if $\lambda_{\max}(R(0)) \le \frac{1}{\gamma}$ (respectively $< \frac{1}{\gamma}$).

Proof: The proof is the same as the proof of Corollary 6 of [4]. ■

Corollary 2: Given $\gamma > 0$ and a lossless negative imaginary transfer function $R(s)$. Then the positive feedback interconnection $[\Delta(s), R(s)]$ is internally stable for all strictly negative imaginary transfer functions $\Delta(s)$ satisfying $\Delta(\infty)R(\infty) = 0$, $\Delta(\infty) \ge 0$ and $\lambda_{\max}(\Delta(0)) < \gamma$ (respectively $\le \gamma$) if and only if $\lambda_{\max}(R(0)) \le \frac{1}{\gamma}$ (respectively $< \frac{1}{\gamma}$).

Proof: The proof is the same as the proof of Corollary 6 of [4]. ■

Remark 3: The results in this section are in fact special cases of the corresponding results in [4], [5] with one system being lossless negative imaginary. The readers are referred to [4], [5] for more details about strictly negative imaginary transfer functions.

Remark 4: Given a lossless negative imaginary system, a strictly negative imaginary controller can be used to stabilize the system as long as the conditions stated in Theorem 1 are satisfied. The design of such a strictly negative imaginary controller will be illustrated in the next section.

VI. Illustrative Example

To illustrate the main results of this paper, let us consider a lossless LC electrical circuit plant as depicted in Fig. 2 which consists of two capacitors, two inductors and two sources. The plant is a two-input, two-output, fourth-order linear system. To obtain a lossless negative imaginary transfer function, the measured voltage across the capacitor C_1 and the measured current in the inductor L_2 have to be scaled by C_1 and L_2, respectively. Let

$$p(s) \triangleq s^4 C_1 L_1 C_2 L_2 + s^2(C_1L_1 + C_2L_2 + C_1L_2) + 1.$$

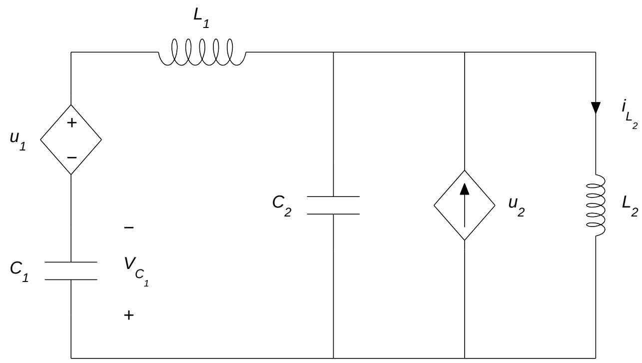

Fig. 2. LC circuit plant

Then the transfer function of the circuit plant from the source input vector $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ to the scaled measurement output vector $y = \begin{bmatrix} C_1 V_{C_1} \\ L_2 i_{L_2} \end{bmatrix}$ is given by

$$P(s) = \frac{1}{p(s)} \begin{bmatrix} (s^2 C_2 L_2 + 1) C_1 & -s C_1 L_2 \\ s C_1 L_2 & (s^2 C_1 L_1 + 1) L_2 \end{bmatrix}$$

where $C_1 > 0$ and $L_2 > 0$ are assumed to be known while $L_1 > 0$ and $C_2 > 0$ are unknown and represent the uncertainty in the system. In this example, the parameters in the circuit plant are chosen as $C_1 = 0.8$F, $L_1 = 0.1$H, $C_2 = 0.1$F and $L_2 = 0.5$H. With those particular parameter values, the poles of the plant are at $\pm 11.4274j$ and $\pm 1.3836j$.

To illustrate the Lossless Negative Imaginary Lemma, a minimal state-space realization

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}$$

of $P(s)$ is found to be

$$A = \begin{bmatrix} 0 & 0 & -10 & -10 \\ 0 & 0 & 0 & 2 \\ 1.25 & 0 & 0 & 0 \\ 10 & -10 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 10 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 10 \end{bmatrix},$$

$$C = \begin{bmatrix} 0 & 0 & 0.8 & 0 \\ 0 & 0.5 & 0 & 0 \end{bmatrix}, \qquad D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

First, we have that $\det(A) \neq 0$, and $D = D^T$. That is, the condition 1 in Lemma 8 holds. Secondly, a real positive definite matrix

$$Y = \begin{bmatrix} 10 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1.25 & 0 \\ 0 & 0 & 0 & 10 \end{bmatrix}$$

is found to satisfy $AY + YA^T = 0$ and $B + AYC^T = 0$. So the condition 2 in Lemma 8 holds. Applying Lemma 8, we can conclude that $P(s)$ is a lossless negative imaginary transfer function. The lossless negative imaginary property of $P(s)$ can also be confirmed by a direct computation using Definition 5. In fact, the direct computation shows that $P(s)$ is lossless negative imaginary for all $L_1 > 0$ and $C_2 > 0$.

To stabilize the lossless LC circuit plant, a passive RLC circuit controller as depicted in Fig. 3 is used.

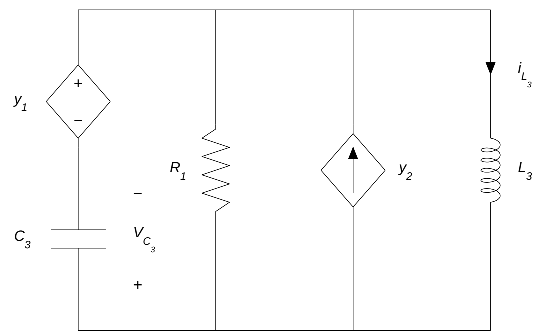

Fig. 3. RLC circuit controller

Let

$$c(s) \triangleq s^2 R_1 C_3 L_3 + s L_3 + R_1.$$

Then the transfer function matrix of the controller from the source input vector $y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ to the scaled measurement output vector $u = \begin{bmatrix} C_3 V_{C_3} \\ L_3 i_{L_3} \end{bmatrix}$ is given by

$$C(s) = \frac{1}{c(s)} \begin{bmatrix} s C_3 L_3 + R_1 C_3 & -s R_1 C_3 L_3 \\ s R_1 C_3 L_3 & R_1 L_3 \end{bmatrix}$$

where $C_3 > 0$, $R_1 > 0$ and $L_3 > 0$ are to be designed. It can be shown that $C(s)$ is a strictly negative imaginary transfer function for any $C_3 > 0$, $R_1 > 0$ and $L_3 > 0$.

When the plant in Fig. 2 is connected to the controller in Fig. 3, the closed loop system can be represented as in Fig. 4. This feedback interconnection can be thought of as being implemented using controlled sources in the plant and controller circuits. Now, the result in Theorem 1 can be used for robustness analysis and synthesis.

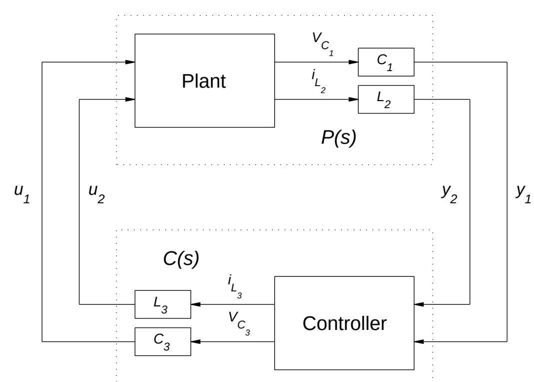

Fig. 4. Closed loop system

Firstly, we have that

$$P(\infty) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad C(\infty) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$
$$P(0) = \begin{bmatrix} C_1 & 0 \\ 0 & L_2 \end{bmatrix}, \quad C(0) = \begin{bmatrix} C_3 & 0 \\ 0 & L_3 \end{bmatrix}.$$

Then, applying Theorem 1, we conclude that the uncertain plant $P(s)$ is robustly stabilized by the controller $C(s)$ if and only if $C_1C_3 < 1$ and $L_2L_3 < 1$ (obtained through the condition $\lambda_{\max}(P(0)C(0)) < 1$). It can be seen that the resistance R_1, the inductor L_1 and the capacitor C_2 do not affect the stability of the closed loop system.

To illustrate the properties of the RLC circuit controller, we now present some simulations. The resistance in the controller is chosen as $R_1 = 0.5\Omega$. The capacitor C_1 in the plant is assumed to be initially charged to 3V and the inductor L_2 is assumed to carry -2A initial current, while the inductor L_1 and the capacitor C_1 has zero initial conditions. The controller is assumed to have zero initial condition. Firstly, The initial condition response of the plant without control is shown in Fig. 5. Secondly, when the capacitor and the inductor are chosen as $C_3 = 0.2$F and $L_3 = 0.4$H so that $C_1C_3 < 1$ and $L_2L_3 < 1$, the closed loop system is stable as illustrated in Fig. 6. Thirdly, when the parameters are chosen as $C_3 = 1.5$F and $L_3 = 2.5$H so that $C_1C_3 > 1$ and $L_2L_3 > 1$, the closed loop system is unstable as shown in Fig. 7.

VII. Conclusions

This paper has studied the lossless negative imaginary properties of square real-rational proper transfer functions. Dynamic systems with lossless negative imaginary transfer functions have applications in control of lossless electrical circuits and position feedback control of undamped flexible structures. The Lossless Negative Imaginary Lemma was derived for transfer functions that are lossless negative imaginary. Moreover, a necessary and sufficient condition was established for the stabilization of a lossless negative imaginary system using a strictly negative imaginary system. Finally, the theory in the paper was illustrated by a numerical example.

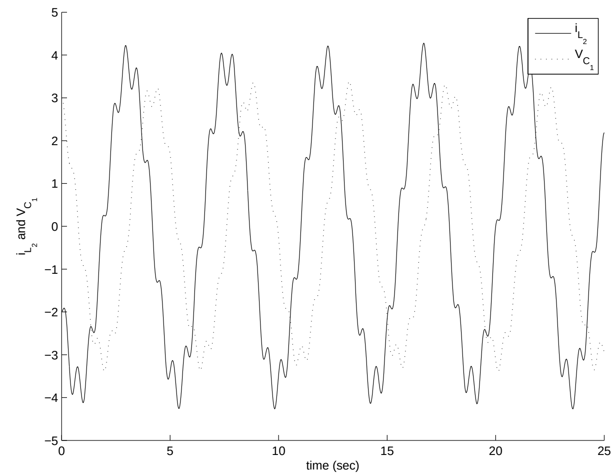

Fig. 5. Initial response of the plant without control

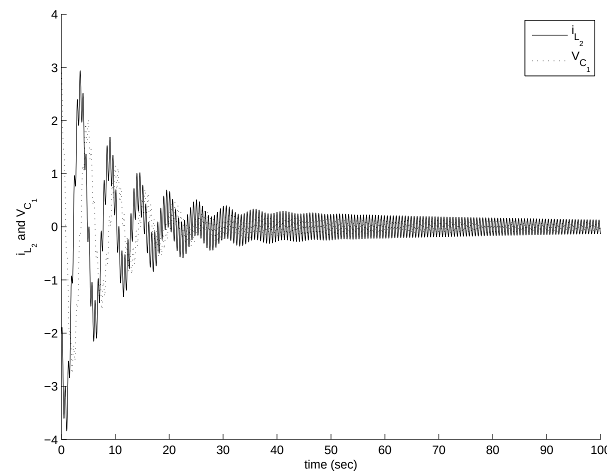

Fig. 6. Initial response of the closed loop system with $C_1C_3 < 1$, $L_2L_3 < 1$

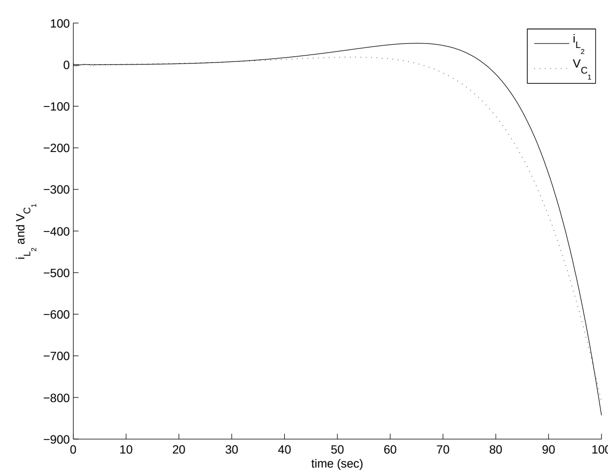

Fig. 7. Initial response of the closed loop system with $C_1C_3 > 1$, $L_2L_3 > 1$

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