

Finite Frequency Negative Imaginary Systems

Junlin Xiong

Ian R. Petersen

Alexander Lanzon

Abstract—This paper is concerned with finite frequency negative imaginary (FFNI) systems. The paper introduces the concept of FFNI transfer function matrices, and the relationship between the FFNI property and the finite frequency positive real (FFPR) property of transfer function matrices is established. The paper also establishes an FFNI lemma which gives a necessary and sufficient condition on the matrices appearing in a minimal state-space realization for a transfer function to be FFNI. Also, a time-domain interpretation of the FFNI property is provided in terms of system input, output and state. An example is presented to illustrate the FFNI concept and the FFNI lemma.

I. INTRODUCTION

Loosely speaking, negative imaginary linear systems are Lyapunov stable dynamical systems whose transfer function matrices satisfying the negative imaginary condition: $j[G(j\omega) - G^*(j\omega)] \geq 0$ for all $\omega \in (0, \infty)$ [1], [2]. In the SISO case, a scalar negative imaginary transfer function $G(s)$ will have non-positive imaginary part when $s = j\omega$ for $\omega \in (0, \infty)$. In other words, the phase of the transfer function satisfies $\angle G(j\omega) \in [-\pi, 0]$ for $\omega \in (0, \infty)$. Many practical physical systems can be modeled as negative imaginary systems. For example, a lightly damped flexible structure with collocated position sensors and force actuators can be modeled by a sum of second-order transfer functions as $G(s) = \sum_{i=0}^{\infty} \frac{\psi_i^2}{s^2 + 2\zeta_i \omega_i s + \omega_i^2}$, where $\omega_i > 0$ is the mode frequency associated with the i -th mode, $\zeta_i > 0$ is the damping coefficient, and ψ_i is determined by the boundary conditions on the underlying partial differential equation [1]. Also, a transfer function of the form $G(s) = \frac{k_1}{s^2 + 2\sigma_1 \omega_1 s + \omega_1^2} + d_1$ has been used to model the voltage subsystem in a piezoelectric tube scanner system [3]; this transfer function turns out to be negative imaginary. The theory of negative imaginary systems is closely related to the theory of positive real systems [4]–[6]. Also, the concept of systems with counterclockwise input-output dynamics [7] is related to the concept of negative imaginary systems.

In [1], a complete state-space characterization of negative imaginary linear systems was established in terms of the solvability of a linear matrix inequality and a linear matrix equation. A necessary and sufficient condition was derived to guarantee the internal stability of a positive feedback

interconnection of negative imaginary linear systems in terms of their DC loop gains. All of the results in [1] have been extended to the case where system poles may be on the imaginary axis [2]. Also, a concept of lossless negative imaginary systems has been developed to model negative imaginary systems with purely imaginary poles only [8]. The corresponding lossless negative imaginary systems theory has been developed as well.

In this paper, a new concept—finite frequency negative imaginary (FFNI) transfer function matrices—will be introduced. Roughly speaking, an FFNI transfer function matrix is a square real-rational proper transfer function which is not only stable in the Lyapunov sense but also possesses the negative imaginary property $j[G(j\omega) - G^*(j\omega)] \geq 0$ for a finite frequency range, say $\omega \in (0, \bar{\omega})$. This concept can be considered as a generalization of the concept of negative imaginary transfer function matrices. The study of FFNI transfer function matrices is motivated by the fact that many such transfer functions arise in practical control problems. For example, the capacitance subsystem of the piezoelectric tube scanner in [3] is modeled by $G(s) = \frac{c_1 s^2 + c_2 s + c_3}{s^2 + 2\sigma_1 \omega_1 s + \omega_1^2}$, with the parameter values obtained through experiment. This transfer function is FFNI; more details of this example are provided in the example section of this paper. Also, for some lightly damped flexible structures, when the position sensors and the force actuators are slightly non-collocated, the resulting transfer functions may be FFNI. The study of FFNI transfer function matrices is also inspired by the work in [9]. In [9], the concept of finite frequency positive real (FFPR) transfer function matrices was proposed. The FFPR theory was developed and applied to design dynamical systems with the FFPR property.

The organization of the paper is as follows. Section II introduces the FFNI concept for square real-rational proper transfer function matrices. Several properties of such matrices are studied. The relationship between the FFNI property and the FFPR property of transfer function matrices is also established in this section. In Section III, the FFNI lemma—the main result of the paper—is established in terms of a linear matrix inequality and two linear matrix equations. The FFNI lemma gives a complete state-space characterization of FFNI linear systems in terms of their minimal realizations. When the bandwidth of an FFNI transfer function matrix approaches infinity, the FFNI lemma is shown to reduce to the negative imaginary lemma of [1], [2]. Moreover, a time-domain interpretation of the FFNI property is presented in terms of the system input, output and state. The validity of the FFNI lemma developed in this paper is verified by

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J. Xiong and I. R. Petersen are with the School of Engineering and Information Technology, University of New South Wales at the Australian Defence Force Academy, Canberra ACT 2600, Australia. junlin.xiong@gmail.com, i.r.petersen@gmail.com

A. Lanzon is with the Control Systems Centre, School of Electrical and Electronic Engineering, University of Manchester, Manchester M60 1QD, U. K. a.lanzon@ieee.org

numerical computations in Section IV. Section V concludes the paper.

II. FINITE FREQUENCY NEGATIVE IMAGINARY TRANSFER FUNCTION MATRICES

In this section, the concept of FFNI transfer function matrices will be introduced. The idea behind the definition of FFNI transfer function matrices is that the negative imaginary conditions, which are used in [2] to define negative imaginary systems, are required to hold on a finite frequency range.

Definition 1: A square real-rational proper transfer function matrix $R(s)$ is said to be finite frequency negative imaginary with bandwidth $\bar{\omega}$ if it satisfies the following conditions:

- 1) $R(s)$ has no poles at the origin and in the open right-half of the complex plane.
- 2) $j[R(j\omega) - R^*(j\omega)] \geq 0$ for all $\omega \in \Omega$, where $\Omega = \{\omega \in \mathbb{R} : 0 < \omega \leq \bar{\omega}, j\omega \text{ is not a pole of } R(s)\}$.
- 3) Every pole of $R(s)$ on $j\bar{\Omega}$, if any, is simple and the corresponding residue matrix of $jR(s)$ is positive semidefinite Hermitian, where $\bar{\Omega} = \{\omega \in \mathbb{R} : 0 < \omega \leq \bar{\omega}\}$.
- 4) $R(\infty) = R^T(\infty)$.

Remark 1: Based on the above definition, an FFNI transfer function matrix $R(s)$ with bandwidth $\bar{\omega}$ has the following properties:

- 1) $j[R(j\omega) - R^*(j\omega)] \leq 0$ for all $\omega \in \Omega_1$, where $\Omega_1 = \{\omega \in \mathbb{R} : -\bar{\omega} \leq \omega < 0, j\omega \text{ is not a pole of } R(s)\}$. A brief proof is as follows: For $0 < \omega \leq \bar{\omega}$ and $j\omega$ is not a pole, we know that $j[R(j\omega) - R^*(j\omega)] \geq 0$. In view of Lemma 2 of [2], we have $0 \leq j[R(j\omega) - R^*(j\omega)] = -j[\overline{R(j\omega)} - R^*(j\omega)] = -j[R(-j\omega) - R^*(-j\omega)]$. That is, $j[R(-j\omega) - R^*(-j\omega)] \leq 0$ for $0 < \omega \leq \bar{\omega}$ with $j\omega$ not a pole. In other words, $j[R(j\omega) - R^*(j\omega)] \leq 0$ for $-\bar{\omega} \leq \omega < 0$ where $j\omega$ is not a pole.
- 2) Every pole of $R(s)$ on $j\bar{\Omega}_1$, if any, is simple and the corresponding residue matrix of $jR(s)$ is negative semidefinite Hermitian, where $\bar{\Omega}_1 = \{\omega \in \mathbb{R} : -\bar{\omega} \leq \omega < 0\}$. A brief proof is as follows: Firstly, note that $R(s)$ can be factored into $R(s) = \frac{1}{(s-j\omega)(s+j\omega)}R_1(s)$ whenever $j\omega$ is a pole. Suppose $j\omega_0$ ($0 < \omega_0 \leq \bar{\omega}$) is a pole of $R(s)$. Then $\lim_{s \rightarrow j\omega_0}(s - j\omega_0)jR(s) = \lim_{s \rightarrow j\omega_0}(s - j\omega_0)\frac{1}{(s-j\omega_0)(s+j\omega_0)}jR_1(s) = \frac{1}{2\omega_0}R_1(j\omega_0)$. Therefore $R_1(j\omega_0)$ is positive semidefinite Hermitian, that is, $R_1(j\omega_0) = R_1^*(j\omega_0) \geq 0$. So $R_1(j\omega_0) = R_1^*(j\omega_0) \geq 0$, that is, $R_1(-j\omega_0) = R_1^*(-j\omega_0) \geq 0$. On the other hand, at the pole of $-j\omega_0$, $\lim_{s \rightarrow -j\omega_0}(s + j\omega_0)jR(s) = \lim_{s \rightarrow -j\omega_0}(s + j\omega_0)\frac{1}{(s-j\omega_0)(s+j\omega_0)}jR_1(s) = \frac{1}{-2\omega_0}R_1(-j\omega_0) \leq 0$.

Similar to the case where the negative imaginary concept is closely related to the positive real concept, the FFNI concept in this paper is closely related to the FFPR concept developed in [9]. Before formally establishing this relationship, let us recall the concept of FFPR transfer function matrices.

Definition 2: [9, Definition 4] A square real-rational proper transfer function matrix $G(s)$ is said to be finite frequency positive real with bandwidth $\bar{\omega}$ if it satisfies the following conditions:

- 1) $G(s)$ has no poles in the open right-half of the complex plane.
- 2) $G(j\omega) + G^*(j\omega) \geq 0$, for all $\omega \in \Omega$, where $\Omega = \{\omega \in \mathbb{R} : |\omega| \leq \bar{\omega}, j\omega \text{ is not a pole of } G(s)\}$.
- 3) Every pole of $G(s)$ on $j\bar{\Omega}$, if any, is simple and the corresponding residue matrix is positive semidefinite Hermitian, where $\bar{\Omega} = \{\omega \in \mathbb{R} : |\omega| \leq \bar{\omega}\}$.

Now, we are ready to state the relationship between FFNI transfer function matrices and FFPR transfer function matrices based on their definitions.

Lemma 1: Given a square real-rational proper transfer function matrix $R(s)$. Suppose $R(s)$ has no poles at the origin, and $R(\infty) = R^T(\infty)$. Then the following statements are equivalent:

- 1) $R(s)$ is FFNI with bandwidth $\bar{\omega}$.
- 2) $\hat{R}(s) \triangleq R(s) - R(\infty)$ is FFNI with bandwidth $\bar{\omega}$.
- 3) $F(s) \triangleq s\hat{R}(s)$ is FFPR with bandwidth $\bar{\omega}$.

Proof: (1 \Leftrightarrow 2) For $R(s)$ and $\hat{R}(s)$, when $j\omega$ is not a pole, we have $j[R(j\omega) - R^*(j\omega)] = j[\{\hat{R}(j\omega) + R(\infty)\} - \{\hat{R}^*(j\omega) + R(\infty)\}^*] = j[\hat{R}(j\omega) - \hat{R}^*(j\omega)]$; when $j\omega_0$ is a pole, $\lim_{s \rightarrow j\omega_0}(s - j\omega_0)jR(s) = \lim_{s \rightarrow j\omega_0}(s - j\omega_0)j\{\hat{R}(s) + R(\infty)\} = \lim_{s \rightarrow j\omega_0}(s - j\omega_0)j\hat{R}(s)$. Then the equivalence follows from Definition 1.

(2 \Leftrightarrow 3) Note that $F(s)$ and $\hat{R}(s)$ have the same set of poles; when $j\omega$ is not a pole, $F(j\omega) + F^*(j\omega) = j\omega\hat{R}(j\omega) - j\omega\hat{R}^*(j\omega) = j\omega[\hat{R}(j\omega) - \hat{R}^*(j\omega)]$; when $j\omega_0$ is a pole, $\lim_{s \rightarrow j\omega_0}(s - j\omega_0)F(s) = \lim_{s \rightarrow j\omega_0}(s - j\omega_0)s\hat{R}(s) = \lim_{s \rightarrow j\omega_0}(s - j\omega_0)j\omega_0\hat{R}(s) = \omega_0 \lim_{s \rightarrow j\omega_0}(s - j\omega_0)j\hat{R}(s)$. Then the equivalence follows from their definitions and the properties in Remark 1. ■

This lemma allows us to translate an FFNI problem to an FFPR problem. In the next section, the FFNI lemma will be developed in this way.

III. FINITE FREQUENCY NEGATIVE IMAGINARY LEMMA

The FFNI Lemma developed in this section establishes a necessary and sufficient condition for a transfer function matrix to be FFNI in terms of the matrices appearing in a minimal state-space realization of the transfer function. This lemma could be considered as a generalization the Negative Imaginary Lemma [1], [2] and is analogous to the FFPR Lemma [9].

Theorem 1 (Finite Frequency Negative Imaginary Lemma): Consider a real-rational proper transfer function matrix $R(s)$ with a minimal state-space realization (A, B, C, D) . Suppose all poles of $R(s)$ are in the closed left-half of the complex plane, and the poles on the imaginary axis, if any, are simple. Let a positive scalar $\bar{\omega}$ be given. If A has eigenvalues $j\omega_i$ ($i \in \{1, \dots, q\}$) such that $0 < \omega_i \leq \bar{\omega}$, the residue of $(sI - A)^{-1}$ at $s = j\omega_i$ is given by $\Phi_i \triangleq \lim_{s \rightarrow j\omega_i}(s - j\omega_i)(sI - A)^{-1}$. The following statements are equivalent:

- 1) The transfer function matrix $R(s)$ is FFNI with bandwidth $\bar{\omega}$.
- 2) $\det(A) \neq 0$, $D = D^T$, and the transfer function matrix $F(s)$ with a minimal state-space realization (A, B, CA, CB) is FFPR with bandwidth $\bar{\omega}$.
- 3) $\det(A) \neq 0$, $D = D^T$, $CA\Phi_i B = (CA\Phi_i B)^* \geq 0$ for all $i \in \{1, \dots, q\}$ if A has any eigenvalues on $j\bar{\Omega}$ where $\bar{\Omega} = \{\omega \in \mathbb{R} : 0 < \omega \leq \bar{\omega}\}$, and there exist real symmetric matrices $P = P^T$ and $Q = Q^T \geq 0$ such that

$$PA + A^T P - A^T Q A + \bar{\omega}^2 Q \leq 0, \quad (1)$$

$$C + B^T A^{-T} P = 0, \quad (2)$$

$$Q A^{-1} B = 0. \quad (3)$$

- 4) $\det(A) \neq 0$, $D = D^T$, $CA\Phi_i B = (CA\Phi_i B)^* \geq 0$ for all $i \in \{1, \dots, q\}$ if A has any eigenvalues on $j\bar{\Omega}$ where $\bar{\Omega} = \{\omega \in \mathbb{R} : 0 < \omega \leq \bar{\omega}\}$, and there exist real symmetric matrices $Y = Y^T$ and $X = X^T \geq 0$ such that

$$AY + Y A^T - A X A^T + \bar{\omega}^2 X \leq 0, \quad (4)$$

$$B + A Y C^T = 0, \quad (5)$$

$$C X = 0. \quad (6)$$

Proof: (1 \Leftrightarrow 2) Because $R(s) = C(sI - A)^{-1}B + D$, $\hat{R}(s) = C(sI - A)^{-1}B = R(s) - R(\infty)$ and $F(s) = CA(sI - A)^{-1}B + CB = s\hat{R}(s)$, this equivalence follows from their definitions and Lemma 1.

(2 \Leftrightarrow 3) In view of Theorem 3 of [9], the transfer function matrix $F(s)$ with a minimal realization (A, B, CA, CB) is FFPR with bandwidth $\bar{\omega}$ if and only if $CA\Phi_i B = (CA\Phi_i B)^* \geq 0$ for all $i \in \{1, \dots, q\}$ if A has any eigenvalues on $j\bar{\Omega}$, and there exist real symmetric matrices $P = P^T$ and $Q = Q^T \geq 0$ such that

$$\begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^T \begin{bmatrix} -Q & P \\ P & \bar{\omega}^2 Q \end{bmatrix} \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} \leq \begin{bmatrix} 0 & A^T C^T \\ CA & CB + B^T C^T \end{bmatrix}. \quad (7)$$

So we only need to prove that the inequality (7) is equivalent to the inequality in (1) and the equations in (2), (3).

Note that the inequality in (7) can be re-written as

$$\begin{bmatrix} \Xi_{11} & \Xi_{12} \\ \Xi_{12}^T & \Xi_{22} \end{bmatrix} \leq 0,$$

where

$$\Xi_{11} = -A^T Q A + P A + A^T P + \bar{\omega}^2 Q,$$

$$\Xi_{12} = -A^T Q B + P B - A^T C^T,$$

$$\Xi_{22} = -B^T Q B - C B - B^T C^T.$$

Pre- and post-multiplying this inequality by

$$\begin{bmatrix} I & 0 \\ -B^T A^{-T} & I \end{bmatrix}$$

and its transpose, respectively, we obtain

$$\begin{bmatrix} \Xi_{11} & \bar{\Xi}_{12} \\ \bar{\Xi}_{12}^T & \bar{\Xi}_{22} \end{bmatrix} \leq 0,$$

where $\bar{\Xi}_{12} = -A^T P A^{-1} B - \bar{\omega}^2 Q A^{-1} B - A^T C^T$. Therefore we must have $\bar{\omega}^2 B^T A^{-T} Q A^{-1} B = 0$, which is equivalent to (3). Furthermore, the above inequality becomes

$$\begin{bmatrix} \Xi_{11} & -A^T P A^{-1} B - A^T C^T \\ -B^T A^{-T} P A - C A & 0 \end{bmatrix} \leq 0. \quad (8)$$

Therefore, the inequality (8) is equivalent to (1), (2) as the matrix A is nonsingular. Now we can conclude that the inequality (7) is equivalent to the inequality in (1) and the equations in (2), (3).

(2 \Leftrightarrow 4) In view of Theorem 3 of [9], the transfer function matrix $F(s)$ with a minimal realization (A, B, CA, CB) is FFPR with bandwidth $\bar{\omega}$ if and only if $CA\Phi_i B = (CA\Phi_i B)^* \geq 0$ for all $i \in \{1, \dots, q\}$ if A has any eigenvalues on $j\bar{\Omega}$, and there exist real symmetric matrices $Y = Y^T$ and $X = X^T \geq 0$ such that

$$\begin{bmatrix} A & I \\ CA & 0 \end{bmatrix} \begin{bmatrix} -X & Y \\ Y & \bar{\omega}^2 X \end{bmatrix} \begin{bmatrix} A & I \\ CA & 0 \end{bmatrix}^T \leq \begin{bmatrix} 0 & B \\ B^T & CB + B^T C^T \end{bmatrix}. \quad (9)$$

So we only need to prove that the inequality (9) is equivalent to (4), (5), (6).

The inequality in (9) can be re-written as

$$\begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{12}^T & \Theta_{22} \end{bmatrix} \leq 0,$$

where

$$\Theta_{11} = -A X A^T + Y A^T + A Y + \bar{\omega}^2 X,$$

$$\Theta_{12} = -A X A^T C^T + Y A^T C^T - B,$$

$$\Theta_{22} = -C A X A^T C^T - C B - B^T C^T.$$

Pre- and post-multiplying the above inequality by

$$\begin{bmatrix} I & 0 \\ -C & I \end{bmatrix}$$

and its transpose, respectively, we obtain

$$\begin{bmatrix} \Theta_{11} & \bar{\Theta}_{12} \\ \bar{\Theta}_{12}^T & \bar{\Theta}_{22} \end{bmatrix} \leq 0,$$

where $\bar{\Theta}_{12} = -A Y C^T - \bar{\omega}^2 X C^T - B$. Therefore we must have $\bar{\omega}^2 C X C^T = 0$, which is equivalent to (6). Furthermore, the above inequality becomes

$$\begin{bmatrix} \Theta_{11} & -A Y C^T - B \\ -C Y A^T - B^T & 0 \end{bmatrix} \leq 0.$$

Therefore, the above inequality is equivalent to (4), (5), and we conclude that the inequality (9) is equivalent to (4), (5), (6). This completes the proof. \blacksquare

Remark 2: In Theorem 1, another method to compute the residue matrix Φ_i is to use the formula $\Phi_i = r_i l_i^*$ where r_i and l_i are column vectors such that $A r_i = j\omega_i r_i$, $l_i^* A = j\omega_i l_i^*$, and $l_i^* r_i = 1$; see Lemma 6 of [9] for more details.

It follows from the definitions (i.e., Definition 3 of [2] and Definition 1) that when the bandwidth $\bar{\omega} \rightarrow \infty$, the FFNI transfer function matrix $R(s)$ reduces to a normal negative imaginary transfer function matrix. In the next result,

the conditions in the Finite Frequency Negative Imaginary Lemma are proved to reduce to the conditions in the Negative Imaginary Lemma in [2].

Corollary 1: Under the same assumptions as in Theorem 1. Let the bandwidth $\bar{\omega} \rightarrow \infty$. Then the finite frequency negative imaginary lemma reduces to the negative imaginary lemma. That is, the following two statements are equivalent:

- 1) The transfer function matrix $R(s)$ is negative imaginary.
- 2) $\det(A) \neq 0$, $D = D^T$, and there exists a real symmetric matrix $Y = Y^T > 0$ such that

$$AY + YA^T \leq 0, \quad \text{and} \quad B + AYC^T = 0. \quad (10)$$

Proof: To complete the proof, we need to show, under the assumptions of Theorem 1, that (a) the inequality in (4) and the equations in (5) and (6) are reduced to (10); (b) the real symmetric matrix $Y = Y^T$ is positive definite; and (c) the matrix $CA\Phi_i B$ is always positive semidefinite Hermitian. The proof is accordingly divided into three steps.

Step 1: We show that the inequality in (4) with the equality constraints in (5) and (6) reduces to the inequality and the equation in (10).

Since the bandwidth $\bar{\omega}$ approaches infinity, the parameter X in (4) must approach zero. So we have $X = 0$. Then the inequality in (4) and the equations in (5) and (6) reduce to the inequality and the equation in (10).

Step 2: Under the condition that all the eigenvalues of A are in the closed left-half of the complex plane, we will prove that the real symmetric matrix $Y = Y^T$ satisfying (10) is positive definite.

Because all of the poles of $R(s)$ are assumed to be in the closed left-half plane, that is, $\Re[\lambda_i(A)] \leq 0$, it follows from the inequality in (10) that $Y = Y^T \geq 0$. Next we prove that Y is nonsingular by contradiction.

Suppose that Y is singular. Then an unitary congruence transformation can be used to give

$$U^*YU = \begin{bmatrix} Y_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad U^*AU = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

$$U^*B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad CU = [C_1 \quad C_2],$$

where $Y_1 = Y_1^T > 0$ is nonsingular and U is an unitary matrix, that is, $U^*U = I$. Hence we can assume that the matrices Y , A , B and C are of the following forms without loss of generality:

$$Y = \begin{bmatrix} Y_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad Y_1 = Y_1^T > 0, \quad A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

$$B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad C = [C_1 \quad C_2].$$

With those forms, the inequality in (10) can be re-written as

$$\begin{bmatrix} A_{11}Y_1 + Y_1A_{11}^T & Y_1A_{21}^T \\ A_{21}Y_1 & 0 \end{bmatrix} \leq 0.$$

Because the (2, 2) block of the above LMI is zero, we must have $A_{21}Y_1 = 0$. Furthermore, the non-singularity of Y_1

leads to $A_{21} = 0$. Therefore, the matrix A is of the form

$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}. \quad (11)$$

Similarly, the equation in (10) can be re-written as

$$\begin{bmatrix} B_1 + A_{11}Y_1C_1^T \\ B_2 + A_{21}Y_1C_2^T \end{bmatrix} = 0.$$

Because $A_{21} = 0$, we have $B_2 = 0$. Therefore, the matrix B is of the form

$$B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}. \quad (12)$$

It follows from the matrix forms in (11) and (12) that the matrix pair (A, B) is not controllable. This contradicts the controllability of (A, B) . Hence Y must be nonsingular.

In summary, we have that $Y = Y^T \geq 0$ and that Y is nonsingular. So $Y = Y^T > 0$.

Step 3: Under the assumption that the purely imaginary poles of $R(s)$, if any, are simple, we will prove that the matrix $CA\Phi_i B$ is always positive semidefinite Hermitian.

Firstly, in view of the equation in (10), we have that $CA\Phi_i B = -CA\Phi_i AYC^T$. In the sequel, it suffices to show that $A\Phi_i AY$ is negative semidefinite Hermitian.

Suppose that $R(s)$ has a purely imaginary pole pair at $\pm j\omega_i$, $\omega_i > 0$; that is, the matrix A has eigenvalues at $\pm j\omega_i$. Then there exists a unique nonsingular real matrix T (e.g., by considering the real canonical form of the matrix A) such that

$$TAT^{-1} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}.$$

where $A_1 \in \mathbb{R}^{(n-2) \times (n-2)}$ has no eigenvalues at $\pm j\omega_i$, and $A_2 \in \mathbb{R}^{2 \times 2}$ is of the form

$$A_2 = \begin{bmatrix} 0 & -\omega_i \\ \omega_i & 0 \end{bmatrix}.$$

Hence, we can assume that the matrices Y , A , B and C are of the following forms without loss of generality:

$$Y = \begin{bmatrix} Y_1 & Y_3 \\ Y_3^T & Y_2 \end{bmatrix} = Y^T > 0, \quad A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix},$$

$$B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad C = [C_1 \quad C_2],$$

where A_1 is nonsingular and has no eigenvalues at $\pm j\omega_i$, and $A_2 = \begin{bmatrix} 0 & -\omega_i \\ \omega_i & 0 \end{bmatrix}$.

Now, we calculate

$$\begin{aligned} \Psi_i &\triangleq \lim_{s \rightarrow j\omega_i} (s - j\omega_i)(sI - A_2)^{-1} \\ &= \lim_{s \rightarrow j\omega_i} (s - j\omega_i) \frac{1}{s^2 + \omega_i^2} \begin{bmatrix} s & -\omega_i \\ \omega_i & s \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} 1 & j \\ -j & 1 \end{bmatrix}. \end{aligned}$$

Therefore,

$$\Phi_i \triangleq \lim_{s \rightarrow j\omega_i} (s - j\omega_i)(sI - A)^{-1}$$

$$\begin{aligned}
&= \lim_{s \rightarrow j\omega_i} (s - j\omega_i) \begin{bmatrix} (sI - A_1)^{-1} & 0 \\ 0 & (sI - A_2)^{-1} \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 \\ 0 & \Psi_i \end{bmatrix}.
\end{aligned}$$

On the other hand, the inequality in (10) can be re-written as

$$\begin{bmatrix} A_1 Y_1 + Y_1 A_1^T & A_1 Y_3 + Y_3 A_2^T \\ Y_3 A_1^T + A_2 Y_3^T & A_2 Y_2 + Y_2 A_2^T \end{bmatrix} \leq 0. \quad (13)$$

Hence, we have $A_2 Y_2 + Y_2 A_2^T \leq 0$ with $Y_2 = Y_2^T > 0$. Let Y_2 be

$$Y_2 = \begin{bmatrix} y_1 & y_3 \\ y_3 & y_2 \end{bmatrix}.$$

Then

$$A_2 Y_2 + Y_2 A_2^T = \begin{bmatrix} 2\omega_i y_3 & \omega_i(y_2 - y_1) \\ \omega_i(y_2 - y_1) & -2\omega_i y_3 \end{bmatrix} \leq 0.$$

So we must have $y_3 = 0$ and $y_1 = y_2 > 0$. That is, the matrix Y_2 must be of the form

$$Y_2 = \begin{bmatrix} y & 0 \\ 0 & y \end{bmatrix}, \quad y > 0.$$

Now, we have that $A_2 Y_2 + Y_2 A_2^T = 0$; that is, the (2, 2) block of (13) is zero. Therefore,

$$A_1 Y_3 + Y_3 A_2^T = 0. \quad (14)$$

Because the matrices A_1 and $-A_2^T$ have no common eigenvalues, the Sylvester equation (14) has a unique solution which is given by $Y_3 = 0$. Therefore, the matrix Y is of the form

$$Y = \begin{bmatrix} Y_1 & 0 \\ 0 & Y_2 \end{bmatrix} = Y^T > 0.$$

Now, we can calculate

$$\begin{aligned}
A_2 \Psi_i A_2 Y_2 &= \begin{bmatrix} 0 & -\omega_i \\ \omega_i & 0 \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & j \\ -j & 1 \end{bmatrix} \begin{bmatrix} 0 & -\omega_i \\ \omega_i & 0 \end{bmatrix} \begin{bmatrix} y & 0 \\ 0 & y \end{bmatrix} \\
&= -\frac{\omega_i^2 y}{2} \begin{bmatrix} 1 & j \\ -j & 1 \end{bmatrix}.
\end{aligned}$$

Therefore, $A_2 \Psi_i A_2 Y_2 = (A_2 \Psi_i A_2 Y_2)^* \leq 0$. Hence

$$A\Phi_i AY = \begin{bmatrix} 0 & 0 \\ 0 & A_2 \Psi_i A_2 Y_2 \end{bmatrix}.$$

Therefore, $A\Phi_i AY = (A\Phi_i AY)^* \leq 0$. So we have $-CA\Phi_i AY C^T = (-CA\Phi_i AY C^T)^* \geq 0$, that is, $CA\Phi_i B = (CA\Phi_i B)^* \geq 0$. This completes the proof. ■

The following theorem provides a time-domain interpretation of the FFNI properties in terms of the system input, output and state. It is hoped that this result can provide a deeper understanding of FFNI systems.

Theorem 2: Consider a proper stable transfer function matrix $R(s)$ with $R(\infty) = R^T(\infty)$. Let u , y and x be the input, the output and the state of any minimal realization of $R(s)$. Then, the following statements are equivalent:

- 1) $R(s)$ is FFNI with bandwidth $\bar{\omega}$.

- 2) The inequality

$$\int_{-\infty}^{\infty} [\dot{y}(t) - D\dot{u}(t)]^T u(t) dt \geq 0 \quad (15)$$

holds for all square integrable and differentiable inputs u such that

$$\int_{-\infty}^{\infty} \dot{x}(t) \dot{x}^T(t) dt \leq \bar{\omega}^2 \int_{-\infty}^{\infty} x(t) x^T(t) dt. \quad (16)$$

Proof: Let (A, B, C, D) be a minimal state-space realization of $R(s)$. Then the linear system whose transfer function is given by $R(s)$ can be represented as

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t) + Du(t). \end{cases} \quad (17)$$

Let us consider a new transfer function matrix $F(s) \triangleq s[R(s) - R(\infty)]$. Then (A, B, CA, CB) is a minimal state-space realization of $F(s)$ and the corresponding dynamical system is represented by

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ \tilde{y}(t) = CAx(t) + CBu(t). \end{cases} \quad (18)$$

In view of Lemma 1, the transfer function matrix $R(s)$ is FFNI with bandwidth $\bar{\omega}$ if and only if the transfer function matrix $F(s)$ is FFPR with bandwidth $\bar{\omega}$. In view of Theorem 4 of [9], $F(s)$ is FFPR with bandwidth $\bar{\omega}$ if and only if the passivity property

$$\int_{-\infty}^{\infty} \tilde{y}^T(t) u(t) dt \geq 0$$

holds for all square integrable inputs u such that the inequality (16) holds.

On the other hand, it follows from the system equations in (17) and (18) that

$$\tilde{y}(t) = C[Ax(t) + Bu(t)] = C\dot{x}(t) = \dot{y}(t) - D\dot{u}(t)$$

Therefore, $F(s)$ is FFPR with bandwidth $\bar{\omega}$ if and only if the inequality in (15) holds for all square integrable and differentiable inputs u such that (16) holds. This completes the proof. ■

Remark 3: When the transfer function matrix $R(s)$ in Theorem 2 is strictly proper (i.e., $D = 0$), the requirement of differentiability of the inputs can be removed.

IV. ILLUSTRATIVE EXAMPLES

In this section, an illustrative example is presented to illustrate the application of the FFNI concept and the FFNI lemma developed in this paper. The example is the capacitance subsystem of a piezoelectric tube scanner system studied in [3].

Let us consider the piezoelectric tube studied in [3], [10]. Such a piezoelectric tube is used in the scanning unit of scanning tunneling microscopes and atomic force microscopes. The inputs to the piezoelectric tube are two voltage signals: V_{x+} and V_{y+} , which are applied to the “input” ends of the electrodes of the piezoelectric tube. The outputs to the piezoelectric tube are classified into two

groups. The first group includes the voltages V_{x-} and V_{y-} , which are the voltages at the “output” ends of the electrodes. The second output group includes the distances dx (x -axis direction) and dy (y -axis direction) between the aluminum cube being positioned and the capacitive sensors’ heads. These distances are measured by the capacitance sensors in terms of the change in the capacitance between the aluminum cube and the heads of the capacitive sensors. Accordingly, the system transfer function from input $[V_{x+} \ V_{y+}]^T$ to output $[V_{x-} \ V_{y-}]^T$ is called the voltage subsystem of the tube; the system transfer function from input $[V_{x+} \ V_{y+}]^T$ to output $[dx \ dy]^T$ is called the capacitance subsystem of the tube; see [3] for more details.

For the capacitance subsystem of the tube, the experiment in [3] shows that the transfer functions from V_{x+} to dx and from V_{y+} to dy are given by

$$G_{d_x x}^{(v)}(s) = G_{d_y y}^{(v)}(s) = \frac{c_1 s^2 + c_2 s + c_3}{s^2 + 2\sigma_1 \omega_1 s + \omega_1^2}.$$

Note that the equality $G_{d_x x}^{(v)}(s) = G_{d_y y}^{(v)}(s)$ is expected because of the symmetric alignment of the capacitive sensors and the faces of the aluminum cube in the x and y directions [3]. The parameter values of the transfer function are given by $2\sigma_1 \omega_1 = 60.2$, $\omega_1^2 = 2.8488 \times 10^7$, $c_1 = 0.0055$, $c_2 = -112.3$ and $c_3 = 1.807 \times 10^6$; see Table I of [3].

Now, we illustrate that the above transfer function is actually FFNI. We will also determine the corresponding bandwidth. Since the transfer function $G_{d_x x}^{(v)}(s)$ has no purely imaginary poles, we only need to consider the imaginary part of $G_{d_x x}^{(v)}(j\omega)$ on the $(0, \infty)$. It follows from a direct computation that

$$\Im[G_{d_x x}^{(v)}(j\omega)] = \frac{\omega(-3.3080 \times 10^9 + 112.6311\omega^2)}{(2.8488 \times 10^7 - \omega^2)^2 + 3624\omega^2}.$$

Therefore, $G_{d_x x}^{(v)}(s)$ is FFNI with bandwidth $\bar{\omega} = \sqrt{\frac{3.3080 \times 10^9}{112.6311}} = 5419.4$. The Nyquist plot of this transfer function is shown in Figure 1. It can be seen from this

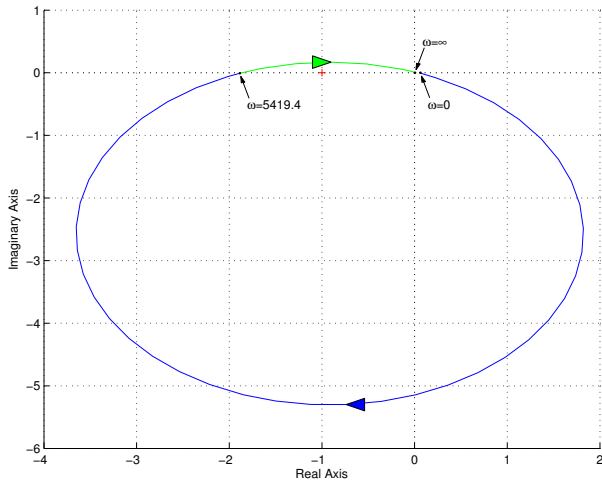


Fig. 1. Nyquist plot of piezo tube frequency response ($\omega \geq 0$)

figure that the imaginary part of $G_{d_x x}^{(v)}(j\omega)$ is negative for $0 < \omega < 5419.4$.

To verify the FFNI Lemma, we first found a minimal state-space realization of $G_{d_x x}^{(v)}(s)$ with

$$A = \begin{bmatrix} -60.2 & -6955.0781 \\ 4096 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 32 \\ 0 \end{bmatrix}, \\ C = [-3.5197 \quad 12.5909], \quad D = 0.0055.$$

Now, solving the linear matrix inequality in (1) and the linear matrix equations in (2), (3) with $\bar{\omega} = 5419.4$, we found a set of feasible solutions. If we set the bandwidth $\bar{\omega}$ in (1) to be a slightly larger number, say 5419.5, then (1), (2), (3) have no feasible solutions. According to the FFNI lemma, the transfer function $G_{d_x x}^{(v)}(s)$ is FFNI with bandwidth $\bar{\omega} = 5419.4$ but not FFNI with bandwidth $\bar{\omega} = 5419.5$. This confirms the above findings via direct computations and the Nyquist plot.

V. CONCLUSIONS

This paper has studied the FFNI properties of dynamical systems. The concept of FFNI transfer function matrices was first introduced. Then an FFNI lemma was derived which gave a condition for dynamical systems to be FFNI in terms of their minimal state-space realizations. A time-domain interpretation of the FFNI property was also proposed in terms of the system input, output and state. Finally, the FFNI lemma was illustrated by examples involving a piezoelectric tube scanner system and a mass-spring-damper system. A area for future research is to develop some stability results for the interconnected FFNI systems.

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