An Algorithm for Joint Identification and Control

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Abstract

This work proposes a unified algorithm for identification and control. Frequency domain data of the plant is weighted to satisfy the given performance specifications. A model is then identified from this weighted frequency domain data and a controller is synthesised using the $\mathcal{H}_\infty$ loopshaping design procedure. The cost function used in the identification stage essentially minimises a tight upper bound on the difference between the achieved and the designed performance in the sense of the $\mathcal{H}_\infty$ loopshaping design paradigm.

1 Introduction

Identification of dynamic models for control has been an active area of research in the past few years. Given an unknown true plant P_0, the designer wishes to maximise some performance criterion as expressed by a function of plant and controller, say, $J(P_0, C)$. Since P_0 is unknown, a model $\hat{P}$ has to be found on the basis of which a controller can be designed. Since

$$J(P_0, C) \geq J(P, C) - |J(P, C) - J(P_0, C)|$$

for any controller C and model P, a promising approach to joint identification and control design will be to solve

$$\max_{P,C} \{ J(P, C) - |J(P, C) - J(P_0, C)| \} \qquad (1)$$

assuming that this term can be somehow captured in a cost function. In practice, this problem is solved iteratively. $i-$th stage of a typical iterative method proceeds as follows:

1. Given a controller C_{i-1}, carry out identification to obtain a new model $\hat{P}_i$ which minimises the mismatch between the designed performance and achieved performance:

$$\hat{P}_i = \arg\min_P |J(P, C_{i-1}) - J(P_0, C_{i-1})| \qquad (2)$$

2. Synthesise a controller C_i for a model $\hat{P}_i$, which maximises the designed performance:

$$C_i = \arg\max_C J(\hat{P}_i, C) \qquad (3)$$

If $J(\hat{P}_i, C_i) - |J(P, C_i) - J(P_0, C_i)|$ is satisfactory, stop; otherwise go back to step 1.

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If the minimum (maximum) is non-unique, it is assumed that any minimising (maximising) argument is chosen. The identification stage in successive iteration may or may not involve new experiments. $|J(P, C_{i-1}) - J(P_0, C_{i-1})|$ is often approximated by a least squares problem over a finite time or frequency domain data. Iterative strategies for identification and control are widely discussed in literature; see [1], [2] and references therein. Despite their intuitive appeal, these iterative schemes suffer from lack of performance guarantees. In these schemes, the cost in (2) (resp. in (3)) is not guaranteed to be non-decreasing (resp. non-increasing).

In this paper, some recent results on identification in the $\nu-$gap metric and on weight selection in $\mathcal{H}_\infty$ loopshaping are combined together to present a unified framework for identification and control. The identification algorithm used here was first proposed in [3]. Instead of using controller C_{i-1} as in (2), this algorithm minimises an upper bound on the (pointwise) mismatch between the designed and achieved performance for *any* controller within a set of controllers. The weight selection stage uses a convex optimisation based procedure, proposed in [4], to synthesise weights and a controller that maximise the $\mathcal{H}_\infty$ loopshaping performance criterion. Further, a new method of model and weight adjustment to minimise the relevant identification cost is presented here.

The rest of the paper is organised as follows. Section 2 introduces the notation used. Section 3 introduces the $\mathcal{H}_\infty$ loopshaping design procedure and its relation with the $\nu-$gap metric. Section 4 outlines the new algorithm and section 5 demonstrates it with a simulation example.

2 Notation

$\mathbb{C}$ and $\mathbb{R}$ represent real and complex numbers respectively. $\mathbb{C}_+$ denote the open right half plane, $\mathbb{C}_+ := \{s \in \mathbb{C} : re(s) > 0\}$. $\mathcal{R}^{m\times n}$ denotes the space of all real rational transfer functions with n inputs and m outputs. The superscript $m \times n$ is dropped whenever the dimension of transfer matrix is irrelevant. $\mathcal{L}_\infty$ denotes the normed space of all functions essentially bounded on $j\mathbb{R}$ and having norm $\|f\|_{\mathcal{L}_\infty} := ess\ \sup_\omega \overline{\sigma}(f(j\omega))$. $\mathcal{H}_\infty$ denotes the subspace of functions in $\mathcal{L}_\infty$ that are analytic and bounded in $\mathbb{C}_+$. $\mathcal{RL}_\infty$ ($\mathcal{RH}_\infty$) represents the subspace of real rational transfer functions in $\mathcal{L}_\infty$ ($\mathcal{H}_\infty$).

For $P \in \mathcal{R}$, $P^{\sim}(s) := P^T(-s)$.

$P = ND^{-1}$ $(= \tilde{D}^{-1}\tilde{N})$ is called a normalised right (left) coprime factorisation of plant $P \in \mathcal{R}$ if N, D are right coprime ($\tilde{D}$, $\tilde{N}$ are left coprime) and $N^*N + D^*D = I$ $(\tilde{N}\tilde{N}^* + \tilde{D}\tilde{D}^* = I)$. $G_i := [N_i^T \; D_i^T]^T$ and $\tilde{G}_i := [-\tilde{D}_i \; \tilde{N}_i]$ are respectively called the normalised right and normalised left graph symbols of plant P_i.

3 $\mathcal{H}_\infty$ loopshaping and the $\nu-$gap metric

$\mathcal{H}_\infty$ loopshaping combines the traditional loopshaping wisdom with robust stabilisation of normalised co-prime factors. Given a model P of a plant, a pre-compensator W_1 and a post-compensator W_2 are selected so that $P_s := W_2\, P\, W_1$ has the desired loopshape. The loopshape is determined from the closed loop performance specifications. Typically, this means choosing compensators (or weights) W_1 and W_2 such that $\underline{\sigma}(P_s)(j\omega) \gg 1$ over some low frequency range, $\overline{\sigma}(P_s) \ll 1$ over some high frequency range and $\sigma_i(P_s)$ have a moderate roll off rate around crossover frequency. Unlike classical loopshaping, the designer need not shape phase explicitly. The loopshaping constraints considered here are

$$\alpha(\omega) < \sigma_i(P_s(j\omega)) < \beta(\omega) \quad \forall\, \omega \in \Omega \tag{4}$$

where α and β are non-negative, real scalar functions and $\Omega = \{\omega_1, \omega_2, \ldots, \omega_m\}$ specifies a set (or a grid) of frequencies of interest. The loopshaping weights W_1 and W_2 are often selected to be diagonal, stable and inversely stable transfer matrices. It is not always easy to see how the weights affect the singular values of P_s. Often the designer has to rely on a process of trial and error to arrive at satisfactory weighting transfer functions. Following an algorithm proposed in [4], a method is suggested in section 4.2 for selection of weights which alleviates these difficulties.

The performance measure chosen for synthesising a controller C stabilising P_s is

$$b\,(P_s, C) = \|H(P_s, C)\|_\infty^{-1} \tag{5}$$

where the closed loop transfer function $H(P_s, C)$ is defined by

$$H(P_s, C) = \begin{bmatrix} P_s \\ I \end{bmatrix} (I - CP_s)^{-1} \begin{bmatrix} -C & I \end{bmatrix} \tag{6}$$

$b\,(P_s, C)$ represents the robustness of the closed loop against bounded perturbations of normalised coprime factors of the shaped plant P_s. The best achievable robust stability margin is defined by

$$b_{opt}(P_s) := \max_{C \text{ stabilising}} b(P_s, C)$$

and can be explicitly computed [5]. The same reference also provides a characterisation of all controllers C_∞ achieving $b\,(P_s, C_\infty) = \epsilon < b_{opt}(P_s)$. $b_{opt}(P_s)$ is an indicator of success of the loopshaping design stage; a large (resp. small) $b_{opt}(P_s)$ indicates compatibility (resp. incompatibility) between the designed loopshape and robust stability. As a rule of thumb, $b\,(P_s, C_\infty) > 0.3$ would be considered adequate in most cases. The final controller is given by re-aligning weights with the controller; $C_s = W_1 C_\infty W_2$. Besides robust stability, another motivation for the use of $b\,(P_s, C)$ as a performance measure comes from the fact that the size of each of the four closed loop transfer matrices in $H(P, C_s)$ can be bounded from above at each frequency ω in terms of $b\,(P_s, C_\infty)$, $\overline{\sigma}(P_s)(j\omega)$, $\underline{\sigma}(P_s)(j\omega)$ and the condition numbers of $W_1(j\omega)$, $W_2(j\omega)$. More details on $\mathcal{H}_\infty$ loopshaping may be found in [6]. $\mathcal{H}_\infty$ loopshaping has been used successfully in a variety of applications; see [7] and references therein.

A metric called $\nu-$gap metric was suggested in [8] as a natural dual to $b\,(P_s, C)$. The $\nu-$gap between two plants P_1, $P_2 \in \mathcal{R}$ can be defined as

$$\begin{aligned} \delta_\nu(P_1, P_2) &= \inf_{Q, Q^{-1} \in \mathcal{L}_\infty} \| G_1 - G_2 Q \|_\infty \text{ if } I(P_1, P_2) = 0 \\ &= \quad 1 \quad \text{otherwise} \end{aligned} \tag{7}$$

where $I(P_1, P_2) :=$ wno $\det(G_2^* G_1) =$ wno $\det(\tilde{G}_1 \tilde{G}_2^*)$ and wno (g) denotes the winding number of $g(s)$ evaluated on the standard Nyquist contour indented to the right around any poles on $j\mathbb{R}$. For a real rational transfer matrix X such that $X, X^{-1} \in \mathcal{RL}_\infty$, winding number wno $\det(X)$ is the excess of number of zeros of X in $\mathbb{C}_+$ over the number of poles of X in $\mathbb{C}_+$. When $I(P_1, P_2) = 0$, $\delta_\nu(P_1, P_2)$ equals $\mathcal{L}_2$ *-gap*, defined by

$$\delta_{\mathcal{L}_2}(P_1, P_2) := \|\tilde{G}_2 G_1\|_\infty = \sup_\omega \kappa(P_1, P_2) \tag{8}$$

where $\kappa(P_1, P_2)(j\omega)$ is the pointwise chordal distance,

$$\kappa(P_1, P_2)(j\omega) := \overline{\sigma}\Big((I + P_2 P_2^*)^{-\frac{1}{2}} (P_2 - P_1)(I + P_1^* P_1)^{-\frac{1}{2}}\Big)(j\omega)$$

It is known that [8] any controller stabilising P_1 and achieving $b\,(P_1, C) > \alpha$ stabilises the plant set

$$\{P_2 : \delta_\nu(P_1, P_2) \leq \alpha\}$$

More importantly, $\delta_\nu(P_1, P_2)$ is a measure of the 'closeness' of the closed loop performance of P_1 and P_2 for a given controller. The following result can be easily derived from the proof of theorem 3.8 in [9]:

Lemma 1 *Suppose a controller C stabilises a given pair of plants P_1, P_2. Then*

$$\frac{1}{\overline{\sigma}(H(P_1, C))(j\omega)} \geq \frac{1}{\overline{\sigma}(H(P_2, C))(j\omega)} - \kappa(P_1, P_2)(j\omega) \tag{9}$$

From (5)-(9), it follows that any controller C that stabilises P_2 with a good $b\,(P_2, C)$ also stabilises P_1, *without any significant deterioration in performance* (in terms of $b\,(P_1, C)$), provided $\delta_\nu(P_1, P_2)$ is small.

To pose a control oriented identification problem with finite data, some relevant quantities need to be defined. As *a posteriori* information in the identification process, suppose that a block matrix of (not necessarily uniformly spaced) frequency response samples of the true plant $P_0(s) \in \mathcal{R}^{p\times n}$ at measurement frequencies ω_i, $i = 1, 2, \ldots, m$ is given:

$$P_\Omega := \begin{bmatrix} P_0(j\omega_1) & P_0(j\omega_2) & \ldots & P_0(j\omega_m) \end{bmatrix} \tag{10}$$

$$\begin{aligned} \text{Define } \delta_\Omega(P_1, P_2) &:= \max_{i\in[1,m]} \kappa(P_1, P_2)(j\omega_i) \text{ if } I(P_1, P_2) = 0 \\ &= 1 \qquad otherwise \end{aligned} \tag{11}$$

Next, for any model P_1 and a controller C stabilising both the plant P_0 and model P_1, define a performance measure over finite frequency set,

$$b_\Omega(P_k, C) := \left\{ \max_{i\in[1,m]} \overline{\sigma}\left(H(P_k, C)\right)(j\omega_i) \right\}^{-1}, \; k = 0, 1 \tag{12}$$

Then from (9), it is easy to show that

$$b_\Omega(P_0, C) \geq b_\Omega(P_1, C) - \delta_\Omega(P_0, P_1) \tag{13}$$

holds.

4 A unified Algorithm for identification and $\mathcal{H}_\infty$ loopshaping control

4.1 Outline of the algorithm

From (13), a sensible - although intractable - joint identification and control problem would be

$$\max_{C, W_1, W_2, \hat{P}} b_\Omega(\hat{P}, C) - \delta_\Omega(W_2 P_0 W_1, \hat{P}) \tag{14}$$

where the weighting transfer function matrices W_1, W_2 and the model $\hat{P}$ are constrained to appropriate sets, C belongs to the set of controllers stabilising $\hat{P}$ and $W_2 P_0 W_1(j\omega)$ satisfies the loopshaping constraints (4). Comparing with (1), note that the second term in (14) is independent of controller.

Here, an algorithm which minimises a cost similar to (14) is outlined. Details of its numerical implementation are discussed in the subsequent sections.

Given: P_Ω as in (10), a controller C_{init} stabilising P_0 (possibly 0 for a stable plant) and the loopshaping specifications (4).

1. Find stable, minimum phase diagonal weighting transfer matrices $W_{1,0}, W_{2,0}$ such that the 'shaped' frequency response samples $W_{2,0}P_0W_{1,0}(j\omega_i)$ satisfy (4) for all ω_i, $i \in [1, m]$.
2. Solve
$$\min_{P\in\mathcal{S}} \delta_\Omega(W_{2,0} P_0 W_{1,0}, P) \tag{15}$$
where $\mathcal{S}$ is an appropriate model set. Let λ_1 be the achieved minimum cost and let $\hat{P}_1 \in \mathcal{S}$ be any model which achieves it.
3. Given $\hat{P}_1$, solve
$$\min_{\substack{W_1\in\mathcal{W}_1 \\ W_2\in\mathcal{W}_2}} \max_i \kappa(W_2 P_0 W_1, -(W_1^{-1}\overline{P}^{\sim} W_2^{-1})^{\sim})(j\omega_i) \tag{16}$$
subject to (4) being satisfied. Here $\overline{P} = -(W_{1,0}\hat{P}_1^{\sim} W_{2,0})^{\sim}$ and $\mathcal{W}_1, \mathcal{W}_2$ are appropriate model sets such that $W_{1,0} \in \mathcal{W}_1$, $W_{2,0} \in \mathcal{W}_2$. Let $W_{2,1}, W_{1,1}$ be the weights obtained on solving (16) and let λ_2 be the achieved minimum cost.
4. Find the controller C_∞ which achieves $b(\hat{P}_2, C_\infty) = b_{opt}(\hat{P}_2)$. Here, $\hat{P}_2 = -(W_{1,1}^{-1}\overline{P}^{\sim} W_{2,1}^{-1})^{\sim}$. The final controller is given by $C_s = W_{1,1} C_\infty W_{2,1}$.

Assuming that global minimum exists (and is found) for each of the optimisation problems (15)-(16), a nice property of the above algorithm is non-increasing cost:

Lemma 2 *Let λ_1 and λ_2 be the achieved cost in the optimisations (15) and (16) respectively. Then*

$$\lambda_2 \leq \lambda_1$$

Proof : Since, $W_{1,0} \in \mathcal{W}_1$, $W_{2,0} \in \mathcal{W}_2$,

$$\begin{aligned} \lambda_2 &= \min_{\substack{W_1\in\mathcal{W}_1 \\ W_2\in\mathcal{W}_2}} \max_i \kappa(W_2 P_0 W_1, -(W_1^{-1}\overline{P}^{\sim} W_2^{-1})^{\sim})(j\omega_i) \\ &\leq \max_i \kappa(W_{2,0} P_0 W_{1,0}, -(W_{1,0}^{-1}\overline{P}^{\sim} W_{2,0}^{-1})^{\sim})(j\omega_i) \\ &= \max_i \kappa(W_{2,0} P_0 W_{1,0}, \hat{P}_1)(j\omega_i) = \lambda_1 \end{aligned}$$

where the last step uses $\overline{P} = -(W_{1,0}\hat{P}_1^{\sim} W_{2,0})^{\sim}$. ∎

If λ_1 is deemed sufficiently small, step 3 is not required. On the other hand, if λ_2 is deemed to be too large at the end of step 3, it is possible to iterate through steps 2 and 3 till the cost becomes sufficiently small. Similarly, if $b_{opt}(\hat{P}_2)$ is too small for the final model $\hat{P}_2$, it may be necessary to relax the loopshaping specifications and return to step 1 again.

Note that the controller design and model identification stages in the above procedure are interleaved, since the weights form part of the final controller C_s. Step 4 of this procedure is standard and is described in many robust control textbooks, e.g. [10]. The choice of model sets and numerical implementation in the first three steps is described in the subsequent sections.

4.2 Weight Selection

Given a true plant P_0, a controller C_{init} stabilising P_0 and *any* stable minimum phase transfer functions W_1 and W_2, recall that

$$H(P_0, C_{init}) \in \mathcal{H}_\infty \Leftrightarrow H(W_2 P_0 W_1, W_1^{-1} C_{init} W_2^{-1}) \in \mathcal{H}_\infty$$

In the weight selection procedure of [4], the aim is to find stable, minimum phase weights W_1 and W_2 such that

1. $b_\Omega(W_2P_0W_1, W_1^{-1}C_{init}W_2^{-1})$ is maximised and
2. the 'shaped' plant $P_s(s) := (W_2P_0W_1)(s)$ satisfies the loopshaping specifications (4).

The following procedure is taken from [4].

1. Let Γ_q denote the set of real diagonal $q \times q$ matrices. For ease of notation, let $P_{\omega_i} = P_0(j\omega_i) \in \mathbb{C}^{p\times q}$ and let $C_{\omega_i} = C_{init}(j\omega_i)$. It is assumed that $p \geq q$. The case when $p < q$ can be handled using a dual problem; see [4]. Given P_Ω as in (10) and the loopshaping constraints (4), solve the following quasiconvex optimisation problem at each frequency ω_i, $i = 1, 2, \ldots, m$:

$$\inf_{X_i \in \Gamma_q,\, Y_i \in \Gamma_p} \gamma_i \tag{17}$$

subject to

$$\begin{bmatrix} 0 & P_{\omega_i} \\ 0 & I \end{bmatrix}^* \begin{bmatrix} X_i & 0 \\ 0 & Y_i \end{bmatrix} \begin{bmatrix} 0 & P_{\omega_i} \\ 0 & I \end{bmatrix} < \gamma_i \begin{bmatrix} I & P_{\omega_i} \\ C_{\omega_i} & I \end{bmatrix}^* \begin{bmatrix} X_i & 0 \\ 0 & Y_i \end{bmatrix} \begin{bmatrix} I & P_{\omega_i} \\ C_{\omega_i} & I \end{bmatrix} \tag{18}$$

$$\alpha^2(\omega_i)Y_i < P_{\omega_i}^* X_i P_{\omega_i} < \beta^2(\omega_i)Y_i \tag{19}$$

$$X_i > 0,\, Y_i > 0 \tag{20}$$

Let $\hat{X}_i$, $\hat{Y}_i$ $i = 1, 2, \ldots, m$ be the solutions of the pointwise optimisation problems (17)-(20) and let $\hat{\gamma}_i$ be the optimum cost at each i. Let $\hat{\gamma} = \max_i \hat{\gamma}_i$.

2. Construct diagonal transfer function matrices $W_{1,0}(s)$, $W_{2,0}(s)$ that are units in $\mathcal{RH}_\infty$ by fitting minimum phase stable transfer function to each magnitude function on the diagonal of $\hat{Y}_i^{-\frac{1}{2}}$ and $\hat{X}_i^{\frac{1}{2}}$ respectively.

If $W_{1,0}$ and $W_{2,0}$ interpolate $\hat{Y}_i^{-\frac{1}{2}}$ and $\hat{X}_i^{\frac{1}{2}}$ exactly, it can be easily shown that

$$\alpha(\omega_i) < \sigma_k((W_{2,0}P_0W_{1,0})(j\omega_i)) < \beta(\omega_i) \quad \forall\, i \in [1, m] \quad \forall\, k \in [1, q]$$

Note that, with $W_{1,0}(j\omega_i) = \hat{Y}_i^{-\frac{1}{2}}$ and $W_{2,0}(j\omega_i) = \hat{X}_i^{\frac{1}{2}}$, (18) is equivalent to

$$\overline{\sigma}(H(W_{2,0}P_0W_{1,0}, W_{1,0}^{-1}C_{init}W_{2,0}^{-1}))(j\omega_i) < \sqrt{\gamma_i} \forall\, i \in [1, m] \tag{21}$$

Proof of this fact may be found in [4]. In practice, an approximate, low order fit to $\hat{Y}_i^{-\frac{1}{2}}$ and $\hat{X}_i^{\frac{1}{2}}$ should still ensure that $P_s = W_{2,0}P_0W_{1,0}$ adheres to the loopshaping specifications. While approximating $\hat{Y}_i^{-\frac{1}{2}}$ and $\hat{X}_i^{\frac{1}{2}}$, it must be kept in mind that $P_s(j\omega)$ should have a moderate roll-off rate around crossover frequency.

It is possible to include additional constraints on optimisation (17) for better numeric conditioning of X_i, Y_i; see [4] for details.

4.3 Identification in the $\nu-$gap metric

A method for identification in $\nu-$gap metric was presented in [3]. An outline of the same is given here for easy reference. SISO case is discussed here for simplicity; extension to MIMO case is straightforward. Let S_n denote the set of Finite Impulse Response (FIR) models of degree less than n. Next, define a candidate model set for approximation of coprime factors

$$S_{1,2} = \{f : f = [f_1\ f_2]^T, f_1 \in S_{n_1},\ f_2 \in S_{n_2}\}$$

Lastly, let $\mathcal{R}_n$ denote the set of real rational transfer functions of order less than n.

1. Let $\omega_{\max} = \max_i \omega_i$. Take $\omega_s = \frac{2\pi}{T_s} > 2\,\omega_{\max}$[1] and let

$$e^{j\theta_i} = \frac{1 + \frac{j\omega_i T_s}{2}}{1 - \frac{j\omega_i T_s}{2}}$$

Let $P_{s,\,\omega_i} = W_{2,0}(j\omega_i)P_{\omega_i}W_{1,0}(j\omega_i)$ and define $F_i = \begin{bmatrix} P_{s,\,\omega_i}\,(1 + P_{s,\,\omega_i}^* P_{s,\,\omega_i})^{-\frac{1}{2}} \\ (1 + P_{s,\,\omega_i}^* P_{s,\,\omega_i})^{-\frac{1}{2}} \end{bmatrix}$.
Here, $W_{1,0}(s)$ and $W_{2,0}(s)$ are stable, minimum phase weights obtained using the algorithm described in section 4.2.

2. Solve $\mathcal{L}_2$-gap approximation problem:

$$\min_{P\in\mathcal{R}_n} \max_i \kappa(W_{2,0}P_0W_{1,0}, P)(e^{j\theta_i}) = \min_{f\,\in\, S_{1,2}} \max_i \inf_{Q_i} \overline{\sigma}\left(F_i - f(e^{j\theta_i})Q_i\right) \tag{22}$$

Note that, for a fixed $\hat{Q}_i \in \mathbb{C}$, $i = 1, \cdots, m$,

$$\min_{f\,\in\, S_{1,2}} \max_i \overline{\sigma}\left(F_i - f(e^{j\theta_i})\hat{Q}_i\right) \tag{23}$$

is an LMI optimisation in parameters of f. On the other hand, for a fixed $\hat{f} \in S_{1,2}$, at each θ_i,

$$\inf_{Q_i} \overline{\sigma}\left(F_i - \hat{f}(e^{j\theta_i})Q_i\right) \tag{24}$$

is a linear least squares problem in $Q_i \in \mathbb{C}$ and has a (pointwise) closed form solution. Using these facts, (22) may be solved iteratively in f and Q_i and the cost is non-increasing through iterations; see [3] for details.

3. Let $\hat{f} = \begin{bmatrix} f_1 & f_2 \end{bmatrix}^T$ be the result of $\mathcal{L}_2$-gap approximation. Then the discrete time model is given by $\hat{P}_d = f_1\, f_2^{-1}$. Note that Q doesn't appear in $\hat{P}_d$. Hence it is not parameterised and is evaluated only pointwise in (24). The continuous time model is obtained by bilinear transformation:

$$\hat{P}_x = \hat{P}_d\left(\frac{1 + \frac{s\,T_s}{2}}{1 - \frac{s\,T_s}{2}}\right)$$

[1] ω_s should not be too large as compared to ω_{max} as this may place the poles and zeros of model too close to each other and may cause numerical difficulties in optimisation.

4. The procedure for approximation does not guarantee that the true 'shaped' plant and the model will satisfy the winding number condition, *i.e.* a controller stabilising the shaped plant with an adequate stability margin may still fail to stabilise $\hat{P}_x$. A model $\hat{P}_1$ such that $I(W_{2,0}P_0W_{1,0}, \hat{P}_1) = 0$ can be obtained from $\hat{P}_x$ and any controller C_x stabilising the shaped plant by a procedure described in [3]. Note that this procedure is not specific to the identification algorithm described so far; it may be used even if a model $\hat{P}_x$ obtained by *any other* identification method is de-stabilised by a controller that stabilises the true plant. If the true plant and $\hat{P}_x$ are both stable or are both stabilised by the same controller, this procedure is not required and $\hat{P}_1 = \hat{P}_x$. See [3] for details of this procedure.

At the end of this identification procedure described above, a model $\hat{P}_1$ is obtained which is a suboptimal solution to

$$\inf_{\substack{P\in\mathcal{R} \\ I(P,\hat{P}_0)=0}} \delta_\Omega(W_{2,0}P_0W_{1,0}, P) \tag{25}$$

It is instructive to compare this with (14).

4.4 Weight and Model Re-adjustment

This section outlines a procedure to adjust weights and model simultaneously to reduce cost $\delta_\Omega(W_2P_0W_1, P)$ further, with the adjusted weights still satisfying loop-shaping constraints (4). Let $\hat{P}_1$ be the model and $W_{1,0}$, $W_{2,0}$ be the weights obtained at the end of the 2nd step of the algorithm in section 4.1. The solution to (16) rests on the following result:

Lemma 3 *Given* $\overline{P} = -(W_{1,0}\hat{P}_1^{\sim}W_{2,0})^{\sim}$ *and frequency response samples* $P_0(j\omega_i)$*, suppose* $W_{1,1} \in \mathcal{W}_1$ *and* $W_{2,1} \in \mathcal{W}_2$ *is any pair of diagonal transfer function matrices which is solution to*

$$\min_{\substack{W_1\in,\mathcal{W}_1 \\ W_2\in,\mathcal{W}_2}} \max_i \kappa(W_2P_0W_1, -(W_1^{-1}\overline{P}^{\sim}W_2^{-1})^{\sim})(j\omega_i) \tag{26}$$

subject to (4) being satisfied. Then $W_{1,1}$ *and* $W_{2,1}$ *also solve*

$$\min_{\substack{W_1\in,\mathcal{W}_1 \\ W_2\in,\mathcal{W}_2}} \max_i \overline{\sigma}(\,H(W_2P_0W_1, W_1^{-1}\overline{P}^{\sim}W_2^{-1})\,)(j\omega_i) \tag{27}$$

subject to (4) being satisfied. Here $H(P, C)$ *is as defined in (6).*

Proof : The proof follows from the following relation from [9]: Given P, C at any frequency ω,

$$\frac{1}{\overline{\sigma}(\,H(P,C))(j\omega)} = \sqrt{1-(\,\kappa(P,-C^{\sim})(j\omega)\,)^2}$$

■

Let $C_{\omega_i} = \overline{P}^{\sim}(j\omega_i) = -(W_{1,0}\hat{P}_1^{\sim}W_{2,0})(j\omega_i)$. Consider the following problem at each frequency ω_i, $i = 1, 2, \ldots, m$:

$$\inf_{X_i\in\Gamma_n,\, Y_i\in\Gamma_p} \gamma_i \tag{28}$$

subject to

$$\begin{bmatrix} 0 & P_{\omega_i} \\ 0 & I \end{bmatrix}^* \begin{bmatrix} X_i & 0 \\ 0 & Y_i \end{bmatrix} \begin{bmatrix} 0 & P_{\omega_i} \\ 0 & I \end{bmatrix} < \gamma_i \begin{bmatrix} I & P_{\omega_i} \\ C_{\omega_i} & I \end{bmatrix}^* \begin{bmatrix} X_i & 0 \\ 0 & Y_i \end{bmatrix} \begin{bmatrix} I & P_{\omega_i} \\ C_{\omega_i} & I \end{bmatrix} \tag{29}$$

$$\alpha^2(\omega_i)Y_i < P_{\omega_i}^*X_iP_{\omega_i} < \beta^2(\omega_i)Y_i \tag{30}$$

$$X_i > 0,\, Y_i > 0 \tag{31}$$

Similar to (21), (29) can be shown to be equivalent to the constraint

$$\overline{\sigma}(\,H(W_2P_0W_1, W_1^{-1}\overline{P}^{\sim}W_2^{-1})\,)(j\omega_i) < \sqrt{\gamma_i}\,\forall\, i \in [1, m]$$

with $X_i = W_2^*W_2(j\omega_i)$, $Y_i = W_1^{-1}W_1^{-*}(j\omega_i)$. Thus the optimisation (28) subject to constraints (29)-(31) is simultaneously affine in $W_1^{-1}W_1^{-*}(j\omega_i)$ and $W_2^*W_2(j\omega_i)$. Hence (28) may be solved as an LMI optimisation problem in X_i, Y_i at each ω_i and then minimum phase stable diagonal weights could be fitted to $X_i^{\frac{1}{2}}$, $Y_i^{-\frac{1}{2}}$, as in section 4.2. Alternatively, the matrix functions $W_1^{-1}W_1^{-*}$ and $W_2^*W_2$ may be affinely parameterised. If the parameterisation includes $W_{1,0}^{-1}W_{1,0}^{-*}$ and $W_{2,0}^*W_{2,0}$, non-increasing cost from (15) to (16) is ensured, as stated in lemma 2.

Let $W_{1,1}$ and $W_{2,1}$ be the weights obtained by this procedure. Then $W_{1,1}$ and $W_{2,1}$ also solve (26), as mentioned in lemma 3. The final model is given by $\hat{P}_2 = -(W_{1,1}^{-1}\overline{P}^{\sim}W_{2,1}^{-1})^{\sim}$. Note that, in this step, both the weights and the model are adjusted to reduce the worst case chordal distance between the weighted plant and the model, while the changed weights are such that the weighted plant still satisfies (4).

If $\delta_\Omega(W_{2,1}P_0W_{1,1}, \hat{P}_2)$ is still deemed too large, the next logical step would be to minimise

$$\min_{P\in\mathcal{S}} \delta_\Omega(\,W_{2,1}\,P_0\,W_{1,1},\,P\,) \tag{32}$$

using the procedure outlined in section 4.3. Provided $\hat{P}_2 \in \mathcal{S}$, solution of (32) will not increase worst case chordal distance. Further iterations of steps 2 and 3 of the procedure in section 4.1 are possible, though simulation experience indicates that any iterations beyond (32) will be rarely required.

4.5 Effect of Noise

So far in this discussion, noise-free frequency response samples of the true plant P_0 are assumed to be available. In practice, it is far more likely that noisy frequency response samples will be available as a result of

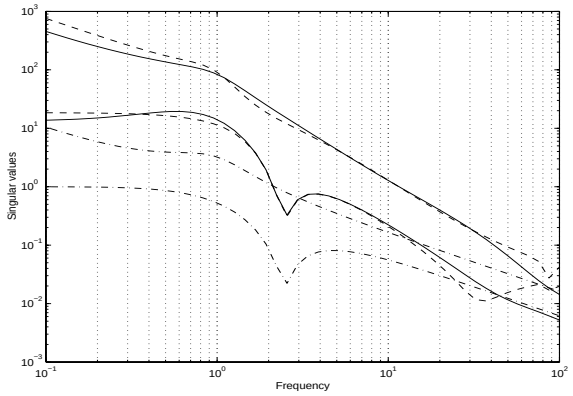

Figure 1: Singular Value Plots - True plant, $\nu-$gap Approximation

an identification experiment. In this case, the procedure outlined above may be carried out using the noisy samples. Suppose, P_{n,ω_i} represents a noisy frequency response sample at frequency ω_i. Let $\hat{P}$ be the model and $W_{1,1}$, $W_{2,1}$ be the weights obtained using the procedure detailed above (with $P_0(\omega_i)$ replaced with P_{n,ω_i}) and let C be the designed controller. Then it is easy to show that

$$b_\Omega(W_{2,1}P_0W_{1,1}, C) \geq$$
$$b_\Omega(\hat{P}, C) - \kappa(W_{2,1}(j\omega_i)P_{n,\omega_i}W_{1,1}(j\omega_i), \hat{P}(j\omega_i)) - \epsilon$$

where $\epsilon = \kappa\left((W_{2,1}P_0W_{1,1})(j\omega_i), W_{2,1}(j\omega_i)P_{n,\omega_i}W_{1,1}(j\omega_i)\right)$.

The term $\kappa(W_{2,1}(j\omega_i)P_{n,\omega_i}W_{1,1}(j\omega_i), \hat{P}(j\omega_i))$ may be minimised using the algorithm in section 4.1. The size of ϵ needs to be controlled at identification experiment stage. If ϵ is small (or equivalently, the effect of noise around the crossover frequencies is small), the possible deterioration of robustness margin due to noise is small.

5 Simulation Example

Consider an unstable continuous time plant

$$P_0(s) = \begin{bmatrix} \frac{1}{s+2} & \frac{1}{s} \\ \frac{1}{s-1} & \frac{3}{s^2+s+1} \end{bmatrix}$$

Frequency response samples (matrices) of this plant at 50 frequencies, logarithmically spaced between 0.1 rad/s and 100 rad/s are used for estimation and weight selection. For choosing weights, the loopshaping specifications (4) were

$$\sigma_k(P_s(j\omega_i)) \geq 10 \,\forall\, \omega_i \leq 1\,\text{rad/s},\ k = 1, 2$$
$$\sigma_k(P_s(j\omega_i)) \leq 0.1 \,\forall\, \omega_i \geq 50\,\text{rad/s},\ k = 1, 2$$

Given an initial stabilising controller, weights $W_{1,1}, W_{2,1}$ each of degree 4 and a model $\hat{P}_2$ of order 9 is obtained using the first 3 steps of the algorithm in section 4.1. This yields $\max_i \kappa(W_{2,1}P_0W_{1,1}, \hat{P})(j\omega_i) = 0.0550$. Then (32) is solved to obtain a model $\hat{P}_3$ of order 9 which yields a further reduction in worst case chordal distance. The final $\max_i \kappa(W_{2,1}P_0W_{1,1}, \hat{P}_3)(j\omega_i) = 0.0296$ and $\delta_\nu(W_{2,1}P_0W_{1,1}, \hat{P}_3) = 0.0334$. Also, $b_{opt}(\hat{P}_3) = 0.2982$ and the controller C_{final} which achieves $b_{opt}(\hat{P}_3)$ yields $b(W_{2,1}P_0W_{1,1}, C_{final}) = 0.2761$.

Figure 5 shows the singular values of shaped plant frequency response samples (solid line) and the singular values of model (dashed line). dash-dot line indicates the singular values of the unweighted plant.

6 Conclusion

An algorithm for identification and control using $\mathcal{H}_\infty$ loopshaping is presented. The implementation of the final, controller design stage in the algorithm is available in commercial software. The earlier steps of weight selection, identification and weight and model adjustment are based on LMI optimisation and can as such be implemented easily using any LMI solver. It is believed that this algorithm has a potential to reduce substantially the time normally required to identify a model from data and then to synthesise a controller which yields satisfactory closed loop performance with the true system.

References

[1] P. M. J. van den Hof and R. J. P. Schrama, "Identification and control - closed loop issues," *Automatica*, vol. 31, pp. 1751–1770, 1995.

[2] J. M. Böling and P. M. Mäkilä, "On control relevant criteria in $\mathcal{H}_\infty$ identification," *IEEE Trans. Automat. Contr.*, vol. 43, pp. 694–698, 1998.

[3] P. Date and G. Vinnicombe, "An algorithm for identification in the $\nu-$gap metric," in *Proc. of 38th Conf. on Dec. and Contr.*, (Phoenix, USA), 1999.

[4] A. Lanzon, "Simultaneous synthesis of weights and controllers in $\mathcal{H}_\infty$ loop-shaping," in *Proc. of 40th IEEE Conf. on Dec. and Contr.*, (Orlando, USA), 2001.

[5] K. Glover and D. McFarlane, "Robust stabilisation of normalised coprime factor plant descriptions with $\mathcal{H}_\infty$-bounded uncertainty," *IEEE Trans. Automat. Contr.*, 1989.

[6] D. C. McFarlane and K. Glover, "A loop shaping design procedure using $\mathcal{H}_\infty$ synthesis," *IEEE Trans. Automat. Contr.*, vol. 37, pp. 759–769, 1992.

[7] G. Papageorgiou and K. Glover, "$\mathcal{H}_\infty$ loopshaping: Why is it a sensible procedure for designing robust flight controllers?," in *Proc. of AIAA conf. on Guidance, Navigation and Control*, 1999.

[8] G. Vinnicombe, "Frequency domain uncertainty and the graph topology," *IEEE Trans. Automat. Contr.*, vol. 38, pp. 1371–1383, 1993.

[9] G. Vinnicombe, *Uncertainty and Feedback: $\mathcal{H}_\infty$ loop-shaping and the $\nu-$gap metric*. Imperial College Press, 2000.

[10] K. Zhou, J. Doyle, and K. Glover, *Robust and Optimal Control*. Prentice Hall, 1996.