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# Eigenvalue-based Algorithms for Testing Negative Imaginariness of Scalar Transfer Functions

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***Abstract*—This paper proposes several algorithms based on eigenvalue computations for testing negative imaginariness of scalar transfer functions of single-input single-output time-invariant linear systems. The input of the proposed eigenvalue-based algorithms can be any state-space realization, not necessarily minimal. The algorithms can test negative imaginariness of descriptor systems whose transfer functions may be non-proper, and only involve eigenvalues computation of matrices or matrix pencils with fewer restrictions. Finally, illustrative examples are provided to show how the proposed eigenvalue-based algorithms are applied for negative imaginariness test.**

***Index Terms*—Negative imaginariness, non-proper transfer functions, state-space realization, eigenvalue computations.**

## I. Introduction

Positive realness, as a fundamental concept in system and control theory [1], [2], has been widely applied in the study of circuit analysis [1], robust control [3], adaptive control [4] and absolute stability [5], [6]. An important research topic of the study of positive real systems theory is how to test positive real properties of a given transfer function in terms of the state-space representation. Methods for checking positive real properties of a given transfer function based on the frequency conditions, positive real lemma, Riccati equation and Hamiltonian matrix have been developed in [1], [5], [7]–[9]. However, for large dimensional systems or high order systems, the computational cost of these methods may be expensive. Several eigenvalue-based algorithms in [10], [11], which have less computational cost, provide an efficient method to test positive realness of transfer functions of single-input single-output (SISO) systems.

Although the research on positive real systems theory has achieved fruitful results, one limitation of positive real systems theory is that the relative degree of positive real transfer functions must be unity or zero [2]. Such restriction has inspired researchers to study negative imaginary (NI) systems whose relative degree is between zero and two [12]–[19]. The concept of NI systems was firstly established in [12], where NI systems were required to be stable. Furthermore, the definition of NI systems has been extended to allow poles on the imaginary axis [13]–[17]. Particularly, articles [16] and [17] respectively studied the non-proper irrational symmetric NI systems and the non-proper rational NI systems by allowing poles at infinity. In addition, the study of NI systems theory was extended to descriptor systems [18], networked NI systems [20]–[23], fractional-order systems [24], and nonlinear systems [25], [26].

A key result of NI theory is the stability result developed in [12], [13], [27]–[30]. Article [12] first presented the internal stability result, by characterizing the gain condition at zero and infinite frequencies of positive feedback interconnections, between a NI system and a strictly negative imaginary (SNI) system. Then, the internal stability result was extended in [13], [27] when NI systems have poles on the imaginary axis. The internal stability results of NI systems have important application in the study of robust control. Articles [28], [29] derived converse negative imaginary theorems and established necessary and sufficient conditions for robust feedback stability against diverse NI uncertainties. Moreover, the robust stability of feedback interconnections via frequency-dependent constraints has been studied in [30], which can also be applicable to NI systems. Several application examples of the NI stability results could be found in the control of trajectory tracking of a quadcopter unmanned aerial vehicle [31], the robust cooperative control of multi-agent systems [20] and the positioning problem of atomic force microscopes [15], [32]. Moreover, the research on designing feedback controllers for non-NI systems such that the resulting closed-loop systems are NI, referred to as NI synthesis problem, has been addressed in [15], [33]–[35].

Along this line of research, another main issue on NI systems theory is how to test NI properties of a given transfer function. The definition of NI systems was first characterized by frequency response conditions [12], [13]. Under the assumption of minimal state-space realizations, articles [12], [13] and [15] established NI lemmas based on the linear matrix inequalities (LMIs) to test negative imaginariness. By removing the assumption of minimal realizations, NI lemmas in [12], [13] were proved to be a sufficient condition for NI systems [36]. A method based on spectral conditions was studied in [32] to test NI or SNI properties of a given transfer function, by computing the spectrum of Hamiltonian matrices. [15] and [37] developed NI or SNI test conditions in terms of Riccati equations, and [35] further established several necessary and sufficient conditions to test SNI or strongly SNI properties based on Riccati equations. A time-domain

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characterization of NI systems was proposed in [38].

The objective of this paper is to provide a novel method based on eigenvalue computations of matrix or matrix pencil to determine negative imaginariness of SISO scalar transfer functions. The results of this paper are also applicable to descriptor systems whose transfer functions may be non-proper. Motivations for studying the NI test method based on eigenvalue computations are threefold.

1) Many practical systems with large dimension or high order can be modelled as NI systems, such as RLC networks [39] and lightly damped flexible structures with a collocated force actuator and position sensor [13]. When determining NI properties of large-scale systems, the checking of frequency conditions and solving the LMIs are computationally expensive. Moreover, if solutions of the LMIs exist at the boundary of the convex sets, it is hard to compute the values of these solutions [32]. Motivated by the method of eigenvalue-based algorithms in [10], [11], this paper proposes several eigenvalue-based algorithms to test NI properties of a given transfer function.
2) The existing methods of judging negative imaginariness need to satisfy technical assumptions. For example, a minimal state-space realization is needed in [12], [13], [15], [32] and the product of input matrix and output matrix is required to be positive in [15], [32]. Removing those assumptions, the eigenvalue-based methods provided in this paper are more extensive than the existing methods for testing NI properties of a given transfer function.
3) Descriptor systems have been applied in many engineering fields, such as electric power engineering [40] and integrated circuit design [41]. Recently, the study of NI properties of descriptor systems has been addressed in [18]. However, the results based on LMIs in [18] were only applicable to proper descriptor systems, while some engineering systems in practice may be modeled as descriptor systems with non-proper transfer functions [42]. Inspired by this, the eigenvalue-based algorithms proposed in this paper provide a new negative imaginariness test approach, which is also applicable to descriptor systems with non-proper transfer functions.

The remainder of this paper is as follows. In Section II, definitions of NI transfer functions of SISO systems are presented. In Section III, the eigenvalue-based characterization of negative imaginariness is provided. In Section IV, an algorithm based on eigenvalue computations is proposed for testing NI properties of scalar transfer functions. In Section V, the NI test algorithm is extended to be applicable to descriptor systems. In Section VI, three examples are provided to verify the proposed eigenvalue-based algorithms. Section VII draws the conclusion.

*Notation:* $\mathbb{R}^{m\times n}$ denotes the set of $m \times n$ real matrices. $\mathbb{R}$ and $\mathbb{C}$ denote the sets of real and complex numbers, respectively. $\mathrm{Re}[\cdot]$ and $\mathrm{Im}[\cdot]$ are the real and imaginary part of complex numbers, respectively. $A^T$ presents the transpose of a matrix $A$. $A > 0$ or $A \geq 0$ denotes a real symmetric positive definite or real symmetric positive semidefinite matrix. $\mathbf{0}$ denotes the zero vector with appropriate dimensions.

## II. Preliminaries

Considering the following class of SISO linear time-invariant system

$$\begin{cases} E\dot{x}(t) = Ax(t) + bu(t) \\ \quad y(t) = c^T x(t) + du(t), \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}$ is the control input, $y(t) \in \mathbb{R}$ is the system output and $E, A \in \mathbb{R}^{n\times n}$, $b, c \in \mathbb{R}^n$, $d \in \mathbb{R}$. If $E$ is singular, the system (1) is called a singular system or a descriptor system [43]. The matrix pencil $sE - A$ is called regular if $\det(sE - A) \neq 0$ for some $s \in \mathbb{C}$. The transfer function of system (1) is given by

$$h(s) = c^T(sE - A)^{-1}b + d. \tag{2}$$

To make the results meaningful, assume that $h(s) \not\equiv 0$ and the matrix pencil $sE - A$ is regular throughout of rest of the paper. One important case of system (1) is that $E = I$. In this case, the form in (1) is a normal SISO system with the transfer function

$$h(s) = c^T(sI - A)^{-1}b + d.$$

The definition of negative imaginary (NI) transfer functions is introduced in the following to determine whether a SISO real-rational transfer function is NI.

*Definition 1:* [17] A real-rational transfer function $h(s)$ is NI if

1) $h(s)$ has no poles in $\mathrm{Re}[s] > 0$;
2) For all $\omega > 0$ such that $j\omega$ is not a pole of $h(s)$, $\mathrm{Im}[h(j\omega)] \leq 0$;
3) If $s = j\omega_0$, $\omega_0 > 0$ is a pole of $h(s)$, then it is a simple pole, and the residue $K_0 = \lim_{s\to j\omega_0}(s - j\omega_0)jh(s) \geq 0$;
4) If $s = 0$ is a pole of $h(s)$, then $\lim_{s\to 0} s^k h(s) = 0$ for all $k \geq 3$ and $\lim_{s\to 0} s^2 h(s) \geq 0$;
5) If $s = j\infty$ is a pole of $h(s)$, then $\lim_{\omega\to\infty} \frac{h(j\omega)}{(j\omega)^k} = 0$ for all $k \geq 3$ and $\lim_{\omega\to\infty} \frac{h(j\omega)}{(j\omega)^2} \leq 0$.

*Remark 1:* Article [17] extended the definition of NI systems to the case where transfer functions may be non-proper and have poles at $j\infty$. The non-proper transfer function in this paper means that the order of numerator polynomial is higher than the order of denominator polynomial. If only one proper transfer function $h(s)$ is considered, the conditions 1-4 of Definition 1 are used to determine negative imaginariness of $h(s)$ [15]. The condition $\mathrm{Im}[h(j\omega)] \leq 0$ is equivalent to $j[h(j\omega) - h(-j\omega)] \geq 0$.

The definitions of the lossless NI transfer function and strictly negative imaginary (SNI) transfer function are introduced in the following. To avoid the ambiguity in determining SNI properties of non-proper transfer functions in the case of poles at infinity, we only consider proper SNI transfer function in this article.

*Definition 2:* [17], [44] A real-rational transfer function $h(s)$ is lossless NI if

1) $h(s)$ is NI;
2) $\mathrm{Im}[h(j\omega)] = 0$ for all $\omega > 0$ except values of $\omega$ where $j\omega$ is a pole of $h(s)$.

*Definition 3:* [13] A real-rational proper transfer function $h(s)$ is SNI if

1) $h(s)$ has no poles in $\mathrm{Re}[s] \geq 0$;
2) $\mathrm{Im}[h(j\omega)] < 0$ for all $\omega > 0$.

## III. Eigenvalue-based Characterization

In this section, characterizations of negative imaginariness based on eigenvalue computations are developed. In Subsection III-A, the transfer function $h(s)$ is firstly transformed into a new form of (3). In Subsection III-B, the transfer function $h(s)$ is further transformed into the form of Lemma 1, which is characterized in terms of determinants. The poles of $h(s)$ are transformed into eigenvalue computations of matrix pencils in Lemma 2. Moreover, Lemma 3 is established to test the condition $\mathrm{Im}[h(j\omega)] \leq 0$ in NI definitions. Combining the results of Lemmas 1-3, eigenvalue-based characterizations of negative imaginariness are developed in Theorems 1-2.

### A. Transformation of Transfer Functions

In this subsection, a transformation is introduced such that the transfer function $h(s)$ in (2) can be transformed into a simplified form (3) with the unit vector.

If $h(s) = c^T(sE - A)^{-1}b + d \equiv d$, it follows that either $b = 0$ or $c = 0$. Then, $h(s) = d$ is lossless NI according to Definition 2.

For $h(s) \not\equiv d$, which implies that $b \neq 0$ and $c \neq 0$, two cases of $E = I$ and $E \neq I$ are studied to transform the transfer function $h(s) = c^T(sE - A)^{-1}b + d$ into the form

$$h(s) = \gamma e^T(sT - S)^{-1}b_1 + d, \tag{3}$$

where $TS = ST$, $T, S \in \mathbb{R}^{n\times n}$, $b_1 \in \mathbb{R}^n$, $\gamma > 0$, $\gamma \in \mathbb{R}$, $e$ is the first unit vector of length $n$ (i.e., $e = [1\ 0 \cdots 0]^T_{1\times n}$).

First, consider the case $E = I$. In view of the Householder transformation [45], since $c \neq 0$, there exists an orthogonal matrix $Q \in \mathbb{R}^{n\times n}$ satisfying $Q = Q^T = Q^{-1}$ such that

$$Qc = \|c\|_2 e, \tag{4}$$

where $\|c\|_2$ stands for 2-norm of the vector $c$. In view of (4), $h(s) = c^T(sI - A)^{-1}b + d$ is rewritten as the form $h(s) = \gamma e^T(sT - S)^{-1}b_1 + d$ in (3), where

$$\gamma = \|c\|_2, T = I, S = Q^{-T}AQ^T \text{and } b_1 = Q^{-T}b.$$

Second, consider the case $E \neq I$. Select any $s_0 \in \mathbb{R}$ such that $\det(s_0E - A) \neq 0$. Then, a factorization of $s_0E - A$ is given as

$$s_0E - A = LU, \tag{5}$$

where $L, U \in \mathbb{R}^{n\times n}$. The factorization in (5) can be any formal factorization, such as, a LU factorization with a lower triangular matrix $L$ and a upper triangular matrix $U$, or a commonly used factorization with $L = s_0E - A$ and $U = I$. Obviously, matrices $L, U$ are non-singular and $L^{-1}(s_0E - A)U^{-1} = I$ holds. In view of (5) and $L^{-1}(s_0E - A)U^{-1} = I$, the transfer function $h(s) = c^T(sE - A)^{-1}b + d$ can be rewritten as

$$\begin{aligned} h(s) &= c^TU^{-1}(sL^{-1}EU^{-1} - L^{-1}AU^{-1})^{-1}L^{-1}b + d \\ &= c^TU^{-1}(sG - s_0G + I)^{-1}L^{-1}b + d, \end{aligned} \tag{6}$$

where

$$G = L^{-1}EU^{-1}.$$

Moreover, it follows from $c \neq 0$ that $U^{-T}c \neq 0$. Thus, there exists a Householder transformation $\bar{Q} \in \mathbb{R}^{n\times n}$ such that

$$\bar{Q}U^{-T}c = \|U^{-T}c\|_2 e. \tag{7}$$

According to (6) and (7), the transfer function $h(s) = c^T(sI - A)^{-1}b + d$ has the form $h(s) = \gamma e^T(sT - S)^{-1}b_1 + d$ in (3), where

$$\begin{aligned} &\gamma = \|U^{-T}c\|_2, T = \bar{Q}^{-T}G\bar{Q}^T \\ &S = s_0T - I, b_1 = \bar{Q}^{-T}L^{-1}b. \end{aligned}$$

### B. Characterization of Negative Imaginariness

In this subsection, the eigenvalue-based characterization is introduced for testing NI properties of scalar transfer functions.

According to Subsection III-A, the transfer function $h(s)$ of system (1) can be written as the form $h(s) = \gamma e^T(sT - S)^{-1}b_1 + d$ in (3). Then, $h(s)$ is further transformed into a form with determinant in the following lemma.

*Lemma 1:* The transfer function $h(s) = \gamma e^T(sT - S)^{-1}b_1 + d$ in (3) can be transformed into the form

$$h(s) = \gamma\frac{\det(s\tilde{T} - \tilde{S})}{\det(sT - S)} + d, \tag{8}$$

where

$$\tilde{T} = T(I - ee^T),\ \tilde{S} = S(I - ee^T) - b_1e^T. \tag{9}$$

*Proof*: Let $\mathbf{adj}_{(sT-S)}$ be the adjoint matrix of $sT - S$. In terms of the algebraic cofactor in linear algebra, the following condition holds.

$$e^T(sT - S)^{-1}b_1 = \frac{e^T\mathbf{adj}_{(sT-S)}b_1}{\det(sT - S)} = \frac{\det(s\tilde{T} - \tilde{S})}{\det(sT - S)}, \tag{10}$$

where $s\tilde{T} - \tilde{S}$ is the matrix pencil which replaces the first column of matrix pencil $sT - S$ by the vector $b_1$. Furthermore, $\tilde{T}$ and $\tilde{S}$ can be given by

$$\tilde{T} = T(I - ee^T), \quad \tilde{S} = S(I - ee^T) - b_1e^T.$$

Thus, the transfer function $h(s)$ in (3) is rewritten as the form in (8). □

In view of (8), the poles of transfer function $h(s)$ are characterized in Lemma 2, which indicates that the poles of $h(s)$ can be obtained by computing eigenvalues of matrix pencils $sT - S$ and $s\tilde{T} - \tilde{S}$.

*Lemma 2:* Considering $h(s) = \gamma e^T(sT - S)^{-1}b_1 + d$ in (3). The poles of $h(s)$ are the finite eigenvalues of the matrix pencil $sT - S$ which are not the finite eigenvalues of the matrix pencil $s\tilde{T} - \tilde{S}$, where $\tilde{T}$ and $\tilde{S}$ are given by (9).

*Proof*: According to the form (8) in Lemma 1, the proof is trivial. □

To test negative imaginariness of the transfer function $h(s)$, we also need to check the condition $\mathrm{Im}[h(j\omega)] \leq 0$ in Definition 1. The following lemma shows that the checking condition $\mathrm{Im}[h(j\omega)] \leq 0$ can be transformed into a problem of eigenvalue computations.

*Lemma 3:* Considering $h(s) = \gamma e^T(sT - S)^{-1}b_1 + d$ in (3), $\mathrm{Im}[h(j\omega)] \leq 0$ holds for all $\omega > 0$ except values of $\omega_0$, where $j\omega_0$ is a pole of $h(s)$, if and only if

$$\det(\lambda M + N) \geq 0, \tag{11}$$

for all $\lambda > 0$, where

$$M = T^2(I - ee^T),\ N = S^2(I - ee^T) + Tb_1e^T. \tag{12}$$

*Proof*: According to $h(s) = \gamma e^T(sT - S)^{-1}b_1 + d$ with $TS = ST$, it follows that

$$\begin{aligned}\mathrm{Im}[h(j\omega)] &= -\frac{1}{2}j[h(j\omega) - h(-j\omega)] \\ &= -\frac{1}{2}j\gamma e^T[(j\omega T - S)^{-1} - (-j\omega T - S)^{-1}]b_1.\end{aligned} \tag{13}$$

Due to $TS = ST$, it follows that

$$\begin{aligned}\omega^2T^2 + S^2 &= -(j\omega T - S)(j\omega T + S) \\ &= -(j\omega T + S)(j\omega T - S).\end{aligned} \tag{14}$$

It follows from (14) that

$$(j\omega T - S)^{-1} - (-j\omega T - S)^{-1} = -2j\omega[\omega^2T^2 + S^2]^{-1}T. \tag{15}$$

Thus,

$$\mathrm{Im}[h(j\omega)] = -\omega\gamma e^T(\omega^2T^2 + S^2)^{-1}Tb_1. \tag{16}$$

Similar to Lemma 1, let $\mathbf{adj}_{(\omega^2T^2+S^2)}$ be the adjoint matrix of $\omega^2T^2 + S^2$. In terms of the algebraic cofactor in linear algebra, it follows that

$$\begin{aligned}e^T(\omega^2T^2 + S^2)^{-1}Tb_1 &= \frac{e^T\mathbf{adj}_{(\omega^2T^2+S^2)}Tb_1}{\det(\omega^2T^2 + S^2)} \\ &= \frac{\det(\omega^2M + N)}{\det(\omega^2T^2 + S^2)},\end{aligned} \tag{17}$$

where $\omega^2M + N$ is the matrix pencil which replaces the first column of matrix pencil $\omega^2T^2 + S^2$ by the vector $Tb_1$. Thus, $M$ and $N$ are given by

$$M = T^2(I - ee^T), N = S^2(I - ee^T) + Tb_1e^T.$$

Then, $\mathrm{Im}[h(j\omega)]$ is written as

$$\mathrm{Im}[h(j\omega)] = -\omega\gamma\frac{\det(\omega^2M + N)}{\det(\omega^2T^2 + S^2)}. \tag{18}$$

In view of [10], since $T$, $S$ are real matrices and (14) holds, one has that

$$\det(\omega^2T^2 + S^2) = |\det(S + j\omega T)|^2. \tag{19}$$

Then, (18) is rewritten as

$$\mathrm{Im}[h(j\omega)] = -\omega\gamma\frac{\det(\omega^2M + N)}{|\det(S + j\omega T)|^2}. \tag{20}$$

Since $\gamma > 0$ and $|\det(S+j\omega T)|^2$ is nonnegative for all $\omega > 0$, then $\mathrm{Im}[h(j\omega)] \leq 0$ holds for all $\omega > 0$ except values of $\omega_0$, where $j\omega_0$ is a pole of $h(s)$, if and only if

$$\det(\lambda M + N) \geq 0,$$

for all $\lambda = \omega^2 > 0$ except values of $\lambda = \omega_0^2$, where $\omega_0 > 0$, $j\omega_0$ is a pole of $h(s)$.

Obviously, $\det(\lambda M + N)$ is a real polynomial with degree at most $n-1$. By the continuity of $\det(\lambda M+N)$ with respect to $\lambda$, the condition $\det(\lambda M + N) \geq 0$ also holds for $\lambda = \omega_0^2$. Thus, the condition of $\mathrm{Im}[h(j\omega)] \leq 0$ in Definition 1 for judging negative imaginariness of $h(s)$ is equivalent to judging $\det(\lambda M + N) \geq 0$ for all $\lambda > 0$. □

*Remark 2:* If a real $\lambda$ is a root of $\det(\lambda M+N) = 0$ with odd algebraic multiplicity, the polynomial $\det(\lambda M + N)$ changes its sign at the real $\lambda$. Thus, to ensure $\det(\lambda M+N) \geq 0$ for all $\lambda > 0$, any possible real positive root $\lambda$ of $\det(\lambda M + N) = 0$ has even algebraic multiplicity and there exists at least one point $\lambda_0 > 0$ such that $\det(\lambda_0 M + N) > 0$.

According to the transformations in Subsections III-A and III-B, an eigenvalue-based characterization of negative imaginariness is introduced in the following Theorem 1. The idea of Theorem 1 is motivated by the eigenvalue-based characterization of positive realness in [10]. To simplify Theorem 1, let $\Omega$ denote the set of the finite eigenvalues of the matrix pencil $sT-S$ which are not the finite eigenvalues of the matrix pencil $s\tilde{T} - \tilde{S}$. In view of Lemma 2, $\Omega$ is the set of poles of $h(s)$ after possible zero-pole cancellation.

*Theorem 1:* Consider the transfer function $h(s)$ in (3), and the matrices $\tilde{T}$, $\tilde{S}$ in (9), $M, N$ in (12). Then, $h(s)$ is NI if and only if

1) All values in $\Omega$ have no positive real part.
2) For any purely imaginary values in $\Omega$, the conditions 3, 4 and 5 of Definition 1 must hold.
3) The matrix pencil $\lambda M + N$ is either singular, or the following two conditions hold.
   i) Any real positive eigenvalue, $\lambda > 0$, of the matrix pencil $\lambda M + N$ has even algebraic multiplicity;
   ii) There exists at least one point $\lambda_0 > 0$ such that $\det(\lambda_0 M + N) > 0$.

Furthermore, if $\lambda M + N$ is singular, then $\mathrm{Im}[h(j\omega)] = 0$ holds, and hence the transfer function $h(s)$ is lossless NI.

*Proof*: In view of Lemma 2, the conditions 1) and 2) are equivalent to the conditions 1, 3, 4 and 5 of Definition 1. According to Lemma 3, the condition 2 of Definition 1 holds if and only if $\det(\lambda M + N) \geq 0$ for all $\lambda > 0$, which is equivalent to the condition 3).

Furthermore, according to Definition 2, $h(s)$ is lossless NI if $\lambda M + N$ is singular. □

*Remark 3:* Theorem 1 is applicable to a non-proper transfer function $h(s)$. If $h(s)$ is proper, then the condition 5 of Definition 1 is not considered.

In view of equation (20), $\mathrm{Im}[h(j\omega)] < 0$ for all $\omega > 0$ if and only if $\det(\lambda M + N) > 0$ for all $\lambda > 0$. Thus, the next theorem provides an eigenvalue-based criterion to check whether $h(s)$ is SNI.

*Theorem 2:* Consider the proper transfer function $h(s)$ in (3), and the matrices $\tilde{T}$, $\tilde{S}$ in (9), $M, N$ in (12). Then, the proper transfer function $h(s)$ is SNI if and only if

1) All values in $\Omega$ have negative real part.
2) All the finite eigenvalues of the matrix pencil $\lambda M + N$ are not real positive eigenvalues.
3) There exists at least one point $\lambda_0 > 0$ such that $\det(\lambda_0 M + N) > 0$.

*Proof*: In view of Lemma 2, the condition 1) in this theorem is equivalent to the condition 1 of Definition 3. In view of equation (20), the condition 2 of Definition 3 holds if and only if $\det(\lambda M + N) > 0$ for all $\lambda > 0$, which is equivalent to the conditions 2) and 3). □

## IV. Eigenvalue-based Algorithms for $E = I$

In this section, an algorithm based on eigenvalue computations is developed for testing negative imaginariness of normal SISO systems with $E = I$.

According to Theorems 1 and 2, the eigenvalue-based test of negative imaginariness involves computations of eigenvalues of matrix pencils $sT - S$, $s\tilde{T} - \tilde{S}$ and $\lambda M + N$. In view of [10], [11], [45], the eigenvalue computations of matrix pencils cost less than the eigenvalue computations of matrices. Thus, in this section, the eigenvalue-based algorithm only involving eigenvalue computations of matrices is proposed to check NI properties of system (1) with $E = I$.

In the case of $E = I$, due to

$$T = I \quad \text{and} \quad S = Q^{-T} A Q^T,$$

it can be verified that

$$\begin{aligned}\det(sT - S) &= \det(Q^{-T})\det(sI - A)\det(Q^T)\\ &= \det(sI - A)\end{aligned}$$

holds. Thus, the eigenvalues of the matrix $A$ and the matrix pencil $sT - S$ are equivalent. Then, the following Lemma 4, which only involves the eigenvalue computations of matrices, provides a method to address the eigenvalues of the matrix pencil $s\tilde{T} - \tilde{S}$.

*Lemma 4:* Let $\tilde{T} = T(I - ee^T)$ and $\tilde{S} = S(I - ee^T) - b_1 e^T$ as in (9). Partition $\tilde{S}$ as $\tilde{S} = \begin{bmatrix} s_{11} & s_{12}^T \\ s_{21} & S_1 \end{bmatrix} \in \mathbb{R}^{n\times n}$, where $s_{11} \in \mathbb{R}$, $S_1 \in \mathbb{R}^{(n-1)\times(n-1)}$, $s_{12}, s_{21} \in \mathbb{R}^{n-1}$. Then, the following three cases are considered to compute the eigenvalues of the matrix pencil $s\tilde{T} - \tilde{S}$.

1) If $s_{11} \neq 0$, the eigenvalues of the matrix pencil $s\tilde{T} - \tilde{S}$ are the eigenvalues of the matrix $\hat{S} = S_1 - (1/s_{11}) s_{21} s_{12}^T$.
2) If $s_{11} = 0$ and $s_{21} = \mathbf{0}$, then $\det(s\tilde{T} - \tilde{S}) = 0$ holds for all values of $s \in \mathbb{C}$.
3) If $s_{11} = 0$ and $s_{21} \neq \mathbf{0}$, compute a Householder transformation matrix $Q_1$ such that $Q_1 s_{21} = \|s_{21}\|_2 e$. Let $\tilde{S} = (I - ee^T) Q_1 S_1 Q_1^{-1} + e s_{12}^T Q_1^{-1}$ and repartition $\tilde{S}$ as $\begin{bmatrix} s_{11} & s_{12}^T \\ s_{21} & S_1 \end{bmatrix}$. Then, Cases 1-3 are considered again to compute the eigenvalues of the matrix pencil $s\tilde{T} - \tilde{S}$.

Moreover, a special case is that $\tilde{S} \in \mathbb{R}^{2\times 2}$ and $s_{11} = 0$. In this case, the above Cases 2 and 3 are not applicable, and $\det(s\tilde{T} - \tilde{S})$ is a constant for all values of $s \in \mathbb{C}$.

*Proof*: In view of $\tilde{T}$, $\tilde{S}$ in (9) and $T = I$, let

$$\det(s\tilde{T} - \tilde{S}) = \det\left(\begin{bmatrix} -s_{11} & -s_{12}^T \\ -s_{21} & sI - S_1 \end{bmatrix}\right), \tag{21}$$

where $s_{11} \in \mathbb{R}$, $s_{12}, s_{21} \in \mathbb{R}^{n-1}$, $S_1 \in \mathbb{R}^{(n-1)\times(n-1)}$.

If $s_{11} \neq 0$, it follows that

$$\det(s\tilde{T} - \tilde{S}) = -s_{11}\det(sI - S_1 + (1/s_{11}) s_{21} s_{12}^T)$$

Thus, the eigenvalues of the matrix pencil $s\tilde{T} - \tilde{S}$ are the eigenvalues of the matrix $S_1 - (1/s_{11}) s_{21} s_{12}^T$.

Then, consider the case of $s_{11} = 0$. If $s_{21} = \mathbf{0}$, one has that $\det(s\tilde{T} - \tilde{S}) = 0$ and $h(s)$ is lossless NI. If $s_{21} \neq \mathbf{0}$, there exists a Householder transformation $Q_1 \in \mathbb{R}^{(n-1)\times(n-1)}$ such that

$$Q_1 s_{21} = \|s_{21}\|_2 e, \tag{22}$$

where $e$ is the first unit vector of length $n - 1$. Let

$$\hat{Q} = \begin{bmatrix} 1 & 0 \\ 0 & Q_1 \end{bmatrix}.$$

Then, it follows that

$$\begin{aligned}&\det(s\tilde{T} - \tilde{S})\\ =&\det(\hat{Q})\det(s\tilde{T} - \tilde{S})\det(\hat{Q}^{-1})\\ =&\det\left(\begin{bmatrix} 0 & -s_{12}^T Q_1^{-1} \\ -\|s_{21}\|_2 e & sI - Q_1 S_1 Q_1^{-1} \end{bmatrix}\right)\\ =&\det\left(\begin{bmatrix} 0 & -s_{22} & -s_{13}^T \\ -\|s_{21}\|_2 & s - s_1 & -s_2 \\ \mathbf{0} & -s_{31} & sI - S_2 \end{bmatrix}\right)\\ =&\|s_{21}\|_2 \det\left(\begin{bmatrix} -s_{22} & -s_{13}^T \\ -s_{31} & sI - S_2 \end{bmatrix}\right),\end{aligned} \tag{23}$$

where

$$s_{12}^T Q_1^{-1} = [s_{22}\ s_{13}^T],$$
$$Q_1 S_1 Q_1^{-1} = \begin{bmatrix} s_1 & s_2 \\ s_{31} & S_2 \end{bmatrix},$$

and $s_1$, $s_{22} \in \mathbb{R}$, $s_2^T$, $s_{13}, s_{31} \in \mathbb{R}^{n-2}$, $S_2 \in \mathbb{R}^{(n-2)\times(n-2)}$. It follows that

$$\begin{bmatrix} s_{22} & s_{13}^T \\ s_{31} & S_2 \end{bmatrix} = (I - ee^T) Q_1 S_1 Q_1^{-1} + e s_{12}^T Q_1^{-1}.$$

Thus, in Case 3 of Lemma 4, we set

$$\tilde{S} = (I - ee^T) Q_1 S_1 Q_1^{-1} + e s_{12}^T Q_1^{-1},$$

and repartition $\tilde{S}$ as

$$\tilde{S} = \begin{bmatrix} s_{11} & s_{12}^T \\ s_{21} & S_1 \end{bmatrix} \in \mathbb{R}^{(n-1)\times(n-1)},$$

where $s_{11} = s_{22} \in \mathbb{R}$, $s_{12} = s_{13}, s_{21} = s_{31} \in \mathbb{R}^{n-2}$, $S_1 = S_2 \in \mathbb{R}^{(n-2)\times(n-2)}$.

Now, according to (23), to compute the eigenvalues of the matrix pencil $s\tilde{T} - \tilde{S}$, the roots of

$$\det(s\tilde{T} - \tilde{S}) = \|s_{21}\|_2 \det\left(\begin{bmatrix} -s_{11} & -s_{12}^T \\ -s_{21} & sI - S_1 \end{bmatrix}\right) = 0,$$

need to be computed. $\|s_{21}\|_2$ is a non-zero constant, which does not change the eigenvalues of the matrix pencil $s\tilde{T}-\tilde{S}$. Thus, we only compute the roots of

$$\det\left(\begin{bmatrix} -s_{11} & -s_{12}^T \\ -s_{21} & sI-S_1 \end{bmatrix}\right)=0,$$

which has the same block structure as equation (21). Thus, by repeating the above analysis process, the eigenvalues of the matrix pencil $s\tilde{T}-\tilde{S}$ or $\det(s\tilde{T}-\tilde{S})=0$ are obtained.

In addition, considering a special case that $\tilde{S}\in\mathbb{R}^{2\times 2}$ and $s_{11}=0$, it follows from (21) that

$$\begin{aligned}\det(s\tilde{T}-\tilde{S}) &= \det\left(\begin{bmatrix} 0 & -s_{12}^T \\ -s_{21} & sI-S_1 \end{bmatrix}\right) \\ &= -s_{21}s_{12}^T,\end{aligned} \tag{24}$$

which is a constant for all values of $s\in\mathbb{C}$. □

*Example 1:* A transfer function is provided in this example to show how Lemma 4 is applied. Consider the following transfer function.

$$h(s)=c^T(sI-A)^{-1}b+d=\frac{4s+3}{s(s+1)(s+3)}, \tag{25}$$

where

$$\begin{aligned} A &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -3 & -4 \end{bmatrix}, b=\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \\ c &= \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix}, d=0. \end{aligned} \tag{26}$$

Thus, $h(s)$ is transformed into the form $h(s)=\gamma e^T(sT-S)^{-1}b_1+d$, where $\gamma=5$, $T=I$ and

$$\begin{aligned} S &= Q^{-T}AQ^T=\begin{bmatrix} 0.48 & -0.36 & 0.8 \\ 0.64 & -0.48 & -0.6 \\ -2.4 & 1.8 & -4 \end{bmatrix}, \\ Q &= \begin{bmatrix} 0.6 & 0.8 & 0 \\ 0.8 & -0.6 & 0 \\ 0 & 0 & 1 \end{bmatrix}, b_1=Q^{-T}b=\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \end{aligned}$$

It follows from Lemma 4 that

$$\tilde{S}=\begin{bmatrix} 0 & -0.36 & 0.8 \\ 0 & -0.48 & -0.6 \\ -1 & 1.8 & -4 \end{bmatrix},$$

with $s_{11}=0$ and $s_{21}\neq\mathbf{0}$. Thus, according to Case 3 of Lemma 4, a new matrix

$$\tilde{S}=\begin{bmatrix} -0.8 & 0.36 \\ -0.6 & -0.48 \end{bmatrix}$$

is established with $s_{11}=-0.8\neq 0$. Then, Case 1 of Lemma 4 is applied and the eigenvalue of $s\tilde{T}-\tilde{S}$ is $-0.75$.

The eigenvalues of the matrix pencil $\lambda M+N$ can be computed by using the similar method in Lemma 4. The specific analysis is as follows.

Partition $N$ as

$$N=\begin{bmatrix} n_{11} & n_{12}^T \\ n_{21} & N_1 \end{bmatrix}, \tag{27}$$

where $n_{11}\in\mathbb{R}$, $n_{12},n_{21}\in\mathbb{R}^{n-1}$, $N_1\in\mathbb{R}^{(n-1)\times(n-1)}$. Then, in view of the matrix $M=T^2(I-ee^T)$ in (12), $T=I$, and the matrix $N$ in (27), let

$$\det(\lambda M+N)=\det\left(\begin{bmatrix} n_{11} & n_{12}^T \\ n_{21} & \lambda I+N_1 \end{bmatrix}\right). \tag{28}$$

If $n_{11}<0$, it follows that

$$\det(\lambda M+N)=n_{11}\det(\lambda I+N_1-(1/n_{11})n_{21}n_{12}^T).$$

Obviously, there exists $\lambda>0$ such that $\det(\lambda M+N)<0$. Thus, $h(s)$ is not NI.

If $n_{11}>0$,

$$\det(\lambda M+N)=n_{11}\det(\lambda I+N_1-(1/n_{11})n_{21}n_{12}^T)$$

holds. Hence, the eigenvalues of the matrix pencil $\lambda M+N$ are the eigenvalues of the matrix $(1/n_{11})n_{21}n_{12}^T-N_1$.

If $n_{11}=0$, we assume that $n_{21}\neq\mathbf{0}$, otherwise $h(s)$ is lossless NI. Due to $n_{21}\neq\mathbf{0}$, there exists a Householder transformation $Q_2\in\mathbb{R}^{(n-1)\times(n-1)}$ such that

$$Q_2n_{21}=-\|n_{21}\|_2e, \tag{29}$$

where $e$ is the first unit vector of length $n-1$. Let

$$\hat{Q}=\begin{bmatrix} 1 & 0 \\ 0 & Q_2 \end{bmatrix}.$$

Then, it follows that

$$\begin{aligned}\det(\lambda M+N) &= \det(\hat{Q})\det(\lambda M+N)\det(\hat{Q}^{-1}) \\ &= \det\left(\begin{bmatrix} 0 & n_{12}^TQ_2^{-1} \\ -\|n_{21}\|_2e & \lambda I+Q_2N_1Q_2^{-1} \end{bmatrix}\right) \\ &= \det\left(\begin{bmatrix} 0 & n_{22} & n_{13}^T \\ -\|n_{21}\|_2 & \lambda+n_1 & n_2 \\ \mathbf{0} & n_{31} & \lambda I+N_2 \end{bmatrix}\right) \\ &= \|n_{21}\|_2\det\left(\begin{bmatrix} n_{22} & n_{13}^T \\ n_{31} & \lambda I+N_2 \end{bmatrix}\right),\end{aligned} \tag{30}$$

where

$$\begin{aligned} n_{12}^TQ_2^{-1} &= [n_{22}\ n_{13}^T], \\ Q_2N_1Q_2^{-1} &= \begin{bmatrix} n_1 & n_2 \\ n_{31} & N_2 \end{bmatrix}, \end{aligned}$$

and $n_1,n_{22}\in\mathbb{R}$, $n_2^T,n_{13},n_{31}\in\mathbb{R}^{n-2}$, $N_2\in\mathbb{R}^{(n-2)\times(n-2)}$. The last line of (30) has the same block structure as equation (28). Thus, by repeating the above analysis, the computation of the eigenvalues of the matrix pencil $\lambda M+N$ is transformed into a eigenvalue computation problem of a matrix $(1/n_{11})n_{21}n_{12}^T-N_1$. Similar to Lemma 4, the special case of $N\in\mathbb{R}^{2\times 2}$ and $n_{11}=0$ also needs to be considered, which is noted in remark 4.

In view of Theorem 1, Theorem 2 and the above analysis for eigenvalue computations of matrix pencils $sT-S$, $s\tilde{T}-\tilde{S}$ and $\lambda M+N$ , the method only involving eigenvalue computations of matrices is summarized in Algorithm 1 for testing NI properties of scalar transfer functions with $E=I$.

*Algorithm 1:* (Test of negative imaginariness for $E=I$)

**INPUT**: The matrices $A \in \mathbb{R}^{n\times n}$, $b, c \in \mathbb{R}^n$ and $d \in \mathbb{R}$ of a transfer function $h(s) = c^T(sI - A)^{-1}b + d$.
**OUTPUT**: not NI, lossless NI, NI or SNI.

1. If $b = 0$ or $c = 0$, then $h(s)$ is lossless NI.
2. Compute $\gamma = \|c\|_2$, and construct a Householder transformation matrix $Q$ such that $Qc = \gamma e$.
3. Set $S = Q^{-T}AQ^T$ and $b_1 = Q^{-T}b$.
4. Compute the eigenvalues of $A$. If all the eigenvalues of $A$ have negative real part, go to Step 5. Otherwise, use Lemma 4 to compute the eigenvalues of the matrix pencil $s\tilde{T} - \tilde{S}$, which is the matrix pencil obtained by (9). If $A$ has eigenvalues with positive real part, check if they are also the eigenvalues of the matrix pencil $s\tilde{T}-\tilde{S}$. If no, stop: $h(s)$ is not NI. If $A$ has purely imaginary eigenvalues which are not the eigenvalues of the matrix pencil $s\tilde{T}-\tilde{S}$, check if they satisfy the conditions 3 and 4 of Definition 1. If no, stop: $h(s)$ is not NI.
5. Set $N = S^2(I - ee^T) + b_1e^T$. Partition $N$ as $\begin{bmatrix} n_{11} & n_{12}^T \\ n_{21} & N_1 \end{bmatrix}$.
6. If $n_{11} < 0$, stop: $h(s)$ is not NI. If $n_{11} = 0$, go to Step 7. If $n_{11} > 0$, set $\hat{N} = (1/n_{11})n_{21}n_{12}^T - N_1$ and go to Step 8.
7. If $n_{21} = \mathbf{0}$, stop: $h(s)$ is lossless NI. Otherwise, compute a Householder transformation matrix $Q_2$ such that $Q_2n_{21} = -\|n_{21}\|_2e$. Set $N = (I - ee^T)Q_2N_1Q_2^{-1} + en_{12}^TQ_2^{-1}$. Repartition $N$ as $\begin{bmatrix} n_{11} & n_{12}^T \\ n_{21} & N_1 \end{bmatrix}$.Then, go to Step 6.
8. Compute the eigenvalues of $\hat{N}$. If there has any real positive eigenvalue with odd multiplicity, stop: $h(s)$ is not NI.
9. Select any $\lambda_0 > 0$ that is not an eigenvalue of $\hat{N}$. If $f(\lambda_0) = \det(\lambda_0M + N) < 0$, then $h(s)$ is not NI. Otherwise, if all the eigenvalues of $A$ which are not the eigenvalues of the matrix pencil $s\tilde{T} - \tilde{S}$ have negative real part, and if all the eigenvalues of $\hat{N}$ are not real positive eigenvalues, $h(s)$ is SNI. Otherwise, $h(s)$ is NI.

*Remark 4:* When the matrix $N \in \mathbb{R}^{2\times 2}$ and $n_{11} = 0$ hold in Algorithm 1, $\det(\lambda M + N) = -n_{21}n_{12}^T$ is a constant in view of (28). In this case, Steps 7, 8 and 9 in Algorithm 1 can be omitted. Furthermore, if $\det(\lambda M + N) = -n_{21}n_{12}^T = 0$, $h(s)$ is lossless NI. If $n_{21}n_{12}^T > 0$, $h(s)$ is not NI. If $n_{21}n_{12}^T < 0$, whether $h(s)$ is NI or SNI depends on the judgment of the pole condition of $h(s)$ in Step 4.

*Remark 5:* In Algorithm 1, Steps 1-3 transform $h(s)$ into the form (3). Step 4 is used to test the conditions 1 and 2 in Theorem 1. The condition 3 in Theorem 1 is tested in Steps 5-8. Step 9 identified SNI properties according to Theorem 2. The frameworks of Algorithms 2 and 3 in next section are similar to Algorithm 1. The Householder transformation matrix $Q$ is an orthogonal matrix satisfying $Q = Q^T = Q^{-1}$, which can reduce some unnecessary calculations in Algorithm 1. Moreover, [45, Chap.5] provides a method to compute the matrix $Q$. In addition to obtaining the eigenvalues of the matrix pencil $s\tilde{T}-\tilde{S}$, Lemma 4 may also output that $\det(s\tilde{T}-\tilde{S}) = 0$ or $\det(s\tilde{T} - \tilde{S})$ is equal to a non-zero constant for all values of $s \in \mathbb{C}$. If $\det(s\tilde{T} - \tilde{S}) = 0$ holds for all values of $s \in \mathbb{C}$, it follows that $h(s) = d$ is lossless NI. If $\det(s\tilde{T} - \tilde{S})$ is a non-zero constant, the poles of $h(s)$ are equivalent to the eigenvalues of the matrix $A$ and the discussion of zero-pole cancellation in Step 4 of Algorithm 1 is not required, in view of Lemma 2 and Theorem 1.

## V. Eigenvalue-based Algorithms for $E \neq I$

In this section, two eigenvalue-based algorithms are developed for testing negative imaginariness of scalar transfer functions with $E \neq I$.

### *A. Algorithms based on eigenvalue computations of matrix pencils*

According to Theorems 1 and 2, a method based on eigenvalue computations of matrix pencils $sT - S$, $s\tilde{T} - \tilde{S}$ and $\lambda M + N$ is summarized in Algorithm 2 to test NI properties of scalar transfer functions with $E \neq I$, which may be non-proper transfer functions of descriptor systems. Moreover, since this article only studies proper SNI transfer functions, the SNI properties test is limited to proper case in Step 9 of Algorithm 2.

*Algorithm 2:* (Test of negative imaginariness for $E \neq I$)

**INPUT**: The matrices $E, A \in \mathbb{R}^{n\times n}$, $b, c \in \mathbb{R}^n$ and $d \in \mathbb{R}$ of a transfer function $h(s) = c^T(sE - A)^{-1}b + d$.
**OUTPUT**: not NI, lossless NI, NI or SNI.

1. If $b = 0$ or $c = 0$, then $h(s)$ is lossless NI.
2. Choose a scalar $s_0 \in \mathbb{R}$ such that $\det(s_0E - A) \neq 0$. Compute a formal factorization $s_0E - A = LU$. Compute $G = L^{-1}EU^{-1}$. Then, compute $\gamma = \|U^{-T}c\|_2$, and construct a Householder transformation matrix $\bar{Q} \in \mathbb{R}^{n\times n}$ such that $\bar{Q}U^{-T}c = \gamma e$.
3. Set $T = \bar{Q}^{-T}G\bar{Q}^T$, $S = s_0T - I$ and $b_1 = \bar{Q}^{-T}L^{-1}b$.
4. Compute the finite eigenvalues of the matrix pencil $sT - S$. If all the finite eigenvalues of $sT - S$ have negative real part, go to Step 5. Otherwise, compute the finite eigenvalues of the matrix pencil $s\tilde{T} - \tilde{S}$, which is the matrix pencil obtained by (9). If $sT - S$ has eigenvalues with positive real part, check if they are also the eigenvalues of the matrix pencil $s\tilde{T} - \tilde{S}$. If no, stop: $h(s)$ is not NI. If $sT-S$ has purely imaginary eigenvalues which are not the eigenvalues of the matrix pencil $s\tilde{T} - \tilde{S}$, check if they satisfy the conditions 3-5 of Definition 1. If no, stop: $h(s)$ is not NI.
5. Set $M = T^2(I - ee^T)$ and $N = S^2(I - ee^T) + Tb_1e^T$.
6. If the matrix pencil $\lambda M + N$ is singular, then stop: $h(s)$ is lossless NI.
7. Compute the finite eigenvalues of the matrix pencil $\lambda M + N$. If there has any real positive eigenvalue with odd multiplicity, stop: $h(s)$ is not NI.
8. Select any $\lambda_0 > 0$ that is not an eigenvalue of $\lambda M + N$. If $f(\lambda_0) = \det(\lambda_0M + N) < 0$, then $h(s)$ is not NI. Otherwise, if the order of the numerator polynomial

of $h(s)$ is higher than the order of the denominator polynomial of $h(s)$, $h(s)$ is NI. Otherwise, go to Step 9.

9. If all the eigenvalues of the matrix pencil $sT-S$ which are not the eigenvalues of the matrix pencil $s\tilde{T}-\tilde{S}$ have negative real part, and if all the finite eigenvalues of the matrix pencil $\lambda M+N$ are not real positive eigenvalues, $h(s)$ is SNI. Otherwise, $h(s)$ is NI.

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*Remark 6:* If $\det(E)\neq 0$, $h(s)$ can be transformed into the form $h(s)=c^T(sI-E^{-1}A)^{-1}E^{-1}b+d$. Therefore, Algorithm 1 is also applicable to the case of $\det(E)\neq 0$. Compared with Algorithm 2, the method in Algorithm 1 only involves the eigenvalue computations of matrices, and hence Algorithm 1 is preferred to test negative imaginariness of transfer functions when $\det(E)\neq 0$. Compared with Algorithm 1, an advantage of Algorithm 2 is that it is applicable to descriptor systems, which may be non-proper.

### *B. Algorithms based on eigenvalue computations of matrices*

Algorithm 2 in Subsection V-A calculates the eigenvalues of matrix pencils, which is more expensive than calculating the eigenvalues of matrices. Thus, under the assumption of $\det(A)\neq 0$, an algorithm based on eigenvalue computations of matrices is provided in this subsection to check NI properties of scalar transfer functions with $E\neq I$.

Since $\det(A)\neq 0$, the transfer function $h(s)=c^T(sE-A)^{-1}b+d$ is rewritten as

$$h(s)=c^T(sA^{-1}E-I)^{-1}A^{-1}b+d. \tag{31}$$

Similar to Section III, $h(s)$ in (31) can also be transformed into the form $h(s)=\gamma e^T(sT-S)^{-1}b_1+d$ in (3) and further into the form

$$h(s)=\gamma\frac{\det(s\tilde{T}-\tilde{S})}{\det(sT-S)}+d$$

in Lemma 1, where

$$\begin{aligned}&\gamma=\|c\|_2, T=Q^{-T}A^{-1}EQ^T,\\&S=I, b_1=Q^{-T}A^{-1}b,\\&\tilde{T}=T(I-ee^T), \tilde{S}=S(I-ee^T)-b_1e^T.\end{aligned}$$

Using the same analysis in Lemmas 1-3, Theorems 1 and 2 also hold. Since $\det(A)\neq 0$, $h(s)$ has no poles at the origin, and hence the condition 4 of Definition 1 is not considered in Theorem 1. Then, the eigenvalue computations of matrix pencils $sT-S$ and $s\tilde{T}-\tilde{S}$ can be transformed into the eigenvalue computations of matrices, respectively.

Since

$$\det(sT-S)=\det(sT-I)$$

holds, it is verified that $sT-S$ has no eigenvalues at zero. Furthermore, for $s\neq 0$,

$$\det(sT-S)=\det(sT-I)=(-s)^n\det(s^{-1}I-T). \tag{32}$$

It is obvious that the eigenvalues of the matrix pencil $sT-S$ are the inverse of the non-zero eigenvalues of $T$.

Since $sT-S$ has no eigenvalues at zero, only the non-zero eigenvalues of the matrix pencil $s\tilde{T}-\tilde{S}$ need to be computed. For $s\neq 0$,

$$\det(s\tilde{T}-\tilde{S})=(-s)^{n-1}\det(s^{-1}\bar{T}-\bar{S}), \tag{33}$$

where

$$\bar{T}=I-ee^T,\ \bar{S}=T(I-ee^T)-b_1e^T. \tag{34}$$

Thus, the non-zero eigenvalues of the matrix pencil $s\tilde{T}-\tilde{S}$ are the inverse of the non-zero eigenvalues of the matrix pencil $s\bar{T}-\bar{S}$.

Since $\bar{T}$ in (34) is equivalent to $\tilde{T}$ in Lemma 4, the similar method of computing the eigenvalues of the matrix pencil $s\tilde{T}-\tilde{S}$ in Lemma 4 can be used to compute the non-zero eigenvalues of the matrix pencil $s\bar{T}-\bar{S}$ by partitioning $\bar{S}$ as

$$\bar{S}=\begin{bmatrix} s_{11} & s_{12}^T\\ s_{21} & S_1\end{bmatrix}. \tag{35}$$

Thus, the following Lemma 5 is applied to compute the non-zero eigenvalues of the matrix pencil $s\bar{T}-\bar{S}$, in view of the method in Lemma 4.

*Lemma 5:* Considering the matrices $\bar{T}=I-ee^T$ and $\bar{S}=T(I-ee^T)-b_1e^T$ in (34). Partitioning $\bar{S}$ as the form in (35), then the three cases and the special case in Lemma 4 are also applicable to address the non-zero eigenvalues of the matrix pencil $s\bar{T}-\bar{S}$, by replacing matrices $\tilde{T}$, $\tilde{S}$ into matrices $\bar{T}$, $\bar{S}$.

*Proof*: The proof is the same with that proof in Lemma 4. □

Next, we analyze how the eigenvalue computation of $\lambda M+N$ is replaced by the eigenvalue computation of a matrix. For $\lambda>0$,

$$\det(\lambda M+N)=\lambda^{n-1}\det(\lambda^{-1}\bar{M}+\bar{N}), \tag{36}$$

where

$$\bar{M}=I-ee^T,\ \bar{N}=T^2(I-ee^T)+Tb_1e^T. \tag{37}$$

Thus, the real positive eigenvalues of the matrix pencil $\lambda M+N$ are the inverse of the real positive eigenvalues of the matrix pencil $\lambda\bar{M}+\bar{N}$. $\bar{M}$ in (37) is equivalent to $M$ in Section IV. Then, by partitioning $\bar{N}$ as

$$\bar{N}=\begin{bmatrix} n_{11} & n_{12}^T\\ n_{21} & N_1\end{bmatrix},$$

the same method of computing $\lambda M+N$ in Section IV can be applied to compute the eigenvalues of the matrix pencil $\lambda\bar{M}+\bar{N}$. Therefore, in Algorithm 3, the similar steps in Algorithm 1 can be used to test NI properties of scalar transfer functions of descriptor systems, which avoids the eigenvalue computations of matrix pencils.

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*Algorithm 3:* (Test of negative imaginariness for $E\neq I$)

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**INPUT**: The matrices $E, A\in\mathbb{R}^{n\times n}$, $b, c\in\mathbb{R}^n$ and $d\in\mathbb{R}$ of a transfer function $h(s)=c^T(sE-A)^{-1}b+d$ with $\det(A)\neq 0$.

**OUTPUT**: not NI, lossless NI, NI or SNI.

1. If $b = 0$ or $c = 0$, then $h(s)$ is lossless NI.
2. Compute $\gamma = \|c\|_2$, and construct a Householder transformation matrix $Q$ such that $Qc = \gamma e$.
3. Set $T = Q^{-T}A^{-1}EQ^T$ and $b_1 = Q^{-T}A^{-1}b$.
4. Compute the non-zero eigenvalues of $T$. If all the non-zero eigenvalues of $T$ have negative real part, go to Step 5. Otherwise, use Lemma 5 to compute the non-zero eigenvalues of the matrix pencil $s\bar{T} - \bar{S}$, which is the matrix pencil obtained by (34). If $T$ has the eigenvalues with positive real part, check if they are also the non-zero eigenvalues of the matrix pencil $s\bar{T} - \bar{S}$. If no, stop: $h(s)$ is not NI. If $T$ has purely imaginary eigenvalues which are not the non-zero eigenvalues of the matrix pencil $s\bar{T} - \bar{S}$, check if the inverse of these purely imaginary eigenvalues satisfies the conditions 3 and 5 of Definition 1. If no, stop: $h(s)$ is not NI.
5. Set $\bar{N} = T^2(I - ee^T) + Tb_1e^T$. Partition $\bar{N}$ as $\begin{bmatrix} n_{11} & n_{12}^T \\ n_{21} & N_1 \end{bmatrix}$.
6. If $n_{11} < 0$, stop: $h(s)$ is not NI. If $n_{11} = 0$, go to Step 7. If $n_{11} > 0$, set $\hat{N} = (1/n_{11})n_{21}n_{12}^T - N_1$ and go to Step 8.
7. If $n_{21} = \mathbf{0}$, stop: $h(s)$ is lossless NI. Otherwise, compute a Householder transformation matrix $Q_2$ such that $Q_2n_{21} = -\|n_{21}\|_2 e$. Set $\bar{N} = (I - ee^T)Q_2N_1Q_2^{-1} + en_{12}^TQ_2^{-1}$. Repartition $\bar{N}$ as $\begin{bmatrix} n_{11} & n_{12}^T \\ n_{21} & N_1 \end{bmatrix}$.Then, go to Step 6.
8. Compute the eigenvalues of $\hat{N}$. If there has any real positive eigenvalue with odd multiplicity, stop: $h(s)$ is not NI.
9. Select any $\lambda_0 > 0$ that is not an eigenvalue of $\hat{N}$. If $f(\lambda_0) = \det(\lambda_0^{-1}\bar{M} + \bar{N}) < 0$, then $h(s)$ is not NI. Otherwise, if the order of the numerator polynomial of $h(s)$, is higher than the order of the denominator polynomial of $h(s)$, $h(s)$ is NI. Otherwise, go to Step 10.
10. If all the non-zero eigenvalues of $T$ which are not the non-zero eigenvalues of the matrix pencil $s\bar{T} - \bar{S}$ have negative real part, and if all the eigenvalues of $\hat{N}$ are not real positive eigenvalues, $h(s)$ is SNI. Otherwise, $h(s)$ is NI.

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*Remark 7:* Similar to Remark 4, if the matrix $\bar{N} \in \mathbb{R}^{2\times 2}$ and $n_{11} = 0$ in Algorithm 3, $\det(\lambda^{-1}\bar{M} + \bar{N}) = -n_{21}n_{12}^T$ is a constant. In this case, Steps 7, 8 and 9 in Algorithm 3 can be omitted. If $\det(\lambda^{-1}\bar{M} + \bar{N}) = -n_{21}n_{12}^T = 0$, $h(s)$ is lossless NI. If $n_{21}n_{12}^T > 0$, $h(s)$ is not NI. If $n_{21}n_{12}^T < 0$, whether $h(s)$ is NI or SNI depends on the judgment of the pole condition of $h(s)$ in Step 4 of Algorithm 3 and the properness determination of $h(s)$.

*Remark 8:* If $\det(E) \neq 0$, both Algorithm 1 and Algorithm 3 are applicable to test NI properties of $h(s)$. Compared with Algorithm 1, an advantage of Algorithm 3 is the availability for non-proper descriptor systems. Compared with Algorithm 2, the method in Algorithm 3 only involves the eigenvalue computations of matrices which cost less. Obviously, Algorithm 3 also has a limitation on $\det(A) \neq 0$.

*Remark 9:* If the given transfer function $h(s)$ is stable, all the poles of $h(s)$ lie in $\mathrm{Re}[s] < 0$, and hence Step 4 of Algorithms 1-3 can be omitted. If the realization $(E, A, b, c)$ is minimal, the discussion of zero-pole cancellation is not required, and hence Step 4 of Algorithms 1-3 can be simplified.

*Remark 10:* The eigenvalues of matrices in Algorithms 1 and 3 can be computed by the QR algorithm, and the eigenvalues of matrix pencils in Algorithm 2 can be computed by the QZ algorithm. The costs of the QZ and QR algorithms are $30n^3 + \mathcal{O}(n^2)$ and $10n^3 + \mathcal{O}(n^2)$ operations, respectively [45]. Similar to the computational complexity analysis of the positive real test algorithms in [10], [11], the overall operation count involving eigenvalue computations of matrices in Algorithm 1 or 3 is approximately $30n^3 + \mathcal{O}(n^2)$, while the overall operation count involving eigenvalue computations of matrix pencils in Algorithm 2 is approximately $90n^3 + \mathcal{O}(n^2)$. Moreover, the eigenvalue computations in Algorithms 1-3 based on the QR or QZ algorithm, can be implemented in MATLAB or by using off-the-shelf linear algebra libraries, such as LAPACK [45].

*Remark 11:* A key feature of this paper is the transformation of the NI transfer function definitions conditions into eigenvalue computations, leading to the development of Algorithms 1-3. However, for very large systems, particularly those with high-dimensional or poorly conditioned matrices, eigenvalue computations can be impractical. This impracticality arises from several factors. For instance, the availability of computational resources, such as memory and processing power, can limit eigenvalue computations. Additionally, for extremely high-dimensional or poorly conditioned matrices, QR and QZ algorithms may become infeasible and numerically unstable, resulting in failures in MATLAB and LAPACK computations.

## VI. Illustrative Examples

In this section, three examples are presented to determine negative imaginariness of transfer functions by using eigenvalue-based algorithms. Example 2 is provided to show how the eigenvalue computation is applied for testing negative imaginariness. Example 3 is a n-stage RLC circuit network, which shows that the eigenvalue computation is useful to determine negative imaginariness of large dimension systems. The last example is used to test negative imaginariness of non-proper descriptor systems.

*Example 2:* Consider the transfer function (25) and the minimal realization (26) in Example 1. Algorithm 1 is applied to test NI properties of (25), where $\gamma$, $Q$, $S$ and $b_1$ are given by Example 1. Then, according to Step 4 of Algorithm 1, the eigenvalues of matrix $A$ are $0, -1, -3$. Obviously, system (25) has a pole at the origin and the condition 4 of Definition 1 holds. Then, Step 5 of Algorithm 1 establishes the following matrix $N$ with $n_{11} = 0$ and $n_{21} = \begin{bmatrix} 0 & 1 \end{bmatrix}^T \neq \mathbf{0}$,

$$N = \begin{bmatrix} 0 & 1.44 & -2.6 \\ 0 & -1.08 & 3.2 \\ 1 & -7.2 & 13 \end{bmatrix}.$$

Thus, Step 7 establishes a new matrix $N$ with $n_{11} = 2.6 > 0$:

$$N = \begin{bmatrix} 2.6 & -1.44 \\ 3.2 & -1.08 \end{bmatrix}.$$

Fig. 1. Plot of the imaginary part of the transfer function $h(j\omega)$.

Fig. 2. A n-stage RLC circuit network.

Then, $\hat{N} = -0.6923$ is obtained with multiplicity 1. In view of Step 9 of Algorithm 1, the transfer function (25) is NI, but not SNI. To illustrate this, the imaginary part of the transfer function (25) is plotted in Fig. 1, where

$$\begin{aligned}\mathrm{Im}[h(j\omega)] &= -\frac{1}{2}j[h(j\omega) - h(-j\omega)] \\ &= -\frac{13\omega^2 + 9}{\omega^5 + 10\omega^3 + 9\omega} < 0,\end{aligned}$$

for all $\omega > 0$.

Moreover, since $\det(A) = 0$ and $c^T b = 0$ hold in this example, the existing methods of judging negative imaginariness, such as, LMIs [12], [13], spectral conditions [32] and Riccati equations [15], [35], [37], are not applicable to the test of system (25). Compared with the existing methods in [12], [13], [15], [32], [35], [37], Example 2 shows that Algorithm 1 has less restrictions in determining NI properties of SISO system. In addition, if the state-space realization in (26) is changed to an arbitrary non-minimal realization, Algorithm 1 is also applicable while the LMI condition in [15, Lemma 2] requires a minimal realization. Additionally, the NI properties of (25) can be tested directly by using Definition 1 with plotting the imaginary part in Fig. 1. However, for a given state-space realization of high-dimensional systems, the test using NI definitions becomes complex. For example, the transfer function in following Example 3 is difficult to be obtained for a 2001-dimensional system, and the determination of the imaginary part is very complex.

*Example 3:* Consider a n-stage RLC circuit network as shown in Fig. 2, which is taken from [39]. Such ladder RLC networks are practically relevant, as they arise in real-world engineering modeling tasks like long transmission lines. The input is the voltage $V(t)$. The output is the total charge $Q(t)$ in the capacitances, where

$$Q(t) = \sum_{i=0}^{k} Q_i(t).$$

Let the state variable be

$$x(t) = \begin{bmatrix} u_0(t) & i_{L_1}(t) & u_1(t) & \cdots & i_{L_k}(t) & u_k(t) \end{bmatrix}^T$$

where $u_i(t)$ is the voltage of the capacitance $C_i$, and $i_{L_i}(t)$ is the current through the inductor $L_i$. Thus, the following system (38) with $u(t) = V(t)$ and $y(t) = Q(t)$ is obtained.

$$\begin{cases} \dot{x}(t) = Ax(t) + bu(t) \\ y(t) = c^T x(t), \end{cases} \tag{38}$$

where

$$A = \begin{bmatrix} -\frac{1}{C_0R_0} & -\frac{1}{C_0} & 0 & 0 & 0 & \cdots & 0 \\ \frac{1}{L_1} & -\frac{R_1}{L_1} & -\frac{1}{L_1} & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{C_1} & 0 & -\frac{1}{C_1} & 0 & \cdots & 0 \\ 0 & 0 & \frac{1}{L_2} & -\frac{R_2}{L_2} & -\frac{1}{L_2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{L_k} & -\frac{R_k}{L_k} & -\frac{1}{L_k} \\ 0 & 0 & 0 & \cdots & 0 & \frac{1}{C_k} & 0 \end{bmatrix},$$

$$b = \begin{bmatrix} \frac{1}{C_0R_0} & 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}^T,$$

$$c = \begin{bmatrix} C_0 & 0 & C_1 & 0 & C_2 & \cdots & C_k \end{bmatrix}^T.$$

Consider a 2001-dimensional system (38) with $k = 10^3$. Let $L_i = 1H$, $R_i = 0.5\Omega$ and $C_i = 1F$. By running Algorithm 1, it can be found that all the eigenvalues of $A$ have negative real part and $n_{11} > 0$. Then, compute the eigenvalues of $\hat{N}$. It turns out that the eigenvalues of $\hat{N}$ are $2 \times 10^3$ different complex numbers. Thus, Step 9 of Algorithm 1 is used to found that the system (38) with $k = 10^3$ is SNI.

We also further test SNI properties by solving the LMIs conditions in [13]. In view of Lemma 8 in [13], we need to find a matrix $Y > 0$ such that

$$AY + YA^T \leq 0, \text{and } b + AYc = 0, \tag{39}$$

and check whether $\mathrm{rank}(M(j\omega)) = m$ for any $\omega \in (0, \infty)$, where $M(s) = LY^{-1}A^{-1}(sI - A)^{-1}b$, $m = \mathrm{rank}(b) = 1$, $L^T L = -AY - YA^T$. However, the solutions of the LMIs (39) are not found by using the solver SDPT3 or MOSEK, and the MATLAB toolbox YALMIP [46] which is designed for modeling and solving optimization problems. Moreover, the information "Maximum iterations or time limit exceeded" is returned. Obviously, the number of iterations and the solution time will increase with the expansion of the matrix dimension, which may lead to no solution or difficulty in solving. In contrast, Algorithm 1 in this paper only involves the computation of matrix eigenvalues which can be applied to determine negative imaginariness of large dimensional systems.

Moreover, when the SNI lemma based on Riccati equations in [35], [37] which can be converted to an eigenvalue problem

for a Hamiltonian matrix, is taken, there exist 4002 eigenvalues of the Hamiltonian matrix and the system (38) is verified to be SNI. However, testing of SNI properties via Riccati equations or the eigenvalue computations for the Hamiltonian matrix in [35], [37] is more expensive than Algorithm 1, which only involves the calculation of $2 \times 10^3$ eigenvalues.

In Example 3 with $k = 10^3$, the runtime of Algorithm 1 is 5.0477 seconds. Additionally, for system (38), we further increased the value of $k$, resulting in system dimensions of 6001 and 10001, with corresponding runtime of 174.6259 seconds and 738.8268 seconds, respectively. As the dimension of system (38) increases, the runtime and computational cost increase rapidly. With further increases in the value of $k$, eigenvalue calculations will become difficult and infeasible. In fact, when we increased $k$ to $2.5 \times 10^4$, resulting in a system dimension of 50001, our hardware's memory was insufficient to support the creation of such a large matrix. This also aligns with the limitations of eigenvalue computations analyzed in Remark 11.

*Example 4:* Consider the following quarter-car model, which is taken from [42], [47].

$$m\ddot{q} + t(\dot{q} - \dot{u} - \dot{\omega}) + k(q - u - \omega) = 0, \tag{40}$$

where $u$ is the active displacement of the suspension system, $q$ is the vertical displacement of the vehicle chassis, $\omega$ is the vertical wheel displacement, $m$ represents one quarter of the vehicle mass, $t$ and $k$ respectively represent the damping and stiffness related to the suspension and tires. Considering the case that the input is $(u + \omega)$ and the output is $(-\ddot{q})$, let the state variable be

$$x(t) = \begin{bmatrix} q & \dot{q} & u + \omega & \dot{u} + \dot{\omega} \end{bmatrix}^T.$$

Thus, the car dynamics system (40) has the following non-proper transfer function

$$h(s) = c^T(sE - A)^{-1}b = \frac{-s^2(ts + k)}{(ms^2 + ts + k)}, \tag{41}$$

where

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k}{m} & -\frac{t}{m} & \frac{k}{m} & \frac{t}{m} \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$
$$b = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}, c = \begin{bmatrix} \frac{k}{m} \\ \frac{t}{m} \\ -\frac{k}{m} \\ -\frac{t}{m} \end{bmatrix}.$$

Let $t = k = m = 1$. Then, in view of Algorithm 2, let $s_0 = 0$, $L = s_0E - A = -A$ and $U = I$ hold , where $\det(s_0E - A) \neq 0$. Furthermore, according to Step 2 and Step 3 of Algorithm 2, $\gamma = \|U^{-T}c\|_2 = 2$, and the following matrices are computed.

$$G = \begin{bmatrix} 1 & 1 & -1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix},$$
$$\bar{Q} = \begin{bmatrix} 0.5 & 0.5 & -0.5 & -0.5 \\ 0.5 & 0.5 & 0.5 & 0.5 \\ -0.5 & 0.5 & 0.5 & -0.5 \\ -0.5 & 0.5 & -0.5 & 0.5 \end{bmatrix},$$
$$T = \begin{bmatrix} 0.25 & 0.25 & 0.25 & 0.25 \\ 0.75 & -0.25 & -0.25 & 0.75 \\ -1.25 & -0.25 & 0.75 & -0.25 \\ -0.75 & -0.75 & 0.25 & 0.25 \end{bmatrix},$$
$$b_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, S = -I.$$

According to Step 4 of Algorithm 2, the eigenvalues of $sT - S$ are $-0.5 \pm 0.8660j$ with negative real part. Then, Step 5 implies that the matrix pencil $\lambda M + N$ is

$$\lambda \begin{bmatrix} 0 & -0.25 & 0.25 & 0.25 \\ 0 & -0.25 & 0.25 & 0.25 \\ 0 & -0.25 & 0.25 & -0.75 \\ 0 & -0.25 & 0.25 & -0.75 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}.$$

The eigenvalue of the matrix pencil $\lambda M + N$ is 0 with multiplicity 2. Thus, the transfer function (41) is NI according to Step 8 of Algorithm 2.

On the other hand, since $\det(A) \neq 0$, Algorithm 3 in Subsection V-B is also applicable to test negative imaginariness of system (41). First, according to Step 2 and Step 3 of Algorithm 3, we have $\gamma = \|c\|_2 = 2$, and the following matrices.

$$Q = \begin{bmatrix} 0.5 & 0.5 & -0.5 & -0.5 \\ 0.5 & 0.5 & 0.5 & 0.5 \\ -0.5 & 0.5 & 0.5 & -0.5 \\ -0.5 & 0.5 & -0.5 & 0.5 \end{bmatrix},$$
$$T = Q^{-T}A^{-1}EQ^T = \begin{bmatrix} -0.25 & -0.25 & -0.25 & -0.25 \\ -0.75 & 0.25 & 0.25 & -0.75 \\ 1.25 & 0.25 & -0.75 & 0.25 \\ 0.75 & 0.75 & -0.25 & -0.25 \end{bmatrix},$$
$$b_1 = Q^{-T}A^{-1}b = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}.$$

Then, it can be found that the non-zero eigenvalues of $T$ are $-0.5 \pm 0.8660j$ with negative real part, and hence the matrix $\bar{N}$ in Step 5 is constructed.

$$\bar{N} = \begin{bmatrix} 0 & -0.25 & 0.25 & 0.25 \\ -1 & -0.25 & 0.25 & 0.25 \\ 0 & -0.25 & 0.25 & -0.75 \\ -1 & -0.25 & 0.25 & -0.75 \end{bmatrix},$$

where $n_{11} = 0$ and $n_{21} = \begin{bmatrix} -1 & 0 & -1 \end{bmatrix}^T \neq \mathbf{0}$. By running Steps 6 and 7 of Algorithm 3 twice, a new matrix $\bar{N}$ is computed.

$$\bar{N} = \begin{bmatrix} 0.4083 & -0.1444 \\ 0.2357 & -0.0833 \end{bmatrix},$$

where $n_{11} > 0$. It follows from Steps 8 and 9 that $\hat{N} = 0$ without any real positive eigenvalue and $\det(\lambda_0^{-1}\bar{M} + \bar{N}) > 0$ for $\lambda_0 > 0$. Obviously, the order of the numerator polynomial of the transfer function (41) is higher than the order of the denominator polynomial. Thus, (41) is a non-proper transfer function and (41) is NI according to Step 9 of Algorithm 3.

## VII. Conclusions

The eigenvalue-based characterization of negative imaginariness of scalar transfer functions has been developed in the paper. Based on the characterization, several algorithms which involve eigenvalue computations of matrices or matrix pencils, were proposed for testing NI properties of scalar transfer functions. For a non-proper transfer function of descriptor systems, eigenvalue-based algorithms are also useful to test negative imaginariness. Finally, two numerical examples, a RLC circuit network as well as a car dynamics example were provided to illustrate the effectiveness of the proposed algorithms.

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