Connections between integral quadratic constraints and dissipativity

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***Abstract*—We show that a recent dissipativity approach to feedback stability analysis of potentially open-loop unstable systems, which encompasses the classical soft integral quadratic constraint (IQC) theorem, may be recovered by hard IQC theory. The latter is known to be subsumable by the more general soft IQC theory endowed with homotopies that are continuous in the gap topology. Additionally, we demonstrate how the aforementioned classical soft IQC theorem, initially introduced for the analysis of a feedback interconnection of a nonlinear component and a linear system, may be recast to analyse the stability of a feedback interconnection of two nonlinear systems. This generates a frequency-dependent $(Q(\omega), S(\omega), R(\omega))$-dissipativity result.**

***Index Terms*—Feedback stability, integral quadratic constraints, dissipativity, uncertainty**

I. Introduction

The seminal paper [1] by Megretski and Rantzer introduced a powerful and flexible framework within which robust stability of feedback interconnections of open-loop stable systems can be analysed via the use of *soft* (a.k.a. conditional) integral quadratic constraints (IQCs). A soft IQC differs from a *hard* (a.k.a. unconditional) IQC in that the corresponding time-domain integral is taken from 0 to ∞ with respect to signals lying in the L_2-graph of the system under consideration, whereas a hard IQC involves integrating extended L_2 signals in the graph of the system over all finite intervals starting from 0 [2]. Hard IQC theory for robust stability analysis of feedback interconnections of possibly open-loop unstable systems is discussed in depth in [3], where it is shown it can be established by more general soft IQC theory equipped with homotopies that are continuous in the gap topology from [4]. The latter was first developed in the technical report [5] and later modified in [3] to tailor to the purpose of establishing the link between hard and soft IQC results. Related results in the linear setting may be found in [6], [7], [8], [9], [10], [11].

In recent years, numerous efforts have been invested into proving the soft IQC theorem via dissipativity theory [12], [13], [14], [15], [16]. They rely on the existence of certain canonical factorisations, which may be traced back to the classical multiplier approach [17], [18]. In particular, [15], [16] proposed a dissipativity framework from which it is possible to establish the classical soft (frequency-domain) IQC result in [1] where boundedness of both open-loop components, one of which linear time-invariant (LTI), is required and the components are subject to real-rational multipliers. This provides a beautiful link between classical soft IQCs and dissipativity.

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In this paper, we show that the hard IQC result in [3] can be used to establish the dissipativity result in [15], [16]. In doing so, we tie up loose ends with regards to the connections between dissipativity and hard/soft IQC approaches. In particular, through the results in [15], [16] and this paper, the standard soft IQC theorem in [1] may now be seen to be derivable from the hard IQC theorem described in [3].

Additionally, the soft IQC theorem from [1] was developed for a feedback interconnection of a nonlinear component and an LTI system. We demonstrate in this paper that it can be recast to analyse the stability a feedback interconnection of two nonlinear systems. In doing so, it explicitly leads to a frequency-dependent $(Q(\omega), S(\omega), R(\omega))$-dissipativity result that can be used to recover several known results in the literature, including small-gain, passivity, and static (Q, S, R)-dissipativity [19].

The paper is organised as follows. The next section defines the notation and provide the preliminaries for the paper. Section III recapitulates three known results on feedback stability from the literature — soft IQC, dissipativity, and hard IQC. Section IV is devoted to establishing the dissipativity based feedback stability result using hard IQC theory. The feedback interconnection of two nonlinear systems is studied using soft IQC theory from [1] in Section V. Finally, concluding remarks are provided in Section VII.

II. Notation and preliminaries

Let $\mathbb{R}$, $\mathbb{R}^n$, $\mathbb{R}^{p\times m}$, $\mathbb{C}^{p\times m}$ denote the sets of real numbers, n-dimensional real column vectors, $p \times m$ real matrices, and $p \times m$ complex matrices, respectively. Given a matrix $M \in \mathbb{C}^{p\times m}$, its complex conjugate tranpose is denoted by M^*. The transpose of $M \in \mathbb{R}^{p\times m}$ is denoted by $M^\top$. Let $|x| = (x^\top x)^{\frac{1}{2}}$ for $x \in \mathbb{R}^n$. A matrix $M \in \mathbb{C}^{n\times n}$ is said to be positive (semi)definite, denoted by $M(\geq) > 0$, if $v^* M v (\geq) > 0$ for all $v \neq 0$. $M(\leq) < 0$ is used to denote $-M(\geq) > 0$. An $M \in \mathbb{R}^{n\times n}$ is said to be Hurwitz if the real parts of all its eigenvalues are negative. Denote by I_n the n-dimensional identity matrix. In the sequel, the subscript n is omitted when the dimension is clear from the context.

Denote by L_2^n the set of $\mathbb{R}^n$-valued Lebesgue square-integrable functions:

$$L_2^n = \left\{ v : [0, \infty) \to \mathbb{R}^n \;\middle|\; \int_0^\infty |v(t)|^2 \, dt < \infty \right\}.$$

For $v, w \in L_2^n$, let

$$\langle v, w \rangle = \int_0^\infty v(t)^\top w(t) \, dt \quad \text{and} \quad \|v\|^2 = \langle v, v \rangle.$$

Define the truncation operator $(P_T v)(t) = v(t)$ for $t \leq T$ and $(P_T v)(t) = 0$ for $t > T$, and the extended space

$$L_{2e}^n = \{ v : [0, \infty) \to \mathbb{R}^n \mid P_T v \in L_2 \; \forall T \in [0, \infty) \}.$$

Subsequently, the superscript n is often suppressed for notational convenience. Given $v, w \in L_{2e}$, let

$$\langle v, w \rangle_T = \int_0^T v(t)^\top w(t) \, dt \quad \text{and} \quad \|v\|_T^2 = \langle v, v \rangle_T.$$

A system in this paper is taken to be an operator $\mathbf{\Delta} : L_{2e} \to L_{2e}$. $\mathbf{\Delta}$ is said to be *causal* if $P_T \mathbf{\Delta} P_T = P_T \mathbf{\Delta}$ for all $T \geq 0$. A causal $\mathbf{\Delta}$ is called *bounded* if its bound [20, Section 2.4] is finite, i.e.

$$\|\mathbf{\Delta}\| = \sup_{u \in L_{2e}, T>0: \|u\|_T \neq 0} \frac{\|\mathbf{\Delta} u\|_T}{\|u\|_T} = \sup_{0 \neq u \in L_2} \frac{\|\mathbf{\Delta} u\|}{\|u\|} < \infty.$$

The identity operator $I : L_{2e}^m \to L_{2e}^m$ satisfies $I(f) = f$ for any $f \in L_{2e}^m$. The zero operator $0 : L_{2e}^m \to L_{2e}^p$ satisfies $0(f) = 0$ for any $f \in L_{2e}^m$. Given two nonlinear operators $\mathbf{\Delta}_1, \mathbf{\Delta}_2 : L_{2e}^m \to L_{2e}^p$, the nonlinear operator $(\mathbf{\Delta}_1 + \mathbf{\Delta}_2) : L_{2e}^m \to L_{2e}^p$ satisfies $(\mathbf{\Delta}_1 + \mathbf{\Delta}_2)(f) = \mathbf{\Delta}_1(f) + \mathbf{\Delta}_2(f)$ for any $f \in L_{2e}^m$. Given four nonlinear operators $\mathbf{\Delta}_{ij} : L_{2e}^{m_j} \to L_{2e}^{p_i}$ with $i, j \in \{1, 2\}$, the packed nonlinear operator

$$\begin{bmatrix} \mathbf{\Delta}_{11} & \mathbf{\Delta}_{12} \\ \mathbf{\Delta}_{21} & \mathbf{\Delta}_{22} \end{bmatrix} : L_{2e}^{m_1+m_2} \to L_{2e}^{p_1+p_2}$$

satisfies

$$\begin{bmatrix} \mathbf{\Delta}_{11} & \mathbf{\Delta}_{12} \\ \mathbf{\Delta}_{21} & \mathbf{\Delta}_{22} \end{bmatrix} \left(\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \right) = \begin{bmatrix} \mathbf{\Delta}_{11}(f_1) + \mathbf{\Delta}_{12}(f_2) \\ \mathbf{\Delta}_{21}(f_1) + \mathbf{\Delta}_{22}(f_2) \end{bmatrix}$$

for any $f_i \in L_{2e}^{m_i}$ with $i \in \{1, 2\}$. Given two nonlinear operators $\mathbf{\Delta}_1 : L_{2e}^m \to L_{2e}^p$ and $\mathbf{\Delta}_2 : L_{2e}^p \to L_{2e}^q$, the composition of these two nonlinear operators $\mathbf{\Delta}_2 \circ \mathbf{\Delta}_1 : L_{2e}^m \to L_{2e}^q$ satisfies $\mathbf{\Delta}_2 \circ \mathbf{\Delta}_1(f) = \mathbf{\Delta}_2 (\mathbf{\Delta}_1(f))$ for any $f \in L_{2e}^m$. Given a nonlinear operator $\mathbf{\Delta} : L_{2e}^m \to L_{2e}^m$, the inverse nonlinear operator $\mathbf{\Delta}^{-1} : L_{2e}^m \to L_{2e}^m$ satisfies $\mathbf{\Delta}^{-1}(\mathbf{\Delta}(f)) = f$ for any $f \in L_{2e}^m$. The inverse of a composition of two nonlinear operators satisfies $[\mathbf{\Delta}_2 \circ \mathbf{\Delta}_1]^{-1} = \mathbf{\Delta}_1^{-1} \circ \mathbf{\Delta}_2^{-1}$.

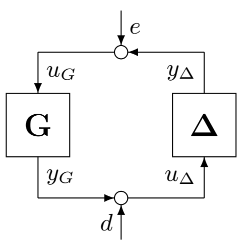

Fig. 1. Standard feedback configuration

The main object of study in this paper is the feedback interconnection of causal systems $\mathbf{\Delta} : L_{2e}^m \to L_{2e}^p$ and $\mathbf{G} : L_{2e}^p \to L_{2e}^m$ described by

$$-e = y_\Delta - u_G; \quad d = u_\Delta - y_G; \quad y_\Delta = \mathbf{\Delta} u_\Delta; \quad y_G = \mathbf{G} u_G. \tag{1}$$

This is denoted by $[\mathbf{\Delta}, \mathbf{G}]$ and illustrated in Fig. 1. The notions of feedback well-posedness and stability defined below are well studied [20], [18].

Definition II.1. $[\mathbf{\Delta}, \mathbf{G}]$ is said to be *well-posed* if the map $(u_\Delta, u_G) \mapsto (d, e)$ defined by (1) has a causal inverse $\mathbf{H}_{\mathbf{\Delta},\mathbf{G}}$ on L_{2e}. $[\mathbf{\Delta}, \mathbf{G}]$ is said to be (finite-gain) *stable* if it is well-posed and

$$\mathbf{H}_{\mathbf{\Delta},\mathbf{G}} = \begin{bmatrix} d \\ e \end{bmatrix} \in L_{2e} \mapsto \begin{bmatrix} u_\Delta \\ u_G \end{bmatrix} \in L_{2e}$$

is bounded, i.e. there exists $C > 0$ such that

$$\int_0^T |u_\Delta(t)|^2 + |u_G(t)|^2 \, dt \leq C \int_0^T |d(t)|^2 + |e(t)|^2 \, dt$$

for all $d, e \in L_{2e}$ and $T > 0$.

Define the graph and extended graph of $\mathbf{\Delta}$ as, respectively, $\mathscr{G}(\mathbf{\Delta}) = \{[\begin{smallmatrix} u \\ y \end{smallmatrix}] \in L_2 : y = \mathbf{\Delta} u\}$ and $\mathscr{G}_e(\mathbf{\Delta}) = \{[\begin{smallmatrix} u \\ y \end{smallmatrix}] \in L_{2e} : y = \mathbf{\Delta} u\}$. Likewise, define the inverse graph and extended inverse graph of $\mathbf{G}$ as, respectively, $\mathscr{G}'(\mathbf{G}) = \{[\begin{smallmatrix} y \\ u \end{smallmatrix}] \in L_2 : y = \mathbf{G} u\}$ and $\mathscr{G}_e'(\mathbf{G}) = \{[\begin{smallmatrix} y \\ u \end{smallmatrix}] \in L_{2e} : y = \mathbf{G} u\}$.

Let

$$\mathcal{B} = \Big\{ \phi : \mathbb{R} \to \mathbb{C}^{n \times n} \;\Big|\; \phi \text{ is piecewise continuous}, \; \phi(\omega) = \phi(\omega)^* \; \forall \omega \in \mathbb{R} \text{ and } \sup_{\omega \in \mathbb{R}} \bar{\sigma}(\phi(\omega)) < \infty \Big\},$$

where $\bar{\sigma}(\cdot)$ denotes the largest singular value. Given $\Pi \in \mathcal{B}$ and $v \in L_2$, define the quadratic form

$$\sigma(\Pi, v) = \int_{-\infty}^{\infty} \hat{v}(j\omega)^* \Pi(\omega) \hat{v}(j\omega) \, d\omega,$$

where $\hat{v}$ denotes the Fourier transform of v, i.e. $\hat{v}(j\omega) = \frac{1}{\sqrt{2\pi}} \int_0^\infty v(t) e^{-j\omega t} \, dt$.

Given a bounded causal system $\mathbf{\Delta}$ and static matrices $Q \geq 0$, $R \leq 0$ and S, $\mathbf{\Delta}$ is said to be ultimately dissipative [19] with respect to (Q, S, R) if $\sigma(\Pi, v) \geq 0$ for all $v \in \mathscr{G}(\mathbf{\Delta})$, where

$$\Pi = \begin{bmatrix} Q & S \\ S^\top & R \end{bmatrix}.$$

In particular, if $S = 0$, $Q = \gamma^2 I$ and $R = -I$, then $\mathbf{\Delta}$ is said to have bound γ. On the other hand, if $S = I$, $Q = -\delta I$ and $R = -\epsilon I$, then $\mathbf{\Delta}$ is said to have input-feedforward passivity index δ and output-feedback passivity index ϵ.

Given a linear time-invariant (LTI) $\mathbf{G} = u \mapsto y$ described by

$$\begin{aligned} \dot{x} &= Ax + Bu, \quad x(0) = 0 \\ y &= Cx + Du, \end{aligned}$$

denote by G its transfer function representation, i.e.

$$G(s) = C(sI - A)^{-1} B + D$$

and define $G^*(s) = G(-s)^\top$.

III. Feedback stability

This section summarises three feedback stability results adopted from the literature. It sets up the preliminaries for the succeeding section, where the relationship between the results is manifested and discussed.

A. Soft IQCs

Consider the feedback interconnection $[\mathbf{\Delta}, \mathbf{G}]$, where $\mathbf{G}$ is LTI with a state-space realisation

$$\begin{aligned} \dot{x} &= Ax + Bu_G, \quad x(0) = x_0 \in \mathbb{R}^n \\ y_G &= Cx + Du_G. \end{aligned} \tag{2}$$

The following feedback stability result based on soft IQCs is a restatement of [1, Theorem 1]. Note that only *rational* multipliers were considered in [1, Theorem 1], but this restriction may be removed without affecting the result, as is done below.

Theorem III.1. *Let $\mathbf{\Delta}$ be a bounded causal (nonlinear) system and $\mathbf{G}$ be of the form* (2) *with A being Hurwitz and $x(0) = 0$. Suppose there exists $\Pi \in \mathcal{B}$ such that*

(i) *$[\tau \mathbf{\Delta}, \mathbf{G}]$ is well-posed for all $\tau \in [0, 1]$;*
(ii) *$\sigma(\Pi, v) \geq 0$ for all $v \in \mathscr{G}(\tau \mathbf{\Delta})$, $\tau \in [0, 1]$;*
(iii) *$\sigma(\Pi, w) \leq -\epsilon \|w\|^2$ for all $w \in \mathscr{G}'(\mathbf{G})$ and some $\epsilon > 0$.*

Then $[\mathbf{\Delta}, \mathbf{G}]$ is stable.

Remark III.2. It may be shown that condition (iii) in Theorem III.1 is equivalent to

$$\begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \Pi(\omega) \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} < 0 \quad \forall \omega \in [0, \infty].$$

See, for example, [21, Proposition 4].

B. Dissipativity

Consider the feedback interconnection $[\mathbf{\Delta}, \mathbf{G}]$ where $\mathbf{G}$ is given by (2), and an LTI operator $\mathbf{\Psi}$ with a state-space realisation

$$\begin{aligned} \dot{\xi} &= A_\Psi \xi + B_\Psi v, \quad \xi(0) = 0 \\ z &= C_\Psi \xi + D_\Psi v. \end{aligned} \tag{3}$$

The following result is taken from [16, Theorem 13 and Lemma 17]. In the statement of the result, we have imposed an additional well-posedness assumption on the feedback interconnection and causality requirements on the open-loop systems so that they are in line with the definition of feedback stability in Definition II.1.

Theorem III.3. *Given a causal (nonlinear) system $\mathbf{\Delta}$ and an LTI system $\mathbf{G}$ of the form (2), suppose there exist $Z = Z^\top$, $X = X^\top = \left[\begin{smallmatrix} X_1 & X_{12} \\ X_{12}^\top & X_2 \end{smallmatrix}\right]$, $P = P^\top$, and $\mathbf{\Psi}$ of the form (3) with a Hurwitz A_Ψ such that*

(i) *$[\mathbf{\Delta}, \mathbf{G}]$ is well-posed when $x(0) = 0$;*
(ii) *$\langle \mathbf{\Psi} v, P \mathbf{\Psi} v \rangle_T - \xi(T)^\top Z \xi(T) \geq 0$ for all $v \in \mathscr{G}_e(\mathbf{\Delta})$, $T > 0$;*
(iii)

$$\begin{bmatrix} \xi(T) \\ x(T) \end{bmatrix}^\top X \begin{bmatrix} \xi(T) \\ x(T) \end{bmatrix} + \langle \mathbf{\Psi} \left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right], P \mathbf{\Psi} \left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right] \rangle_T + \epsilon(\|\xi\|_T^2 + \|x\|_T^2 + \|u\|_T^2) \leq x(0)^\top X_2 x(0)$$

for all $\left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right] \in \mathscr{G}'_e(\mathbf{G})$, $x(0) \in \mathbb{R}^n$, $T > 0$ and some $\epsilon > 0$;
(iv)

$$M = \begin{bmatrix} X_1 + Z & X_{12} \\ X_{12}^\top & X_2 \end{bmatrix} > 0.$$

If $x(0) = 0$, then $[\mathbf{\Delta}, \mathbf{G}]$ is stable.

Even though A_Ψ is not assumed to be Hurwitz in [16, Theorem 13], one can observe from the proof of the theorem that $\xi \in L_2$ for any $x(0) \in \mathbb{R}^n$. This means that even if A_Ψ is not Hurwitz to begin with, the unstable modes in ξ are never excited as far as establishing feedback stability is concerned. Thus, one can assume without loss of generality that A_Ψ is Hurwitz, as is done in Theorem III.3.

Note that the open-loop boundedness of $\mathbf{G}$ and $\mathbf{\Delta}$ is not assumed in Theorem III.3. Moreover, it has been shown in [15] that Theorem III.3 encompasses Theorem III.1 when the multiplier Π therein is restricted to be real-rational proper; see also [16, Theorem 30]. This means Theorem III.3 is more general than Theorem III.1 under the aforementioned real-rational restriction on Π and provides a tight relationship between dissipativity and soft IQC theory of the type described in Theorem III.1.

C. Hard IQCs

The following feedback stability result based on hard IQCs may be specialised from [3, Theorem III.1].

Theorem III.4. *Given causal (nonlinear) systems $\mathbf{\Delta} : L_{2e}^m \to L_{2e}^p$ and $\mathbf{G} : L_{2e}^p \to L_{2e}^m$, suppose $[\mathbf{\Delta}, \mathbf{G}]$ is well-posed and there exist linear bounded causal multipliers $\mathbf{\Theta} : L_{2e}^{m+p} \to L_{2e}^q$ and $\mathbf{\Pi} : L_{2e}^{m+p} \to L_{2e}^q$ such that*

$$\begin{aligned} &\langle \mathbf{\Theta} v, \mathbf{\Pi} v \rangle_T \geq 0 \;\forall v \in \mathscr{G}_e(\mathbf{\Delta}), T > 0 \\ \text{and} \quad &\langle \mathbf{\Theta} w, \mathbf{\Pi} w \rangle_T \leq -\epsilon \|w\|_T^2 \;\forall w \in \mathscr{G}'_e(\mathbf{G}), T > 0. \end{aligned}$$

Then $[\mathbf{\Delta}, \mathbf{G}]$ is stable.

An instance of the hard IQC theorem above may be found in [22], where the multipliers $\mathbf{\Theta}$ and $\mathbf{\Pi}$ take specific forms corresponding to 'mixed' small-gain and passivity properties. Notice that similar to the dissipativity based feedback stability result in Theorem III.3, the open-loop systems in the hard IQC based Theorem III.4 are not required to be bounded. Importantly, the next section shows that Theorem III.3 may be recovered from Theorem III.4, and hence the latter also subsumes the soft IQC based Theorem III.1. It is worth noting that the hard IQC based Theorem III.4 can be recovered by a more general soft IQC based result equipped with a homotopy that is continuous in the directed gap, as detailed in [3].

IV. Dissipativity and hard IQCs

The purpose of this section is to demonstrate that whenever the conditions in the dissipativity-based Theorem III.3 hold, the complementary hard IQC conditions in Theorem III.4 also hold, so that feedback stability may be concluded using the latter. In other words, the latter theorem is more general. First, a couple of lemmas are established.

Lemma IV.1. *Let $\mathbf{\Psi} = v \mapsto z$ be given in (3) and the LTI system $\mathbf{\Phi} = v \mapsto [z_1^\top, z_2^\top, z_3^\top]^\top = [z^\top, \xi^\top, \dot{\xi}^\top]^\top$ be given by*

$$\begin{aligned} \dot{\xi} &= A_\Psi \xi + B_\Psi v, \quad \xi(0) = 0 \\ z_1 &= C_\Psi \xi + D_\Psi v \\ z_2 &= \xi \\ z_3 &= A_\Psi \xi + B_\Psi v. \end{aligned} \tag{4}$$

Also, let $P = P^\top$, $Z = Z^\top$, and

$$Q = \begin{bmatrix} P & 0 & 0 \\ 0 & 0 & -Z \\ 0 & -Z & 0 \end{bmatrix}.$$

Then,

$$\langle \mathbf{\Psi} v, P \mathbf{\Psi} v \rangle_T - \xi(T)^\top Z \xi(T) = \langle \mathbf{\Phi} v, Q \mathbf{\Phi} v \rangle_T$$

for all $v \in L_{2e}$, $T > 0$.

Proof: Note that

$$\begin{aligned} &\langle \mathbf{\Psi} v, P \mathbf{\Psi} v \rangle_T - \xi(T)^\top Z \xi(T) \\ &= \langle \mathbf{\Psi} v, P \mathbf{\Psi} v \rangle_T - \int_0^T \frac{d}{dt}(\xi(t)^\top Z \xi(t))\, dt \\ &= \langle \mathbf{\Psi} v, P \mathbf{\Psi} v \rangle_T - 2\langle \xi, Z\dot{\xi} \rangle_T \\ &= \langle \mathbf{\Phi} v, Q \mathbf{\Phi} v \rangle_T. \end{aligned}$$

The claim thus follows. ■

Lemma IV.2. *Let the suppositions of Lemma IV.1 hold, $X = X^\top = \left[\begin{smallmatrix} X_1 & X_{12} \\ X_{12}^\top & X_2 \end{smallmatrix}\right]$, and $\mathbf{G}$ have the form (2) with $x(0) = 0$. Then, conditions (iii) and (iv) in Theorem III.3 imply that*

$$\langle \mathbf{\Phi} \left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right], Q \mathbf{\Phi} \left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right] \rangle_T \leq -\bar{\epsilon} \| \left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right] \|_T^2$$

for all $\left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right] \in \mathscr{G}'_e(\mathbf{G})$ and some $\bar{\epsilon} > 0$.

Proof: First note that condition (iii) with $x(0) = 0$ implies that

$$\begin{bmatrix} \xi(T) \\ x(T) \end{bmatrix}^\top X \begin{bmatrix} \xi(T) \\ x(T) \end{bmatrix} + \langle \mathbf{\Psi} \left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right], P \mathbf{\Psi} \left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right] \rangle_T + \epsilon(\|x\|_T^2 + \|u\|_T^2) \leq 0$$

for all $\left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right] \in \mathscr{G}'_e(\mathbf{G})$ and $T > 0$. Using condition (iv), this then implies that

$$-\xi(T)^\top Z \xi(T) + \langle \mathbf{\Psi} \left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right], P \mathbf{\Psi} \left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right] \rangle_T \leq -\epsilon(\|x\|_T^2 + \|u\|_T^2).$$

By Lemma IV.1, this is simply

$$\langle \mathbf{\Phi} \left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right], Q \mathbf{\Phi} \left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right] \rangle_T \leq -\epsilon(\|x\|_T^2 + \|u\|_T^2).$$

Since $y = Cx + Du$, it follows that there exists $\bar{\epsilon} > 0$ such that

$$\langle \mathbf{\Phi} \left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right], Q \mathbf{\Phi} \left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right] \rangle_T \leq -\bar{\epsilon} \| \left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right] \|_T^2,$$

as required. ■

The main result of this section is now in order.

Theorem IV.3. *If the suppositions and conditions in Theorem III.3 hold, then the suppositions and conditions in Theorem III.4 hold.*

Proof: This follows by combining Lemmas IV.1 and IV.2, and taking $\mathbf{\Theta} = \mathbf{\Phi}$, $\mathbf{\Pi} = Q\mathbf{\Phi}$ in Theorem III.4. ■

Theorem IV.3 shows that the dissipativity-based Theorem III.3 can be established via the hard IQC based Theorem III.4. In other words, the latter is more general than the former. It is noteworthy that an asymptotic stability version of Theorem III.3 may also be recovered from a more general result on dynamic dissipativity [23].

V. Feedback interconnection of two nonlinear systems

This section examines the feedback interconnection of two bounded nonlinear systems each satisfying a soft IQC that can be selected independently of the other and provides an approach to robust stability analysis using the standard soft IQC Theorem III.1 developed in [1]. The latter was originally established for a feedback interconnection of a nonlinear component and an LTI system, and hence is not directly applicable to the study of a feedback interconnection of two nonlinear systems.

Theorem V.1. *Let $\mathbf{\Delta}_1$ and $\mathbf{\Delta}_2$ be bounded causal (nonlinear) systems. Suppose there exist $\Pi_1, \Pi_2 \in \mathcal{B}$ such that*

(i) $[\tau\mathbf{\Delta}_1, \tau\mathbf{\Delta}_2]$ *is well-posed for all* $\tau \in [0, 1]$*;*
(ii) $\sigma(\Pi_i, v) \geq 0$ *for all* $v \in \mathscr{G}(\tau\mathbf{\Delta}_i)$*,* $\tau \in [0, 1]$*,* $i \in \{1, 2\}$*;*
(iii) *for some* $\alpha > 0$*,*

$$H^\top \Pi_1(\omega) H + \alpha\Pi_2(\omega) < 0 \quad \forall\omega \in [0, \infty],$$

where

$$H = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}.$$

Then $[\mathbf{\Delta}_1, \mathbf{\Delta}_2]$ *is stable.*

Proof: Let

$$\mathbf{\Delta} = \begin{bmatrix} \mathbf{\Delta}_1 & 0 \\ 0 & \mathbf{\Delta}_2 \end{bmatrix}.$$

The equivalence between well-posedness (resp. stability) of $[\tau\mathbf{\Delta}, H]$ and well-posedness (resp. stability) of $[\tau\mathbf{\Delta}_1, \tau\mathbf{\Delta}_2]$ is established via the following chain of equivalent reformulations:[1]

$[\tau\mathbf{\Delta}, H]$ is well-posed (resp. stable)

$\Leftrightarrow$ $[H \circ \tau\mathbf{\Delta}, I]$ is well-posed (resp. stable) {since H is linear and unimodular in $\mathcal{RH}_\infty$; in fact H is static and invertible; see [24, Lemma 3.1]}

$\Leftrightarrow$ the nonlinear operator $\begin{bmatrix} I & -I \\ -H \circ \tau\mathbf{\Delta} & I \end{bmatrix}$ has a casual inverse (resp. casual bounded inverse) on $\mathcal{L}_{2e}$ {via Definition II.1 and feedback equations (1)}

$\Leftrightarrow$ the nonlinear operator

$$\begin{bmatrix} 0 & I \\ I & -I \end{bmatrix} \circ \begin{bmatrix} (I - H \circ \tau\mathbf{\Delta}) & 0 \\ 0 & I \end{bmatrix} \circ \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix}$$

has a casual inverse (resp. casual bounded inverse) on $\mathcal{L}_{2e}$ {since

$$\begin{aligned}
&\begin{bmatrix} 0 & I \\ I & -I \end{bmatrix} \circ \begin{bmatrix} (I - H \circ \tau\mathbf{\Delta}) & 0 \\ 0 & I \end{bmatrix} \circ \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix} \left(\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \right) \\
&= \begin{bmatrix} 0 & I \\ I & -I \end{bmatrix} \circ \begin{bmatrix} (I - H \circ \tau\mathbf{\Delta}) & 0 \\ 0 & I \end{bmatrix} \left(\begin{bmatrix} f_1 \\ f_1 - f_2 \end{bmatrix} \right) \\
&= \begin{bmatrix} 0 & I \\ I & -I \end{bmatrix} \left(\begin{bmatrix} (I - H \circ \tau\mathbf{\Delta})(f_1) \\ f_1 - f_2 \end{bmatrix} \right) \\
&= \begin{bmatrix} 0 & I \\ I & -I \end{bmatrix} \left(\begin{bmatrix} f_1 - H \circ \tau\mathbf{\Delta}(f_1) \\ f_1 - f_2 \end{bmatrix} \right) \\
&= \begin{bmatrix} f_1 - f_2 \\ f_2 - H \circ \tau\mathbf{\Delta}(f_1) \end{bmatrix} \\
&= \begin{bmatrix} I & -I \\ -H \circ \tau\mathbf{\Delta} & I \end{bmatrix} \left(\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \right)
\end{aligned}$$

for any $f_1, f_2 \in L_{2e}$}

$\Leftrightarrow$ the nonlinear operator $(I - H \circ \tau\mathbf{\Delta})$ has a casual inverse (resp. casual bounded inverse) on $\mathcal{L}_{2e}$ {by inspection}

$\Leftrightarrow$ the nonlinear operator $\begin{bmatrix} I & -\tau\mathbf{\Delta}_2 \\ -\tau\mathbf{\Delta}_1 & I \end{bmatrix}$ has a casual inverse (resp. casual bounded inverse) on $\mathcal{L}_{2e}$ {simple rewriting}

$\Leftrightarrow$ $[\tau\mathbf{\Delta}_1, \tau\mathbf{\Delta}_2]$ is well-posed (resp. stable) {via Definition II.1 and feedback equations (1)}.

Therefore, it suffices to establish that $[\mathbf{\Delta}, H]$ is stable. Let Π_i be partitioned conformably as

$$\Pi_i = \begin{bmatrix} \Pi_{i(11)} & \Pi_{i(12)} \\ \Pi_{i(21)} & \Pi_{i(22)} \end{bmatrix}.$$

By hypothesis (ii), it holds that

$$\sigma(\Pi_\alpha, v) \geq 0 \quad \forall v \in \mathscr{G}(\tau\mathbf{\Delta}), \tau \in [0, 1], \alpha > 0,$$

where

$$\Pi_\alpha = \left[\begin{array}{cc|cc} \Pi_{1(11)} & 0 & \Pi_{1(12)} & 0 \\ 0 & \alpha\Pi_{2(11)} & 0 & \alpha\Pi_{2(12)} \\ \hline \Pi_{1(21)} & 0 & \Pi_{1(22)} & 0 \\ 0 & \alpha\Pi_{2(21)} & 0 & \alpha\Pi_{2(22)} \end{array}\right].$$

Therefore, by hypothesis (i), Theorem III.1, and Remark III.2, $[\mathbf{\Delta}, H]$ is stable if there exists $\alpha > 0$ such that

$$\begin{bmatrix} H \\ I \end{bmatrix}^* \Pi_\alpha(\omega) \begin{bmatrix} H \\ I \end{bmatrix} < 0 \quad \forall\omega \in [0, \infty].$$

This is simply hypothesis (iii). ■

Remark V.2. Note that in Theorem V.1, if $\Pi_{i(11)}(\omega) \geq 0$ and $\Pi_{i(22)}(\omega) \leq 0$ for all $\omega \geq 0$, then $\sigma(\Pi_i, v) \geq 0$ for all $v \in \mathscr{G}(\mathbf{\Delta}_i)$ if and only if $\sigma(\Pi_i, v) \geq 0$ for all $v \in \mathscr{G}(\tau\mathbf{\Delta}_i)$, $\tau \in [0, 1]$.

It can be seen that while condition (ii) in Theorem V.1 allows for the IQCs for the open-loop systems to be selected independently, the frequency-dependent multipliers that define the IQCs are coupled in a specific fashion in condition (iii). In the case where $\Pi_i(\omega)$ is expressed as

$$\Pi_i(\omega) = \begin{bmatrix} Q_i(\omega) & S_i(\omega) \\ S_i(\omega)^* & R_i(\omega) \end{bmatrix},$$

Theorem V.1 can be seen as a frequency-dependent $(Q(\omega), S(\omega), R(\omega))$-dissipativity result because conditions (ii) and (iii) in Theorem V.1 are similar (though not identical) to the corresponding conditions in the classical (Q, S, R)-dissipativity result with static Q, S and R matrices (cf. Corollary V.5). Theorem V.1 may be used to characterise a myriad of interesting open-loop properties, such as "mixed" small-gain and passivity in similar spirits to [25].

In what follows, we specialise Theorem V.1 to three classical results.

A. Small gain

First, we show how Theorem V.1 specializes to the classical small-gain theorem [26] for the feedback interconnection of two nonlinear systems.

[1] We do not distinguish between an operator H and a matrix H as the relevant object is clear from the context.

Corollary V.3. *Let $\mathbf{\Delta}_1$ and $\mathbf{\Delta}_2$ be bounded causal (nonlinear) systems. Suppose $[\tau\mathbf{\Delta}_1, \tau\mathbf{\Delta}_2]$ is well-posed for all $\tau \in [0,1]$. For $i \in \{1,2\}$, let $\mathbf{\Delta}_i$ have bound γ_i, i.e. $\sigma(\Pi_i, v) \geq 0$ for all $v \in \mathscr{G}(\mathbf{\Delta}_i)$, where*

$$\Pi_i = \begin{bmatrix} \gamma_i^2 I & 0 \\ 0 & -I \end{bmatrix}.$$

Then $[\mathbf{\Delta}_1, \mathbf{\Delta}_2]$ is stable if $\gamma_1\gamma_2 < 1$.

Proof: Apply Theorem V.1 and Remark V.2 with $\alpha = \gamma_1^2 + \epsilon$ and $\epsilon > 0$ being sufficiently small. ■

B. Passivity indices

Next, we show how Theorem V.1 specializes to the classical passivity theorem [18] for the negative feedback interconnection of two nonlinear systems.

Corollary V.4. *Let $\mathbf{\Delta}_1$ and $\mathbf{\Delta}_2$ be bounded causal (nonlinear) systems. Suppose $[\tau\mathbf{\Delta}_1, -\tau\mathbf{\Delta}_2]$ is well-posed for all $\tau \in [0,1]$. For $i \in \{1,2\}$, let $\mathbf{\Delta}_i$ have input-feedforward passivity index $\delta_i \leq 0$ and output-feedback passivity index $\epsilon_i \geq 0$, i.e. $\sigma(\Pi_i, v) \geq 0$ for all $v \in \mathscr{G}(\mathbf{\Delta}_i)$, where*

$$\Pi_i = \begin{bmatrix} -\delta_i I & I \\ I & -\epsilon_i I \end{bmatrix}.$$

Then, $[\mathbf{\Delta}_1, -\mathbf{\Delta}_2]$ is stable if $\delta_1 + \epsilon_2 > 0$ and $\delta_2 + \epsilon_1 > 0$.

Proof: Define $\tilde{\mathbf{\Delta}}_2 = -\mathbf{\Delta}_2$ and apply Theorem V.1 and Remark V.2 with $\alpha = 1$ to $[\mathbf{\Delta}_1, \tilde{\mathbf{\Delta}}_2]$. ■

C. (Q, S, R)-dissipativity

Lastly, we show how Theorem V.1 specializes to the classical (Q, S, R)-dissipativity theorem with static Q, S, and R matrices [19] for the positive feedback interconnection of two nonlinear systems.

Corollary V.5. *Let $\mathbf{\Delta}_1$ and $\mathbf{\Delta}_2$ be bounded causal (nonlinear) systems. Suppose $[\tau\mathbf{\Delta}_1, \tau\mathbf{\Delta}_2]$ is well-posed for all $\tau \in [0,1]$. For $i \in \{1,2\}$, let $Q_i \geq 0$, $R_i \leq 0$ and $\mathbf{\Delta}_i$ be ultimately dissipative with respect to (Q_i, S_i, R_i), i.e. $\sigma(\Pi_i, v) \geq 0$ for all $v \in \mathscr{G}(\mathbf{\Delta}_i)$, where*

$$\Pi_i = \begin{bmatrix} Q_i & S_i \\ S_i^\top & R_i \end{bmatrix}.$$

Then $[\mathbf{\Delta}_1, \mathbf{\Delta}_2]$ is stable if there exists $\alpha > 0$ such that

$$\begin{bmatrix} R_1 + \alpha Q_2 & S_1^\top + \alpha S_2 \\ S_1 + \alpha S_2^\top & Q_1 + \alpha R_2 \end{bmatrix} < 0.$$

Proof: Straightforward application of Theorem V.1 and Remark V.2. ■

VI. Numerical example

In this section, we provide a numerical example of two uncertain nonlinear systems in a positive feedback interconnection to illustrate Theorem V.1.

Let Δ_1 operate on a scalar input signal u to produce a scalar output signal y, i.e. $y = \Delta_1 u$, according to the nonlinear state-space representation

$$\Delta_1 = \begin{cases} \dot{x}_1 = u - a x_1 - \sum_{i=-N}^{M} b_i x_1^{2i+1}, \\ \dot{x}_2 = u - x_2, \\ \ y = x_1 - x_2, \end{cases}$$

where x_1 and x_2 are states, $x_1(0) = x_2(0) = 0$, $a \geq 1$, $b_i \geq 0 \ \forall i \in \{-N, ..., M\}$ and $N, M \in \mathbb{Z}_{\geq 0}$ (i.e. non-negative integers). It is easy to see that Δ_1 is causal, bounded and satisfies condition (ii) of Theorem V.1 with

$$\Pi_1(\omega) = \begin{bmatrix} \frac{1}{1+\omega^2} & \frac{-j\omega}{1-j\omega} \\ \frac{j\omega}{1+j\omega} & -1 \end{bmatrix}$$

by invoking Remark V.2. It is also clear that Δ_1 is an uncertain nonlinear system since its parameters a and b_i are allowed to take a range of values.

Now, let Δ_2 operate on scalar signals according to the nonlinear map $(\Delta_2 v)(t) = \delta(v(t))$, where δ is an odd function on $\mathbb{R}$ such that $\frac{d\delta}{dx}(x)$ lies in the sector $[0, k]$ for some constant $k > 0$. It is clear that Δ_2 is a static, monotonic, and odd nonlinearity (hence also causal and bounded). Therefore, by Section VI.K in [1], Δ_2 satisfies condition (ii) of Theorem V.1 with

$$\Pi_2(\omega) = \begin{bmatrix} 0 & 1 + Z(j\omega) \\ 1 + Z(-j\omega) & -[2 + Z(j\omega) + Z(-j\omega)]/k \end{bmatrix},$$

where $Z \in \mathcal{B}$ is arbitrary except that $\int_{-\infty}^{\infty} |z(t)| dt \leq 1$ with $z(t)$ being the impulse response of Z. This is known as a Zames-Falb multiplier Z [17].

The well-posedness condition (i) of Theorem V.1 is trivially fulfilled since Δ_1 does not have a direct feedthrough term. What remains to be verified is whether we can find a Z that satisfies condition (iii) of Theorem V.1 for some $k > 0$.

Choosing $Z(s) = \frac{(\frac{1}{\alpha})}{(s+1)}$ with $\alpha \geq 1$ gives

$$H^\top \Pi_1(\omega) H + \alpha \Pi_2(\omega) = \begin{bmatrix} -1 & 1 + \alpha \\ 1 + \alpha & (1 - \frac{2}{k}) \frac{1}{1+\omega^2} - \frac{2\alpha}{k} \end{bmatrix}.$$

By selecting $\alpha = 1$ and any $k < \frac{1}{2}$, condition (iii) of Theorem V.1 is satisfied and hence the positive feedback interconnection of the two uncertain nonlinear systems Δ_1 and Δ_2 is stable via Theorem V.1.

VII. Conclusions

Within the context of input-output feedback stability analysis, we completed the picture involving integral quadratic constraint (IQC) and dissipativity with respect to quadratic supply rate approaches by showing that hard IQC theory generalises the dissipativity-with-terminal-cost approach, which in turn encompasses the classical soft IQC theorem from [1] as shown in [15], [16]. Hard IQC theory is itself subsumed by a more general soft IQC theory equipped with gap-continuous homotopies, as established in [3].

We also demonstrated how the classical soft IQC theorem from [1], originally developed for a feedback interconnection of a nonlinear component and an LTI system, can be recast to investigate the robust stability of a feedback interconnection of two nonlinear systems. This directly yields a frequency-dependent $(Q(\omega), S(\omega), R(\omega))$-dissipativity analysis result which was then shown to reduce to small-gain, passivity, and static (Q, S, R)-dissipativity results.

Interesting future research directions include studying connections between dissipativity, IQC, and Lyapunov theories for input-state-output systems and incremental-type stability results.

References

[1] A. Megretski and A. Rantzer, "System analysis via integral quadratic constraints," *IEEE Trans. Autom. Contr.*, vol. 42, no. 6, pp. 819–830, 1997.

[2] A. Megretski, U. Jönsson, C.-Y. Kao, and A. Rantzer, "The control handbook," W. S. Levine, Ed. Springer-Verlag, 2011, ch. Integral quadratic constraints.

[3] S. Z. Khong, "On integral quadratic constraints," *IEEE Trans. Autom. Contr.*, vol. 67, no. 3, pp. 1603–1608, 2022.

[4] T. T. Georgiou and M. C. Smith, "Robustness analysis of nonlinear feedback systems: An input output approach," *IEEE Trans. Autom. Contr.*, vol. 42, pp. 1200–1221, 1997.

[5] A. Rantzer and A. Megretski, "System analysis via integral quadratic constraints: Part II," Lund Institute of Technology, Tech. Rep. LUTFD2/TFRT–7559-SE, 1997.

[6] M. Cantoni, U. T. Jönsson, and C.-Y. Kao, "Robustness analysis for feedback interconnections of distributed systems via integral quadratic constraints," *IEEE Trans. Autom. Contr.*, vol. 57, no. 2, pp. 302–317, 2012.

[7] M. Cantoni, U. T. Jönsson, and S. Z. Khong, "Robust stability analysis for feedback interconnections of time-varying linear systems," *SIAM J. Control Optim.*, vol. 51, no. 1, pp. 353–379, 2013.

[8] S. Z. Khong and M. Cantoni, "Reconciling ν-gap metric and IQC based robust stability analysis," *IEEE Trans. Autom. Contr.*, vol. 58, no. 8, pp. 2090–2095, 2013.

[9] S. Z. Khong, E. Lovisari, and A. Rantzer, "A unifying framework for robust synchronisation of heterogeneous networks via integral quadratic constraints," *IEEE Trans. Autom. Contr.*, vol. 61, no. 5, pp. 1297–1309, 2016.

[10] S. Z. Khong, E. Lovisari, and C.-Y. Kao, "Robust synchronization in multi-agent networks with unstable dynamics," *IEEE Transactions on Control of Network Systems*, vol. 5, no. 1, pp. 205–214, 2018.

[11] M. S. Akram and M. Cantoni, "Gap metrics for linear time-varying systems," *SIAM J. Control Optim.*, vol. 56, no. 2, pp. 782–800, 2018.

[12] J. Veenman and C. W. Scherer, "Stability analysis with integral quadratic constraints: A dissipativity based proof," in *Proc. 52nd IEEE Conf. Decision Control*, 2013, pp. 3770–3775.

[13] P. Seiler, "Stability analysis with dissipation inequalities and integral quadratic constraints," *IEEE Trans. Autom. Contr.*, vol. 60, no. 6, pp. 1704–1709, 2015.

[14] J. Carrasco and P. Seiler, "Conditions for the equivalence between IQC and graph separation stability results," *International Journal of Control*, vol. 92, no. 12, pp. 2899–2906, 2019.

[15] C. W. Scherer and J. Veenman, "Stability analysis by dynamic dissipation inequalities: On merging frequency-domain techniques with time-domain conditions," *Systems & Control Letters*, vol. 121, pp. 7–15, 2018.

[16] C. W. Scherer, "Dissipativity and integral quadratic constraints: Tailored computational robustness tests for complex interconnections," *IEEE Control Systems Magazine*, vol. 42, no. 3, pp. 115–139, 2022.

[17] G. Zames and P. L. Falb, "Stability conditions for system with monotone and slope-restricted nonlinearites," *SIAM Journal of Control*, vol. 6, no. 1, pp. 89–108, 1968.

[18] C. A. Desoer and M. Vidyasagar, *Feedback Systems: Input-Output Properties.* New York: Academic Press, 1975.

[19] D. Hill and P. Moylan, "Dissipative dynamical systems: Basic input-output and state properties," *Journal of the Franklin Institute*, vol. 309, no. 5, pp. 327–357, 1980.

[20] J. C. Willems, *The Analysis of Feedback Systems*. Cambridge, Massachusetts: MIT Press, 1971.

[21] U. Jönsson, "Lecture notes on integral quadratic constraints," Department of Mathematics, Royal Instutue of Technology (KTH), Stockholm, Sweden, 2001.

[22] W. M. Griggs, B. D. O. Anderson, A. Lanzon, and M. Rotkowitz, "Interconnections of nonlinear systems with "mixed" small gain and passivity properties and associated input-output stability results," *Systems & Control Letters*, vol. 58, no. 4, pp. 289–295, 2009.

[23] S. Z. Khong, C. Chen, and A. Lanzon, "Feedback stability analysis via dissipativity with dynamic supply rates," *Automatica,* under review, 2024.

[24] M. Green and D. J. N. Limebeer, *Linear Robust Control*, ser. Information and System Sciences. Prentice-Hall, 1995.

[25] W. M. Griggs, B. D. O. Anderson, and A. Lanzon, "A "mixed" small gain and passivity theorem in the frequency domain," *Systems & Control Letters*, vol. 56, no. 9–10, pp. 596–602, 2007.

[26] G. Zames, "On the input-output stability of nonlinear time-varying feedback systems – part I: Conditions derived using concepts of loop gain, conicity, and positivity; and part II: Conditions involving circles in the frequency plane and sector nonlinearities," *IEEE Trans. Autom. Contr.*, vol. 11, pp. 228–238; 465–476, 1966.