A Direct Proof of the Equivalence of Side Conditions for Strictly Positive Real Matrix Transfer Functions

Augusto Ferrante, Alexander Lanzon, and Bernard Brogliato

***Abstract*—This brief note proves in a direct way that two different side conditions, which have been used in the literature to characterize strictly positive real matrix transfer functions in the frequency domain, are equivalent.**

***Index Terms*—Control systems analysis, high frequency condition, input-output methods, multivariable linear systems, passivity, strictly positive real systems.**

I. Introduction

The frequency domain conditions characterizing the fact that a matrix transfer function F is strictly positive real involve a positivity constraint at infinite frequency. This constraint—usually referred to as *side condition*—has been a source of confusion and controversy in the literature for more than a decade. As pointed out in [3], the side conditions used in [7]–[9] were incorrect as they had some inconsistencies. To fix the problem, Corless and Shorten [3] proposed a new condition at infinite frequency, i.e.,

$$\lim_{\omega\to\infty} \omega^{2\rho} \det\left(F(j\omega) + F(-j\omega)^\top\right) > 0 \tag{1}$$

where ρ is the dimension of $\ker\left(F(\infty) + F(\infty)^\top\right)$.

On the other hand, a different, but equally valid, condition at infinite frequency was proposed in the second edition of the book by Khalil published in 1996 (see [6, Lemma 10.1]); such a condition, that reads as follows, was recently used in [4] (see [5] for the discrete-time case) to establish a counterpart result for negative imaginary systems (see [4, Remark 1]): $\exists\, \delta > 0, \sigma_0 > 0$ such that

$$\underline{\sigma}\left[\omega^2\left(F(j\omega) + F(-j\omega)^\top\right)\right] \geq \sigma_0 \quad \forall|\omega| \geq \delta. \tag{2}$$

This note is devoted to the analysis of the two side-conditions (1) and (2). We will prove that while they are in general not equivalent at infinite frequency, they are indeed equivalent under the other conditions guaranteeing that F is strictly positive real. Hence, both conditions at infinite frequency are equally valid. While this could be deduced from [3] and [6], our results provide a direct proof of such equivalency.

Notation: Let the set of real (resp. complex) numbers be denoted by $\mathbb{R}$ (resp. $\mathbb{C}$) and the corresponding sets of matrices of dimension $m \times n$ be denoted by $\mathbb{R}^{m\times n}$ (resp. $\mathbb{C}^{m\times n}$). Given $M \in \mathbb{C}^{m\times m}$, $M^\star$ denotes the complex conjugate transpose of matrix M (i.e., if $M = A + jB$ for real matrices A and B, then $M^\star = A^\top - jB^\top$). A matrix M is said to be Hermitian if $M = M^\star$ and $M > 0$ denotes that the matrix M is Hermitian and positive definite. The smallest singular value of M is denoted by $\underline{\sigma}(M)$. We recall that the singular values of a positive semidefinite Hermitian matrix are its nonzero eigenvalues [2, p. 649].

II. Main Result

The following definition, adapted from [4, Definitions 1 and 2], is the standard definition for strictly positive real systems. It essentially states that a transfer function matrix $F(s)$ is strictly positive real if for some $\epsilon > 0$, the transfer function matix $F(s-\epsilon)$ is positive real and $F(s) + F(-s)^\top$ has full normal rank. See also [4, Lemma 2] for an equivalent recharacterization.

Definition 2.1: Let $F : \mathbb{C} \longrightarrow \mathbb{C}^{m\times m}$ be a real transfer function. Then, $F(s)$ is said to be strictly positive real (SPR) if there exists a real scalar $\epsilon > 0$ such that $F(s)$ is analytic in $\{s \in \mathbb{C} : \mathfrak{Re}\{s\} > -\epsilon\}$, $F(s) + F(s)^\star \geq 0$ for all $s \in \{s \in \mathbb{C} : \mathfrak{Re}\{s\} > -\epsilon\}$, and $F(s) + F(-s)^\top$ has full normal rank.

SPR matrix transfer functions can be characterized in the frequency domain by three conditions: the first two conditions are 1 and 2 in the next proposition, the third is the *side condition* and it has been stated in two different manners: side condition (3a) in Proposition 2.1 can be found in [4] and [6], while side condition (3b) can be found in [3]. These side conditions can be interpreted as apparently different conditions on how $\left(F(j\omega) + F(-j\omega)^\top\right)$ approaches zero for sufficiently large $|\omega|$ in directions where it looses rank.

Proposition 2.1: Let $F : \mathbb{C} \longrightarrow \mathbb{C}^{m\times m}$ be a real, rational, proper transfer function such that the following two conditions hold:

1) $F(s)$ has no poles in $\{s \in \mathbb{C} : \mathfrak{Re}\{s\} \geq 0\}$;
2) $F(j\omega) + F(-j\omega)^\top > 0$ for all $\omega \in \mathbb{R}$.

Then, the following two side conditions are equivalent:

3a) $\exists \delta > 0, \sigma_0 > 0:$

$$\underline{\sigma}\left[\omega^2\left(F(j\omega) + F(-j\omega)^\top\right)\right] \geq \sigma_0 \quad \forall|\omega| \geq \delta \tag{3}$$

3b) $$\lim_{\omega\to\infty} \omega^{2\rho} \det[F(j\omega) + F(-j\omega)^\top] \neq 0 \tag{4}$$

where $\rho = \dim\ker(F(\infty) + F(\infty)^\top)$.

Proof: Since $F(s)$ is proper, let $F(s) = C(sI - A)^{-1}B + D$ for some state-space realization (A, B, C, D). Near $s = \infty$, the expansion

$$F(s) = F_0 + F_1/s + F_2/s^2 + \cdots \tag{5}$$

Manuscript received July 17, 2018; revised December 6, 2018; accepted May 10, 2019. Date of publication June 21, 2019; date of current version December 27, 2019. This work was supported by the Engineering and Physical Sciences Research Council under Grant EP/R008876/1. All research data supporting this publication are directly available within this publication. Recommended by Associate Editor L. Menini. *(Corresponding author: Alexander Lanzon.)*

A. Ferrante is with Department of Information Engineering, the University of Padova, 35131 Padova, Italy (e-mail: augusto@dei.unipd.it).

A. Lanzon is with the Control Systems Centre, School of Electrical and Electronic Engineering, University of Manchester, Manchester M13 9PL, U.K. (e-mail: Alexander.Lanzon@manchester.ac.uk).

B. Brogliato is with University Grenoble Alpes, Inria, CNRS, Grenoble INP, LJK, 38000 Grenoble, France (e-mail: bernard.brogliato@inria.fr).

Digital Object Identifier 10.1109/TAC.2019.2918123

holds (with $F_0 = F(\infty) = D$), so that

$$F(j\omega) + F(-j\omega)^\top = Q + \frac{H}{\omega} + \frac{K(\omega)}{\omega^2} \tag{6}$$

with $Q = D + D^\top$, $H = (F_1 - F_1^\top)/j$, and $K(\omega) = \sum_{i=1}^{\infty} (\frac{F_{i+1}}{j^{i+1}\omega^{i-1}} + \frac{F_{i+1}^T}{(-j)^{i+1}\omega^{i-1}})$. Then, it is easy to see that

$$\lim_{\omega\to\infty} K(\omega) = K_0 \in \mathbb{R}^{m\times m}$$

with $K_0 = -F_2 - F_2^T$. If Q is positive definite, in view of the Hermitian symmetry of both H and $K(\omega)$, both conditions 3a) and 3b) are clearly satisfied and the result is obvious.

Assume than that Q is singular with rank $m - \rho$. Since Q is symmetric, there exists an orthogonal matrix U such that $UQU^\top = [\begin{smallmatrix} Q_1 & 0 \\ 0 & 0 \end{smallmatrix}]$ with $Q_1 \in \mathbb{R}^{(m-\rho)\times(m-\rho)}$ nonsingular. Hence, without changing the essence of the problem, we assume that Q has the form

$$Q = \begin{bmatrix} Q_1 & 0 \\ 0 & 0 \end{bmatrix} \tag{7}$$

with $Q_1 \in \mathbb{R}^{(m-\rho)\times(m-\rho)}$ nonsingular, and hence $Q_1 > 0$. We partition H and $K(\omega)$ accordingly, as follows:

$$H = \begin{bmatrix} H_1 & H_{12} \\ H_{21} & H_2 \end{bmatrix}, \qquad K(\omega) = \begin{bmatrix} K_1(\omega) & K_{12}(\omega) \\ K_{21}(\omega) & K_2(\omega) \end{bmatrix}. \tag{8}$$

Thus,

$$[F(j\omega) + F(-j\omega)^\top]_{22} = \frac{1}{\omega}\left(H_2 + \frac{K_2(\omega)}{\omega}\right). \tag{9}$$

Since $H = (F_1 - F_1^\top)/j$, we have that: (i) H is Hermitian, so that H_2 is Hermitian as well, and (ii) the elements on the diagonal of H are zero, so that also the elements on the diagonal of H_2 are zero and hence H_2 is traceless. Therefore, H_2 has only real eigenvalues and the sum of the eigenvalues of H_2 is zero. Hence, either $H_2 = 0$ or H_2 has at least a negative eigenvalue. But for ω sufficiently large, and by continuity of the eigenvalues as functions of ω, the eigenvalues of $H_2 + \frac{K_2(\omega)}{\omega}$ are arbitrarily close to those of H_2, so that if $H_2 \neq 0$, then $[F(j\omega) + F(-j\omega)^\top]_{22}$ has a negative eigenvalue for a sufficiently large ω, and this is against our assumptions because $[F(j\omega) + F(-j\omega)^\top] > 0$ for all $\omega \in \mathbb{R}$ so, in turn, $[F(j\omega) + F(-j\omega)^\top]_{22} > 0$ for all $\omega \in \mathbb{R}$. In conclusion, $H_2 = 0$.

Now let

$$\Phi(\omega) = \begin{bmatrix} \Phi_1(\omega) & \Phi_{12}(\omega) \\ \Phi_{21}(\omega) & \Phi_2(\omega) \end{bmatrix} = F(j\omega) + F(-j\omega)^\top.$$

By continuity, as $\omega \to \infty$, $m - \rho$ of the eigenvalues of Φ, i.e., the eigenvalues of $\Phi_1(j\omega)$, tend to the eigenvalues of Q_1 (that are strictly positive) and the remaining ρ eigenvalues tend to zero. Let $\lambda(\omega)$ be one of the eigenvalues of Φ that tends to zero as $\omega \to \infty$. We now show that $\lambda(\omega)$ tends to zero at least as fast as $1/\omega^2$. In fact, provided that ω is large enough so that $\lambda(\omega)$ is not an eigenvalue of $\Phi_1(\omega)$, then $\lambda(\omega)$ must be an eigenvalue of

$$R(\omega) = \Phi_2(\omega) - \Phi_{21}(\omega)[\Phi_1(\omega) - \lambda(\omega)I]^{-1}\Phi_{12}(\omega) \tag{10}$$

because through Schur complements (e.g., [1])

$$\det[\Phi(\omega) - \lambda(\omega)I] = \det[\Phi_1(\omega) - \lambda(\omega)I]\det[R(\omega) - \lambda(\omega)I].$$

Using the fact that $\Phi_1(\omega) - \lambda(\omega)I = Q_1 + \frac{H_1}{\omega} + \frac{K_1(\omega)}{\omega^2} - \lambda(\omega)I$, and using [1, Fact 9.9.43], $R(\omega)$ in (10) can be written equivalently as

$$R(\omega) = \Phi_2(\omega) - \Phi_{21}(\omega)[Q_1^{-1} + \Delta(\omega)]\Phi_{12}(\omega)$$

where $\Delta(\omega) \to 0$ as $\omega \to \infty$, since $\lambda(\omega) \to 0$ as $\omega \to \infty$. Finally, using (6)–(8), we have $R(\omega) = \frac{1}{\omega^2}[K_2(\omega) - P(\omega)]$ where

$$P(\omega) = \left(H_{21} + \frac{K_{21}(\omega)}{\omega}\right)[Q_1^{-1} + \Delta(\omega)]\left(H_{12} + \frac{K_{12}(\omega)}{\omega}\right).$$

Since for ω sufficiently large $[K_2(\omega) - P(\omega)]$ is bounded, then the eigenvalues of $R(\omega)$ tend to zero at least as fast as $1/\omega^2$. In fact, any eigenvalue $r(\omega)$ of $R(\omega)$ has the form $r(\omega) = \frac{1}{\omega^2}r_0(\omega)$, where $r_0(\omega)$ tends to one of the eigenvalues of the finite matrix $K_2(\infty) - H_{21}Q_1^{-1}H_{12}$.

Then, $\Phi(\omega)$ has m strictly positive eigenvalues for all $\omega \in \mathbb{R}$, ρ of which are going down to zero at least as fast as $1/\omega^2$ as $\omega \to \infty$. The remaining $(m - \rho)$ eigenvalues tend to the strictly positive eigenvalues of Q_1 as $\omega \to \infty$. Now, order the eigenvalues of $\Phi(\omega)$ in nondecreasing size.

Since $\det(\cdot)$ is the product of eigenvalues, then $\omega^{2\rho}\det[\Phi(\omega)]$ is the same as the product of $(\omega^2\lambda_1), \ldots, (\omega^2\lambda_\rho), \lambda_{\rho+1}, \ldots, \lambda_m$. Therefore, by taking into account that for each $i \in \{1, \ldots, \rho\}$, $\lambda_i(\omega)$ tends to zero at least as fast as $1/\omega^2$ and that $\lambda_{\rho+1}(\omega), \ldots, \lambda_m(\omega)$ tend to finite positive values, we have that the side condition (3b)

$$\lim_{\omega\to\infty} \omega^{2\rho}\det[\Phi(\omega)] \neq 0$$

is equivalent to

$$\lim_{\omega\to\infty} (\omega^2\lambda_i) \neq 0 \text{ for all } i \in \{1, \ldots, \rho\}$$

(where the limits exist), which is equivalent to

$$\lim_{\omega\to\infty} \omega^2\lambda_1 > 0$$

which, finally, is equivalent to the first side condition (3a). The proposition is proved. ■

Examples: Consider the case of $F(s) = \frac{1}{s+1}[\begin{smallmatrix} 1 & 0 \\ 0 & s+2 \end{smallmatrix}]$. In this case $\rho = 1$ and, by direct computation, we see that

$$\lim_{\omega\to\infty} \omega^{2\rho}\det[F(j\omega) + F(-j\omega)^\top] = \lim_{\omega\to\infty} \frac{8\omega^2 + 4\omega^4}{(1+\omega^2)^2} = 4 \neq 0.$$

On the other hand, we have

$$\underline{\sigma}\left[\omega^2\left(F(j\omega) + F(-j\omega)^\top\right)\right] = \frac{2\omega^2}{1+\omega^2}$$

which is clearly greater than 1 for all $|\omega| > 1$.

The pathological case, corresponding to the situation in which some of the eigenvalues of the spectrum go to zero faster than $\frac{1}{\omega^2}$, as ω tends to infinity, is more interesting: let $F(s) = [\begin{smallmatrix} \frac{s+2}{(s+1)^2} & 0 \\ 0 & \frac{s+2}{(s+1)} \end{smallmatrix}]$. In this case $\rho = 1$ and, by direct computation, we easily see that

$$\lim_{\omega\to\infty} \omega^{2\rho}\det[F(j\omega) + F(-j\omega)^\top] = \lim_{\omega\to\infty} \frac{\omega^2(16 + 8\omega^2)}{(1+\omega^2)^3} = 0.$$

The same conclusion is obtained with the other side condition which, in this case, reads

$$\underline{\sigma}\left[\omega^2\left(F(j\omega) + F(-j\omega)^\top\right)\right] = \frac{4\omega^2}{(1+\omega^2)^2}$$

which is clearly not bounded away from zero as $|\omega|$ diverges.

Finally, the trivial example $F(s) = [\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}]$ immediately shows that, if we do not assume the first two conditions of Proposition 2.1, the two side conditions are not necessarily equivalent.

III. Conclusion

This note shows that two different side conditions used in the control literature to characterize strictly positive real matrix transfer functions are equivalent.

References

[1] D. S. Bernstein, *Matrix Mathematics. Theory, Facts, and Formulas With Application to Linear Systems Theory*. Princeton, NJ, USA: Princeton Univ. Press, 2005.

[2] S. Boyd and L. Vandenberghe, *Convex Optimization*. Cambridge, U.K.: Cambridge Univ. Press, 2004. 14th printing 2013.

[3] M. Corless and R. Shorten, "On the characterization of strict positive realness for general matrix transfer functions," *IEEE Trans. Autom. Control*, vol. 55, no. 8, pp. 1899–1904, Aug. 2010.

[4] A. Ferrante, A. Lanzon, and L. Ntogramatzidis, "Foundations of not necessarily rational negative imaginary systems theory: Relations between classes of negative imaginary and positive real systems," *IEEE Trans. Autom. Control*, vol. 61, no. 10, pp. 3052–3057, Oct. 2016.

[5] A. Ferrante, A. Lanzon, and L. Ntogramatzidis, "Discrete-time negative imaginary systems," *Automatica*, vol. 79, pp. 1–10, 2017.

[6] H. K. Khalil, *Nonlinear Systems*. 2nd ed. Englewood Cliffs, NJ, USA: Prentice-Hall, 1996.

[7] H. K. Khalil, *Nonlinear Systems*. 3rd ed. Englewood Cliffs, NJ, USA: Prentice-Hall, 2002.

[8] G. Tao and P. A. Ioannou, "Strictly positive real matrices and the Lefschetz-Kalman-Yakubovich lemma," *IEEE Trans. Autom. Control*, vol. 33, no. 12, pp. 1183–1185, Dec. 1988.

[9] G. Tao and P. A. Ioannou, "Necessary and sufficient conditions for strictly positive real matrices," *IEE Proc. G-Circuits, Devices Syst.*, vol. 137, no. 5, pp. 360–366, Oct. 1990.