Foundations of a Bicoprime Factorization Theory

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***Abstract*—Bicoprime factorizations (BCFs) are a generalization of the well known coprime factorizations commonly used in control theory. However they have received negligible attention from the academic community so far. This technical note lays the foundations of a BCF theory. The theory is built from the ground up, starting with the basic characteristics of such factorizations before moving on to state space parameterizations of BCFs and internal stability. Some advantages of BCFs are outlined including the possibility of reduced dimension internal stability tests. An uncertainty structure induced by BCFs is also examined and the associated robust stability analysis tests provided. In multiple instances it is shown how coprime factor results have their roots in the more abstract, and more general, BCFs.**

***Index Terms*—Bicoprime factorizations, coprime factorizations, feedback systems, internal stability, robust stability, stability margin.**

I. Introduction

Coprimeness is a useful property widely exploited in many areas of control theory. Left coprime factorizations (LCFs) and right coprime factorizations (RCFs) find extensive use in various fields of robust control such as $\mathscr{H}_\infty$ loop-shaping [1] and distance measures [2], [3]. The notion of matrix coprimeness is a generalization on that of integers having a greatest common denominator of 1. The polynomial case of this problem was studied by Bézout who showed that two polynomials a and b have greatest common divisor d if there exist polynomials x and y such that the linear Diophantine equation $ax + by = d$ is satisfied. Such an equation is now commonly referred to as Bézout's identity, a version of which is used as a coprimeness test for polynomial matrices.

A coprime factorization is one where a rational object is decomposed into two factors that satisfy the coprimeness condition over some set, usually $\mathscr{RH}_\infty$. One of the most important features of coprime factorizations is the fact that every object in $\mathscr{R}$ admits a coprime factorization over $\mathscr{RH}_\infty$. Hence, any coprime factor results can be directly applied to a wide class of systems.

Bicoprime factorizations (BCFs) are a generalization of the aforementioned coprime factorizations. They were briefly introduced in [4] with only a handful of results given. Two motivating points given therein for the study of BCFs are that they naturally arise in closed loop transfer matrices and the simple fact that a minimal state space representation of a plant is itself a BCF over the ring of polynomials. In fact, such factorizations do arise in many areas of interest such as J-spectral factorizations [5] and chain scattering theory [6].

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The relation between a special set of BCFs (where one of the factors is assumed to be zero—though this condition is lifted for some of the results) and classical coprime factorizations was studied in [7]. A set of simple preliminary results were derived including internal stability for the feedback interconnection of a plant given as a BCF and controller expressed as a RCF or LCF. Those results were extended in [8] and given a decentralized control context.

It has also been shown that BCFs can be useful in the study of decentralized or distributed control problems. For example, in [9], BCFs are used to characterize the location of fixed transmission zeros of a plant which then allows one to deduce whether a decentralized controller exists or not. Furthermore, BCFs are used in the design of a decentralized stabilizing controller for a plant in [10].

BCFs bear many similarities to the polynomial methods extensively studied in the 60's and 70's, particularly the work of Rosenbrock [11] relating to polynomial matrix descriptions (PMDs) of the plant. This seminal work gave rise to state space methods and coprime factor theory both of which proved to be tremendously important and successful in many control related problems. The material developed in this technical note can be viewed as a combination of these two fields of control theory, dealing with the aspects of Rosenbrock's work that were sidelined in favor of the above and thus not advanced in the past few decades.

This technical note provides the foundations to the general study of BCF theory and its applicability to various control related problems. The results presented herein cover a range of topics including internal stability in terms of BCFs of the plant and controller, state space parameterizations of BCFs for a given system and BCF uncertainty characterization. Although BCFs are not a substitute for LCFs or RCFs in control theory, it will become apparent through the course of this technical note that their use can be beneficial.

II. Preliminaries

The sets $\mathbb{R}$ and $\mathbb{C}$ are defined as the real and complex numbers, respectively. $\mathbb{C}_+ = \{s \in \mathbb{C} : \Re(s) > 0\}$ is used to denote the open right half of the complex plane while $\bar{\mathbb{C}}_+ = \mathbb{C}_+ \cup j\mathbb{R}$.

Let $A \in \mathbb{C}^{m\times n}$, then A^* denotes its complex conjugate transpose, while its rank is denoted by $\operatorname{rank} A$. If $m = n$, $\Lambda(A)$ denotes the spectrum of A and $\det A$ its determinant. The geometric multiplicity of $\lambda_i \in \Lambda(A)$ is denoted by $\gamma_A(\lambda_i)$.

The operators $\operatorname{diag}(\cdot)$ and $\operatorname{adiag}(\cdot)$ define block diagonal and anti-diagonal matrices starting from the top left and top right respectively.

$\mathscr{R}$ denotes the set of all real-rational, proper transfer matrices. The subset of $\mathscr{R}$ containing all stable transfer matrices is given by $\mathscr{RH}_\infty$ and the set of units in $\mathscr{RH}_\infty$ is given by $\mathscr{GH}_\infty$ ($f \in \mathscr{GH}_\infty \Leftrightarrow f, f^{-1} \in \mathscr{RH}_\infty$).

Let $P \in \mathscr{R}$, then $P = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ is shorthand notation for the state space realization $P = C\left(sI - A\right)^{-1} B + D$.

The normal rank of a transfer matrix $P(s) \in \mathscr{R}$ is defined as $\max_{s\in\mathbb{C}} \operatorname{rank} P(s)$ and is denoted by $\operatorname{nrank} P$.

Let $H \in \mathscr{R}$ and $\Delta \in \mathscr{R}$, then the lower and upper linear fractional transformations (LFTs) of H with respect to Δ are denoted by $\mathcal{F}_l(H, \Delta)$ and $\mathcal{F}_u(H, \Delta)$ respectively. Furthermore, the Redheffer star product of H and Δ is denoted by $(H \star \Delta)$. See [12] for definitions and details.

As mentioned previously, coprime factorizations are an important part of control theory. The following definition presents, in a formal way, right coprimeness over $\mathscr{RH}_\infty$ as well as RCFs of a plant over $\mathscr{RH}_\infty$.

Definition 1 ([12] Definition 5.3): The ordered pair $\{N, M\}$ is right coprime (RC) in $\mathscr{RH}_\infty$ if $N, M \in \mathscr{RH}_\infty$ and there exist $Y_r, Z_r \in \mathscr{RH}_\infty$ such that $Z_r M + Y_r N = I$. Furthermore, the pair is a RCF of a plant $P \in \mathscr{R}$ over $\mathscr{RH}_\infty$ if, additionally, M is square, $\det M(\infty) \neq 0$ and $P = NM^{-1}$.

Left coprimeness and LCFs of a plant are dually defined.

Definition 2: The set of all RC (resp. LC) pairs in $\mathscr{RH}_\infty$ is defined as $\mathscr{C}_r$ (resp. $\mathscr{C}_l$). Similarly, the set of all RCFs (resp. LCFs) of a plant $P \in \mathscr{R}$ over $\mathscr{RH}_\infty$ is defined as $\mathscr{C}_r(P)$ (resp. $\mathscr{C}_l(P)$).

The following lemma gives necessary and sufficient conditions for well-posedness and internal stability of a standard positive feedback interconnection.

Lemma 1 ([12] Lemma 5.3): Consider the standard positive feedback interconnection of a plant $P \in \mathscr{R}$ and controller $C \in \mathscr{R}$. Then $[P, C]$ is well-posed if and only if $\det(I - CP)(\infty) \neq 0$. Furthermore, $[P, C]$ is internally stable if and only if it is well-posed and

$$\begin{bmatrix} I & -C \\ -P & I \end{bmatrix}^{-1} \in \mathscr{RH}_\infty. \tag{1}$$

III. Bicoprime Factorization Fundamentals

BCFs over $\mathscr{RH}_\infty$ first appeared in literature in [4] where their existence was acknowledged with no significant results given. In the original definition, BCFs of a plant were presented as a quad of objects in $\mathscr{RH}_\infty$ as follows.

Definition 3 ([4] Definition 4.3.1): The ordered quad $\{N, M, L, K\}$ is bicoprime (BC) in $\mathscr{RH}_\infty$ if $\{L, M\} \in \mathscr{C}_l$, $\{N, M\} \in \mathscr{C}_r$ and $K \in \mathscr{RH}_\infty$. Furthermore, the quad is a BCF of a plant $P \in \mathscr{R}$ over $\mathscr{RH}_\infty$ if, additionally, M is square, $\det M(\infty) \neq 0$ and $P = NM^{-1}L + K$.

Similar to LC and RC pairs and factorizations, the following definition presents the notation used for the sets of all BC quads and BCFs of a plant.

Definition 4: The set of all BC quads in $\mathscr{RH}_\infty$ is defined as $\mathscr{B}$. The set of all BCFs of a plant $P \in \mathscr{R}$ over $\mathscr{RH}_\infty$ is defined as $\mathscr{B}(P)$.

It is often convenient to pack a BC quad into a matrix as in the following definition.

Definition 5: The set $\mathscr{B}^m$ is defined as

$$\mathscr{B}^m = \left\{ \begin{bmatrix} M & -L \\ N & K \end{bmatrix} : \{N, M, L, K\} \in \mathscr{B} \right\}$$

When representing a BCF of a plant $P \in \mathscr{R}$, the notation $\mathscr{B}^m(P)$ will be used.

Objects in $\mathscr{B}^m(P)$ will henceforth be referred to as the BCF symbols of P. This naming is chosen to parallel the graph symbols encountered in classical coprime factorizations.

Note that the BCF symbols of a plant are also system matrices as defined by [11], often referred to as Rosenbrock matrices. Thus a BCF is also a PMD of the plant; specifically of the third form. As such, BCF symbols inherit many of the properties of PMDs and thus much of the theory developed in the past for such objects can be readily adapted to BCF theory. However, imposing the bicoprimeness property onto the factors yields additional advantages.

It is a well known result [4, Theorem 4.3.12] that any plant $P \in \mathscr{R}$ with a RCF $\{N, M\} \in \mathscr{C}_r(P)$ is stable if and only if $M \in \mathscr{GH}_\infty$. The following lemma presents an equivalent result for BCFs.

Lemma 2 ([4, Theorem 4.3.12]): Let $P \in \mathscr{R}$ have a BCF $\{N, M, L, K\} \in \mathscr{B}(P)$. Then $P \in \mathscr{RH}_\infty \Leftrightarrow M \in \mathscr{GH}_\infty$.

The following lemma relates the transmission zeros of a plant to those of its BCF symbols.

Lemma 3: Let $P \in \mathscr{R}$ and $G \in \mathscr{B}^m(P)$. Then $z_0 \in \bar{\mathbb{C}}_+$ is a transmission zero of P if and only if it is a transmission zero of G.

Proof: Let the BCF of P associated with G be given by $\{N, M, L, K\} \in \mathscr{B}(P)$. Furthermore let $\{\tilde{N}, \tilde{M}\} \in \mathscr{C}_l(NM^{-1})$ and suppose that Y_r and Z_r is the Bézout factor pair associated with $\{N, M\}$. Then

$$\begin{bmatrix} Z_r & Y_r \\ -\tilde{N} & \tilde{M} \end{bmatrix} \begin{bmatrix} M & -L \\ N & K \end{bmatrix} = \begin{bmatrix} I & Y_r K - Z_r L \\ 0 & \tilde{M} P \end{bmatrix},$$

which implies that G and $\tilde{M}P$ share any transmission zeros. The result then follows by noting that $\{\tilde{N}L + \tilde{M}K, \tilde{M}\} \in \mathscr{C}_l(P)$ [7, Proposition 2.5]. ■

The first advantage of imposing bicoprimeness is now revealed as the properties given by Lemmas 2 and 3 do not hold for Rosenbrock matrices in general. A special case for which these results do hold is when the PMD defines a minimal state space realization of the plant. However, as suggested by [4], this is equivalent to the factorization being BC over the ring of polynomials.

A. Internal Dimension

It is simple to show that the dimensions of the coprime factors of a plant are constant. Suppose that $\{N, M\} \in \mathscr{C}_r(P)$ where $P \in \mathscr{R}^{p \times q}$, then it follows trivially from the definition of RCFs that $N \in \mathscr{RH}_\infty^{p \times q}$ and $M \in \mathscr{RH}_\infty^{q \times q}$.

Such a restriction does not apply to BCFs. Let $P \in \mathscr{R}$ and suppose that $\{N, M, L, K\} \in \mathscr{B}(P)$. Furthermore, define $\tilde{N} = \begin{bmatrix} N & 0 \end{bmatrix}$, $\tilde{M} = \operatorname{diag}(M, I)$, $\tilde{L} = \begin{bmatrix} L^* & 0 \end{bmatrix}^*$. Then it is easy to show that $\{\tilde{N}, \tilde{M}, \tilde{L}, K\} \in \mathscr{B}(P)$ is also a BCF of P with arbitrarily inflated factor dimensions. This fact gives rise to the following definition.

Definition 6: The internal dimension of a BC quad $\{N, M, L, K\} \in \mathscr{B}$ is defined as the number of rows/columns of M. The set of all BC quads of internal dimension $r > 0$ is defined as $\mathscr{B}_r$ (or $\mathscr{B}_r(P)$ if the quad is a BCF of $P \in \mathscr{R}$).

An interesting case arises when the additive term of a BCF is set to zero, as outlined in the following lemma.

Lemma 4: Let $P \in \mathscr{R}^{p \times q}$ and suppose that $\{N, M, L, 0\} \in \mathscr{B}_r(P)$. Then $\operatorname{nrank} P \leq r$.

Before proving the above lemma we need the following result.

Lemma 5: Let $P_1 \in \mathscr{R}^{p \times n}$ and $P_2 \in \mathscr{R}^{n \times q}$ with $n \leq \min\{p, q\}$. Then $\operatorname{nrank}(P_1 P_2) = n$ if and only if $\operatorname{nrank} P_1 = \operatorname{nrank} P_2 = n$.

Proof: ($\Rightarrow$) Suppose that $\operatorname{nrank}(P_1 P_2) = n$, then for some $s_0 \in \mathbb{C}$ $\operatorname{rank}(P_1(s_0) P_2(s_0)) = n$ and the result follows from Sylvester's rank inequality [12, Lemma 2.3].

($\Leftarrow$) Suppose that $\operatorname{nrank}(P_1 P_2) < n$ while $\operatorname{nrank} P_1 = \operatorname{nrank} P_2 = n$, then for all $s \in \mathbb{C}$ $\operatorname{rank}(P_1(s) P_2(s)) < n$. This implies that for all $s_0 \in \mathbb{C}$ where $\operatorname{rank} P_1(s_0) = n$, $\operatorname{rank} P_2(s_0) < n$ and vice versa. By noting that a system can only have a finite number of transmission zeros a contradiction arises which concludes the proof. ■

Proof of Lemma 4: Suppose on the contrary that $r < \operatorname{nrank} P$ and note that $\operatorname{nrank} M = r$ since by definition $\det M(\infty) \neq 0$. Then, using Lemma 5, $\operatorname{nrank}(NM^{-1}L) \leq r < \operatorname{nrank} P$ which is a contradiction since $P = NM^{-1}L$ and hence the proof is complete. ■

Note that as a consequence of Lemma 4, it follows that a BCF can always be chosen to have internal dimension no greater than $\min\{p, q\}$. Now using Lemma 4, a lower bound on the achievable internal dimension for the BCFs of a plant can be stated as in the following theorem.

Theorem 6: Let $P \in \mathscr{R}$ and suppose that $\{N, M, L, K\} \in \mathscr{B}_r(P)$. Then $\inf_{\tilde{P} \in \mathscr{RH}_\infty} \operatorname{nrank}(P - \tilde{P}) \leq r$.

Proof: Since $\{N, M, L, 0\} \in \mathscr{B}_r(P - K)$ it follows from Lemma 4 that $\operatorname{nrank}(P - K) \leq r$. Now suppose that $r < \inf_{\tilde{P} \in \mathscr{RH}_\infty} \operatorname{nrank}(P - \tilde{P})$. Then $\operatorname{nrank}(P - K) < \inf_{\tilde{P} \in \mathscr{RH}_\infty} \operatorname{nrank}(P - \tilde{P})$ which is a contradiction since $K \in \mathscr{RH}_\infty$ and the proof is complete. ∎

B. BCF Parameterization

All LCFs or RCFs of a plant can be simply parameterized by pre- or post-multiplication of the factors by an object in $\mathscr{GH}_\infty$. For example, let $P \in \mathscr{R}$ and suppose that $\{N, M\} \in \mathscr{C}_r(P)$, then it is easy to show that $\{NQ, MQ\} \in \mathscr{C}_r(P)$ for any $Q \in \mathscr{GH}_\infty$ of compatible dimensions.

On the other hand, parameterizing BCFs is not as simple. The following lemma uses a strict system equivalence [13] to parameterize a set of BCFs for a plant $P \in \mathscr{R}$.

Lemma 7: Let $P \in \mathscr{R}$ have the BCF $\{N, M, L, K\} \in \mathscr{B}_n(P)$. Then

$$\begin{bmatrix} \tilde{M} & -\tilde{L} \\ \tilde{N} & \tilde{K} \end{bmatrix} = \begin{bmatrix} Q_l & 0 \\ R_l & I \end{bmatrix} \begin{bmatrix} M & -L \\ N & K \end{bmatrix} \begin{bmatrix} Q_r & -R_r \\ 0 & I \end{bmatrix} \in \mathscr{B}_n^m(P)$$

for all $Q_l, Q_r \in \mathscr{GH}_\infty$ and $R_l, R_r \in \mathscr{RH}_\infty$ with compatible dimensions.

Before proving the above lemma we need the following result. This gives sufficient conditions for a BC quad to retain its bicoprimeness under stable perturbations of the factors.

Lemma 8: Let $\{N, M, L, K\} \in \mathscr{B}$, $Q, R, S, T \in \mathscr{RH}_\infty$ and $U, V \in \mathscr{GH}_\infty$. Then $\{(N - QM)U, V(M - LSN)U, V(L - MR), K + T\} \in \mathscr{B}$ if $[Q, LS]$ and $[SN, R]$ are internally stable.

Proof: Since $U, V \in \mathscr{GH}_\infty$ it follows that they can always be absorbed into the Bézout factors, hence:

$$\{(N - QM)U, V(M - LSN)U\} \in \mathscr{C}_r$$

$$\Leftrightarrow \exists \tilde{Y}_r, \tilde{Z}_r \in \mathscr{RH}_\infty : \tilde{Z}_r(M - LSN) + \tilde{Y}_r(N - QM) = I$$

$$\Leftrightarrow \exists \tilde{Y}_r, \tilde{Z}_r \in \mathscr{RH}_\infty : \begin{bmatrix} \tilde{Z}_r & \tilde{Y}_r \end{bmatrix} \begin{bmatrix} I & -LS \\ -Q & I \end{bmatrix} \begin{bmatrix} M \\ N \end{bmatrix} = I$$

$$\Leftarrow \begin{bmatrix} I & -LS \\ -Q & I \end{bmatrix} \in \mathscr{GH}_\infty$$

$\Leftrightarrow [Q, LS]$ is internally stable.

The fact that $\{V(L - MR), V(M - LSN)U\} \in \mathscr{C}_l$ if $[SN, R]$ is internally stable can be proven similarly. Finally, since $K + T \in \mathscr{RH}_\infty$ the conclusion follows. ∎

Proof of Lemma 7: Using Lemma 8 it can be shown that $\{\tilde{N}, \tilde{M}, \tilde{L}, \tilde{K}\} \in \mathscr{B}$ and then $P = \tilde{N}\tilde{M}^{-1}\tilde{L} + \tilde{K}$ follows from [11, Theorem 3.1]. ∎

Observe that the parameterization of Lemma 7 does not allow for variation in the internal dimension of the BCFs, it is therefore immediate that it does not cover the entire set of BCFs for a given plant.

IV. State Space Formulae and Characterizations

Coprime factorizations can be easily obtained from a state space realization of the plant using the formulae of [14]. A trivial method of obtaining a BCF of a plant is to construct a LCF or RCF and set the remaining factors accordingly. For example, let $P \in \mathscr{R}$ and choose $\{N, M\} \in \mathscr{C}_r(P)$ then it follows trivially that $\{N, M, I, 0\} \in \mathscr{B}(P)$. However, a more systematic state space approach is needed. This problem is addressed in this section.

The following theorem gives a parameterization of BCFs of a plant based on state space data, that generalizes the formulae given by [14].

Theorem 9: Let $P \in \mathscr{R}^{p \times q}$ have a stabilizable and detectable state space realization $P = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$. Furthermore, suppose that $Q \in \mathbb{R}^{n \times r}$, $S \in \mathbb{R}^{r \times r}$ and $R \in \mathbb{R}^{r \times n}$ are such that $A + QSR$ is Hurwitz, where $\det(S) \neq 0$. Finally, let $D_N \in \mathbb{R}^{p \times r}$ and $D_L \in \mathbb{R}^{r \times q}$ be arbitrarily chosen matrices and define

$$\left[\begin{array}{c:c} M & -L \\ \hdashline N & K \end{array}\right] = \left[\begin{array}{c|c:c} A + QSR & QS & B + QSD_L \\ \hline SR & S & SD_L \\ \hdashline C + D_N SR & D_N S & D + D_N SD_L \end{array}\right]. \tag{2}$$

Then $\{N, M, L, K\} \in \mathscr{B}_r(P)$.

Proof: First, it is easy to show that $P = NM^{-1}L + K$. Let $F \in \mathbb{R}^{q \times n}$ and $H \in \mathbb{R}^{n \times p}$ be such that $A + BF$ and $A + HC$ are Hurwitz. Then the following holds after some simple albeit tedious linear algebra:

$$\begin{bmatrix} M & -L \end{bmatrix} \left[\begin{array}{c|c} A + BF & Q \\ \hline -(R + D_L F) & S^{-1} \\ F & 0 \end{array}\right] = I \text{ and}$$

$$\left[\begin{array}{c|cc} A + HC & -(Q + HD_N)H \\ \hline R & S^{-1} & 0 \end{array}\right] \begin{bmatrix} M \\ N \end{bmatrix} = I,$$

which completes the proof. ∎

The BCF presented in Theorem 9 will henceforth be referred to as the QR-BCF parameterization.

Remark 1: The QR-BCF parameterization given in Theorem 9 reduces to the standard LCF and RCF parameterizations of [14] by an appropriate selection of Q, S, R, D_N and D_L. For example, let $P \in \mathscr{R}$ and $\{N, M, L, K\} \in \mathscr{B}(P)$ given by (2) with $Q = B$, $S = I$, $D_N = D$ and $D_L = -I$. Then $L = I$, $K = 0$ and $\{N, M\} \in \mathscr{C}_r(P)$. ♦

Remark 2: The matrices Q, S and R satisfying the conditions of Theorem 9 exist regardless of the stabilizability and detectability of the given state space realization of the plant. However, the assumption is necessary for the resulting factorization to be BC. This can be seen as follows. The pair $\{N, M\}$ is RC if and only if the associated graph symbol has no transmission zeros in $\bar{\mathbb{C}}_+$ or equivalently

$$\operatorname{rank} \begin{bmatrix} M \\ N \end{bmatrix} = r \quad \forall s \in \bar{\mathbb{C}}_+$$

$$\Leftrightarrow \operatorname{rank} \begin{bmatrix} A + QSR - sI & QS \\ SR & S \\ C + D_N SR & D_N S \end{bmatrix} = n + r \quad \forall s \in \bar{\mathbb{C}}_+$$

$$\Leftrightarrow \operatorname{rank} \begin{bmatrix} A - sI & Q \\ 0 & I \\ C & D_N \end{bmatrix} = n + r \quad \forall s \in \bar{\mathbb{C}}_+$$

$$\Leftrightarrow \operatorname{rank} \begin{bmatrix} A - sI \\ C \end{bmatrix} = n \quad \forall s \in \bar{\mathbb{C}}_+$$

$\Leftrightarrow (C, A)$ is detectable,

where n denotes the order of the plant or equivalently the dimension of A. Hence, the detectability of (C, A) is necessary for the QR-BCF parameterization to generate a valid BCF of the plant. It can similarly be shown that $\{L, M\} \in \mathscr{C}_l$ if and only if (A, B) is stabilizable. ♦

An interesting question that arises from the BCF characterization of Theorem 9 is "what is the smallest internal dimension achievable using the QR-BCF parameterization?". This is equivalent to finding the smallest dimension Q such that (A, Q) is stabilizable. This question is answered by the following lemma.

Lemma 10: Let $A \in \mathbb{R}^{n \times n}$, then there exists a matrix $B \in \mathbb{R}^{n \times q}$ such that (A, B) is controllable if and only if

$$\max_{\lambda_i \in \Lambda(A)} \{\gamma_A(\lambda_i)\} \leq q.$$

Proof: This is a consequence of [15, Theorem 1.2]. ■

Then, by a direct application of the above result, it becomes apparent on using Theorem 9 that the minimum BCF internal dimension achievable is given by $r = \max_{\lambda_i \in \Lambda(A) \cap \bar{\mathbb{C}}_+} \{\gamma_A(\lambda_i)\}$.

V. Internal Stability

As is the case for RCFs and LCFs, BCFs can be used to establish the internal stability of the feedback interconnection of two systems. Some internal stability results based on BCFs of the plant and controller are presented in the following theorem.

Theorem 11: Consider the standard positive feedback interconnection of a plant $P \in \mathscr{R}^{p \times q}$ and controller $C \in \mathscr{R}^{q \times p}$. Let $\{N, M, L, K\} \in \mathscr{B}_r(P)$ and $\{U, V, W, X\} \in \mathscr{B}_{\hat{r}}(C)$, with $G_P \in \mathscr{B}_r^m(P)$ and $G_C \in \mathscr{B}_{\hat{r}}^m(C)$ being the associated BCF symbols. Then the following statements are true:

a) $[P, C]$ is internally stable if and only if

$$\begin{bmatrix} G_P & -\mathrm{diag}(0_{r \times \hat{r}}, I_p) \\ -\mathrm{diag}(0_{\hat{r} \times r}, I_q) & G_C \end{bmatrix} \in \mathscr{GH}_\infty. \tag{3}$$

b) Suppose that $[K, X]$ is internally stable. Then $[P, C]$ is internally stable if and only if

$$G_P \star \Big(\mathrm{adiag}(I_q, I_{\hat{r}})\, G_C\, \mathrm{adiag}(I_{\hat{r}}, I_p)\Big) \in \mathscr{GH}_\infty. \tag{4}$$

c) Suppose that $X = 0$. Then $[P, C]$ is internally stable if and only if

$$\begin{bmatrix} M & -LU \\ -WN & V - WKU \end{bmatrix} \in \mathscr{GH}_\infty. \tag{5}$$

d) Suppose that $[K, C]$ is internally stable. Then $[P, C]$ is internally stable if and only if

$$M - LC(I - KC)^{-1} N \in \mathscr{GH}_\infty. \tag{6}$$

Before proving the above theorem, the following useful result is provided.

Lemma 12: Let $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \in \mathscr{RH}_\infty$ and suppose that $A_{22} \in \mathscr{GH}_\infty$. Then $A \in \mathscr{GH}_\infty$ if and only if $A_{11} - A_{12} A_{22}^{-1} A_{21} \in \mathscr{GH}_\infty$.

Proof: The proof follows trivially via a Schur complement decomposition of A. ■

Proof of Theorem 11: Define

$$G_{[P,C]} = \left[\begin{array}{cc:cc} 0 & M & L & 0 \\ V & 0 & 0 & W \\ \hdashline U & 0 & I & -X \\ 0 & N & -K & I \end{array}\right] \tag{7}$$

and note that $G_{[P,C]} \in \mathscr{B}^m\left(\begin{bmatrix} I & -C \\ -P & I \end{bmatrix}\right)$. It then follows from Lemmas 1 and 3 that $[P, C]$ is internally stable if and only if $G_{[P,C]} \in \mathscr{GH}_\infty$. The theorem statements can then be proven via consecutive applications of Lemmas 2 and 12 in addition to some elementary row/column permutations to (7).

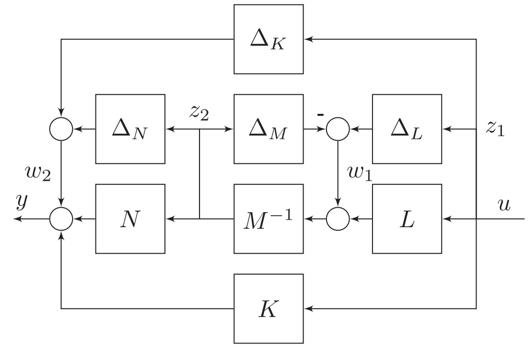

Fig. 1. Perturbed plant block diagram with BC factor uncertainty.

To prove (d) it is also necessary to note that given $\{U, V, W, 0\} \in \mathscr{B}(C)$, then $\{U, V - WKU, W, 0\} \in \mathscr{B}(C(I - KC)^{-1})$. However the supposition that $X = 0$ is not necessary as the term can always be absorbed into the other BC factors of C. ■

Remark 3: It follows from Theorem 11 that the internal stability tests induced by classical coprime factorizations [12, Lemma 5.10] are special cases of their BCF counterparts. ■

VI. Uncertainty and Robust Stability Conditions

Just like RCFs and LCFs, BCFs can be used to define an uncertainty structure and by extent a robust stability margin. In this section, stable additive perturbations on the BC factors of a plant are examined. Following coprime factor convention, a BCF perturbed plant can be defined as

$$P_\Delta = (N + \Delta_N)(M + \Delta_M)^{-1}(L + \Delta_L) + (K + \Delta_K). \tag{8}$$

Fig. 1 shows a block diagram representation of the proposed BCF uncertainty structure given by (8). As expected, the uncertainty structure induced by a plant BCF contains elements from both LCF and RCF uncertainty. Therefore, like coprime factor uncertainty, BCF uncertainty is suitable for capturing low frequency parameter errors, neglected high frequency dynamics and uncertain $\bar{\mathbb{C}}_+$ poles and zeros. Another interesting fact about this structure is that it closely resembles the standard four-block problem commonly studied in robust control as is evident from Fig. 1.

Given a coprime factorization of a plant, any perturbations on the coprime factors of the plant must preserve coprimeness, otherwise the perturbed plant is not robustly stabilizable [16, Remark 4.4]. A similar condition will be imposed herein with $\{N + \Delta_N, M + \Delta_M, L + \Delta_L, K + \Delta_K\} \in \mathscr{B}(P_\Delta)$.

From the very definition of BCF uncertainty it is obvious that this structure will always be at least as good as classical coprime factor uncertainty at capturing modeling errors. This follows by noting that the former forms a superset of the latter. Consider for example the LCF of a plant $\{L, M\} \in \mathscr{C}_l(P)$ being perturbed to $P_\Delta^{LCF} = (M + \Delta_M)^{-1}(L + \Delta_L)$. Now, if we were to allow uncertainty on the induced BCF $\{I, M, L, 0\} \in \mathscr{B}(P)$, the resulting perturbed plant would be given by $P_\Delta^{BCF} = (I + \Delta_N)(M + \Delta_M)^{-1}(L + \Delta_L) + \Delta_K$ which allows for capturing output multiplicative and additive modeling errors [12, Table 9.1] in addition to the coprime factor errors normally represented by LCF uncertainty. Thus it becomes apparent that LCF and RCF uncertainty is a special structured case of BCF uncertainty.

A central part in the study of any uncertainty structure is the construction of a generalized plant. In the case of BCF uncertainty this can be obtained as follows. Define $z = \begin{pmatrix} z_2^* & z_1^* \end{pmatrix}^*$ and $w = \begin{pmatrix} w_1^* & w_2^* \end{pmatrix}^*$. Then from Fig. 1 a generalized plant $\Pi : \begin{pmatrix} w^* & u^* \end{pmatrix}^* \mapsto \begin{pmatrix} z^* & y^* \end{pmatrix}^*$

and uncertainty matrix $\Delta : z \mapsto w$ can be defined as

$$\Pi = \left[\begin{array}{cc:c} M^{-1} & 0 & M^{-1}L \\ 0 & 0 & I \\ \hdashline NM^{-1} & I & P \end{array}\right] \text{ and } \tag{9}$$

$$\Delta = \begin{bmatrix} -\Delta_M & \Delta_L \\ \Delta_N & \Delta_K \end{bmatrix}. \tag{10}$$

It is straightforward to confirm that using the above Π and Δ yields $P_\Delta = \mathcal{F}_u(\Pi, \Delta)$.

Though a BCF robust stability margin can be obtained by directly calculating $\|\mathcal{F}_l(\Pi, C)\|_\infty^{-1}$ (see [17] for details), the procedure is simplified when a BCF of the controller is used. This robust stability result is given in the following theorem.

Theorem 13: Consider the standard positive feedback interconnection of a plant $P \in \mathscr{R}$ and stabilizing controller $C \in \mathscr{R}$ and suppose that $\{N, M, L, K\} \in \mathscr{B}(P)$ and $\{U, V, W, 0\} \in \mathscr{B}(C)$. Furthermore, define $\Delta \in \mathscr{RH}_\infty$ as in (10), $P_\Delta = \mathcal{F}_u(\Pi, \Delta)$ with Π as in (9) and suppose that $\{N + \Delta_N, M + \Delta_M, L + \Delta_L, K + \Delta_K\} \in \mathscr{B}(P_\Delta)$. Then $[P_\Delta, C]$ is internally stable for all $\|\Delta\|_\infty < \gamma$ if and only if

$$\left\| \begin{bmatrix} I & 0 \\ 0 & U \end{bmatrix} \begin{bmatrix} M & -LU \\ -WN & V - WKU \end{bmatrix}^{-1} \begin{bmatrix} I & 0 \\ 0 & W \end{bmatrix} \right\|_\infty \le \frac{1}{\gamma}.$$

Proof: First, define $S = \begin{bmatrix} M & -LU \\ -WN & V - WKU \end{bmatrix}$ which from Theorem 11 (c) belongs to $\mathscr{GH}_\infty$ since $[P, C]$ is internally stable. Then, using the same result again, $[P_\Delta, C]$ is internally stable if and only if

$$\begin{bmatrix} M + \Delta_M & -(L + \Delta_L)U \\ -W(N + \Delta_N) & V - W(K + \Delta_K)U \end{bmatrix} \in \mathscr{GH}_\infty$$

$$\Leftrightarrow \left(S - \begin{bmatrix} I & 0 \\ 0 & W \end{bmatrix} \begin{bmatrix} -\Delta_M & \Delta_L \\ \Delta_N & \Delta_K \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & U \end{bmatrix} \right)^{-1} \in \mathscr{RH}_\infty$$

$$\Leftrightarrow \left(\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} I & 0 \\ 0 & U \end{bmatrix} S^{-1} \begin{bmatrix} I & 0 \\ 0 & W \end{bmatrix} \Delta \right)^{-1} \in \mathscr{RH}_\infty.$$

The conclusion then follows from the small gain theorem. ■

VII. Numerical Example

A numerical example is provided in this section to illustrate how the results presented in this technical note can be used in a practical setting.

Before providing the numerical example we briefly discuss a BCF based controller parameterization. Suppose that a plant $P \in \mathscr{R}$ has the BCF $\{N, M, L, K\} \in \mathscr{B}(P)$. Since $K \in \mathscr{RH}_\infty$, using the Youla parameterization [18] a set of stabilizing controllers for K is given by $\{C = U(I + KU)^{-1} : U \in \mathscr{RH}_\infty, \det(I + KU)(\infty) \neq 0\}$ so that $C(I - KC)^{-1} = U$. Then using Theorem 11 (d), a set of stabilizing controllers for P can be defined as $\mathcal{C}(P) = \{C = U(I + KU)^{-1} : U \in \mathscr{RH}_\infty, \det(I + KU)(\infty) \neq 0, M - LUN \in \mathscr{GH}_\infty\}$.

Consider the plant $P \in \mathscr{R}$ given by

$$P = \left[\begin{array}{cc|cc} 1 & 0 & 1 & 1 \\ 0 & -2 & 1 & 1 \\ \hline 1 & 2 & 0 & 0 \\ 2 & 1 & 0 & 0 \end{array}\right] = \begin{bmatrix} \frac{3s}{(s+2)(s-1)} & \frac{2s+1}{(s+2)(s-1)} \\ \frac{3(s+1)}{(s+2)(s-1)} & \frac{5s+7}{(s+2)(s-1)} \end{bmatrix}.$$

Then with $Q = \begin{bmatrix} 1 & 0 \end{bmatrix}^*$, $S = I$, $R = \begin{bmatrix} -4 & 0 \end{bmatrix}$, $D_N = 0$ and $D_L = 0$, a BCF of P is obtained via Theorem 9 as

$$\left[\begin{array}{c:c} M & -L \\ \hdashline N & K \end{array}\right] = \left[\begin{array}{c:cc} \frac{s-1}{s+3} & -\frac{4}{s+3} & -\frac{8}{s+3} \\ \hdashline \frac{1}{s+3} & \frac{3s+8}{(s+3)(s+2)} & \frac{4s+10}{(s+3)(s+2)} \\ \frac{2}{s+3} & \frac{3s+7}{(s+3)(s+2)} & \frac{5s+11}{(s+3)(s+2)} \end{array}\right] \in \mathscr{B}_1^m(P).$$

A controller can now be synthesized for P using the procedure described above. Since the internal dimension of the given BCF is 1, this is a simple scalar problem; a solution to which given by $U = -\frac{s+3}{s+4}\operatorname{diag}\left(1, \frac{1}{8}\right)$. This yields the controller

$$C = -\frac{1}{d(s)} \begin{bmatrix} 8s^3 + 67s^2 + 182s + 159 & 4s^2 + 22s + 30 \\ 3s^2 + 16s + 21 & s^3 + 6s^2 + 9s \end{bmatrix} \in \mathcal{C}(P),$$

where $d(s) = 8s^3 + 51s^2 + 68s - 35$.

Now consider the reverse problem. That is, given P and C, establish whether or not $[P, C]$ is internally stable. Then using BCF theory, specifically Theorem 11 (d), internal stability can be established by inspection of the transmission zeros of a scalar transfer function [19]. As a comparison, using a coprime factor result would require the inversion of a 2×2 matrix at best. Such a scalar test is possible for any $P \in \mathscr{R}^{p \times q}$ for which $\mathscr{B}_1(P) \neq \emptyset$, regardless of the magnitudes of p and q. One could hence easily imagine benefits in a variety of control problems; for example multi-agent systems, such as those motivated in [20] and [21].

VIII. Conclusion

The foundations of a BCF theory are developed in this technical note. Many fundamental, yet important, results such as state space parameterization and internal stability tests are presented. It is also demonstrated that many RCF or LCF results can be obtained from the more general BCFs via the appropriate restrictions. The QR-BCF parameterization is presented which is shown to capture the standard coprime factor parameterizations given by [14]. Finally, the uncertainty structure induced by a BCF is defined and shown to have an appealing nature that encompasses the classical LCF and RCF uncertainty structures.

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