Distance Measures, Robust Stability Conditions and Robust Performance Guarantees for Uncertain Feedback Systems

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Summary. Given a nominal plant, a perturbed plant, an uncertainty structure and performance weights, we use robust model validation ideas to define and compute a measure of the distance between the nominal and perturbed plants. We also define a stability margin for a feedback system that is related to robust stability and nominal performance, and derive conditions for the stability and bounds for the performance degradation of the perturbed feedback system in terms of the distance measure. These robust stability and robust performance results give the distance measure a feedback interpretation. The simplicity and power of our procedure for computing the distance between two systems is illustrated using a normalized coprime factor uncertainty model to derive results that have already been published in the literature using different techniques. All systems considered in this paper are linear time-invariant.

Key words: Robust stability, robust performance, distance measures, model validation, feedback systems, gap metric, ν-gap matric, $\mathcal{H}_\infty$ control, stability margin

1 Introduction

When asked to design a controller for a plant, a control engineer will typically identify a number of plant models that approximate the true plant dynamics, and design a controller that achieves the desired robustness and performance objectives with all the plant models.

Let us consider an example. The current standard for designing flight controllers is to divide the flight envelope of an aircraft into a number of flight conditions, design a linear time-invariant (LTI) controller for each flight condition and schedule the gains of the LTI controllers to give a full envelope flight controller. The LTI controllers are typically not scheduled on weight and center-of-gravity location, at least for civil aircraft. At each flight condition, the engineer is given a number of aircraft models that correspond to different combinations of weight and center-of-gravity location, and is asked to design a LTI controller for all combinations.

A control engineer can handle more than one plant models in at least two different ways. The first way is current practice in industry:

1. Use engineering experience to choose a nominal plant model;
2. Design a controller that achieves the desired robustness and performance objectives with the nominal plant model;
3. Check robustness and performance with all other plant models; and
4. Iterate if necessary.

This approach relies on the engineer's experience for the choices of a nominal plant model and a measure of robustness. An alternative approach that can be more systematic, is to embed all the plant models into a family of plants and then design a controller for the family. A family of plants is characterized by a nominal plant model, an uncertainty structure and one or more weighting functions that specify the desired performance. It must be emphasized that even though our definition of a family of plants has been taken from $\mathcal{H}_\infty$ control, the controller for the family does not have to be designed using $\mathcal{H}_\infty$ synthesis.

In this paper, given a nominal plant model, an uncertainty structure, weighting functions and a perturbed plant model (all systems are assumed to be LTI), we develop a generic procedure for computing the size of the smallest family that contains both the nominal and perturbed plants. We define the distance between the nominal and perturbed plants to be the size of the smallest family.

We believe that such a distance measure is of practical interest since it can help an engineer choose a suitable family of plants (nominal plant, uncertainty structure and weighting functions). Given a number of plant models, the idea is to choose the smallest family that contains all models, the presumption being that it will be easier for a controller to achieve the desired closed-loop objectives with a small family. Indeed, every distance measure comes with a dual quantity, the so-called stability margin, that captures essential information about the robustness and performance of a feedback system. It is shown in this paper, that the smaller the distance between two plants, the larger the guaranteed residual stability margin and the smaller the guaranteed difference in closed-loop performance when both plants are in feedback with the same controller.

The high-level ideas outlined above are not new. Vidyasagar [1], El-Sakkary [2], Georgiou and Smith [3], Qiu and Davison [4], and Vinnicombe [5, 6] have all defined and worked on similar ideas but specifically for normalized coprime factor or 4-block uncertainty structures. [3] What is new is the generic procedure for computing the distance between two plants and the robust stability and robust performance guarantees which can handle many different uncertainty structures including normalized coprime factor uncertainty that was studied by Georgiou and Smith, and Vinnicombe. In fact, in this paper we show that our procedure captures the relevant results of Vinnicombe.

Our approach has been inspired by robust model validation theory (see Davis [8], Poolla *et al.* [9], Chen and Gu [10], and Newlin and Smith [11]). The objective of robust model validation is, given a family of plants and data from the real plant, to verify that there exists a plant model in the family that interpolates the data. If there doesn't, then we have invalidated the family and any controller designed with the family is not guaranteed to work. In our case, we do not have real plant data but a perturbed plant model; that is, infinite data.

Given a nominal plant, an uncertainty structure, weighting functions and a perturbed plant, the first step of our generic procedure is to solve a so-called consistency equation. For example, in the case of input multiplicative uncertainty, we would solve (if a solution exists)

$$P_{\Delta} = P(I - W\Delta)$$

for all perturbations $\Delta \in \mathcal{RL}_\infty$ that satisfy the equation, where P is the given nominal plant, P_Δ is the given perturbed plant and W is a given weighting function (the uncertainty structure is implied by the structure of the equation). Solving the consistency equation can be thought of as a model validation problem. Once we have parameterized all admissible Δ, the second step is to compute the $\mathcal{L}_\infty$-norm of the smallest perturbation which satisfies consistency and this is our distance measure.

Given a controller that stabilizes the nominal plant, we also determine conditions for stability and bound the degradation in performance of the perturbed plant in feedback with the same controller, all in terms of our distance measure. This step is based on a powerful lemma by Vinnicombe which essentially generalizes the small gain theorem [6, p. 45, Lemma 1.22]. In fact, much of our work has been influenced by the "Cambridge view of feedback control".

In summary, we have used robust model validation ideas to develop a generic procedure for computing the distance between two LTI systems. The procedure works with different uncertainty structures and the defined distance measure has a feedback interpretation because we have also derived robust stability and robust performance guarantees in terms of this distance measure. Compared to other work reported in the literature, although the philosophical

[3] The works of Georgiou and Smith, and Vinnicombe have been used to design a number of experimental controllers, the most notable one being for a Harrier aircraft that was successfully flight tested (see Hyde [7]).

idea is not new, we derive our distance measures in a different way and our procedure can systematically handle different uncertainty structures.

What follows is an outline of the paper. In Section 2, we define our distance measure, the stability margin, show how to compute the distance measure via a systematic procedure and derive robust stability and robust performance results all generic. In Section 3, we apply our tools to a specific uncertainty structure that has been published in the literature. In Section 4, we indicate that our procedure works for many uncertainty structures and in Section 5 we conclude the paper.

1.1 Notation

Let $\mathcal{R}$ denote the set of proper real-rational transfer functions. [4] Also, let $P^*(s)$ denote the adjoint of $P(s) \in \mathcal{R}$ defined by $P^*(s) = P(-s)^T$. Let $\mathcal{RL}_\infty$ denote the space of proper real-rational functions bounded on $j\mathbb{R}$ including ∞ and $\mathcal{RH}_\infty$ denote the space of proper real-rational functions bounded and analytic in the open right half complex plane. Denote also the space of functions that are units in $\mathcal{RH}_\infty$ by $\mathcal{GH}_\infty$ (i.e. $f \in \mathcal{GH}_\infty \Leftrightarrow f, f^{-1} \in \mathcal{RH}_\infty$). Let $\mathcal{F}_l(\cdot,\cdot)$ (resp. $\mathcal{F}_u(\cdot,\cdot)$) denote a lower (resp. upper) linear fractional transformation. Also, let $\cdot \star \cdot$ denote the Redheffer star product of two transfer functions with respect to some partition. For a scalar $p(s) \in \mathcal{R}$, its winding number $\operatorname{wno} p(s)$ is defined as the number of encirclements of the origin made by $p(s)$ as s follows the standard Nyquist D-contour, indented into the right half plane around any imaginary axis poles or zeros of $p(s)$.

The ordered pair $\{N, M\}$, with $M, N \in \mathcal{RH}_\infty$, is a right-coprime factorization (*rcf*) of $P \in \mathcal{R}$ if M is invertible in $\mathcal{R}$, $P = NM^{-1}$ and N and M are right-coprime. Furthermore, the ordered pair $\{N, M\}$ is a normalized *rcf* of P if $\{N, M\}$ is an *rcf* of P and $M^*M + N^*N = I$. Likewise, the ordered pair $\{\tilde{N}, \tilde{M}\}$, with $\tilde{M}, \tilde{N} \in \mathcal{RH}_\infty$, is a left-coprime factorization (*lcf*) of $P \in \mathcal{R}$ if $\tilde{M}$ is invertible in $\mathcal{R}$, $P = \tilde{M}^{-1}\tilde{N}$ and $\tilde{N}$ and $\tilde{M}$ are left-coprime. Furthermore, the ordered pair $\{\tilde{N}, \tilde{M}\}$ is a normalized *lcf* of P if $\{\tilde{N}, \tilde{M}\}$ is an *lcf* and $\tilde{M}\tilde{M}^* + \tilde{N}\tilde{N}^* = I$. Let $\{N, M\}$ be a right and $\{\tilde{N}, \tilde{M}\}$ be a left coprime factorization of a plant P. Also, let $\{U, V\}$ be a right and $\{\tilde{U}, \tilde{V}\}$ be a left coprime factorization of a controller C. Define

$$G := \begin{bmatrix} N \\ M \end{bmatrix}, \tilde{G} := \begin{bmatrix} -\tilde{M} & \tilde{N} \end{bmatrix}, K := \begin{bmatrix} V \\ U \end{bmatrix}, \tilde{K} := \begin{bmatrix} -\tilde{U} & \tilde{V} \end{bmatrix},$$

where G and $\tilde{G}$ will be referred to as the graph symbols of P and K and $\tilde{K}$ will be referred to as the inverse graph symbols of C. [5]

[4] $P \in \mathcal{R}$ implies that $\overline{\sigma}(P(\infty)) < \infty$. Also, $P \in \mathcal{R}$ is invertible in $\mathcal{R}$ if, and only if, P is square and $\det P(\infty) \neq 0$.

[5] For normalized graph symbols,

2 A systematic procedure for robust stability and robust performance analysis

2.1 Objective

Given a nominal plant P, an uncertainty structure with weights, and a controller C, derive conditions for the stability of the feedback interconnection of a perturbed plant P_Δ with C, and bounds for the degradation in performance when P is replaced by P_Δ that depend only on the given data.

2.2 Definitions

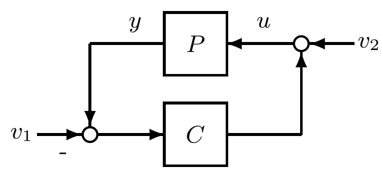

Fig. 1. The standard feedback interconnection.

Let $[P, C]$ denote the standard feedback interconnection illustrated in Fig. 1. From Fig. 1 and after some algebra,

$$\begin{bmatrix} y \\ u \end{bmatrix} = \begin{bmatrix} P \\ I \end{bmatrix} (I - CP)^{-1} \begin{bmatrix} -C & I \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}. \tag{1}$$

Definition 1. *Given a plant $P \in \mathcal{R}$ and a controller $C \in \mathcal{R}$. $[P, C]$ is said to be well-posed if the transfer function in* (1) *belongs to $\mathcal{R}$, that is, if $(I - CP)^{-1} \in \mathcal{R}$. Also, $[P, C]$ is said to be internally stable if, in addition, the transfer function in* (1) *belongs to $\mathcal{RH}_\infty$.*

Let $\langle H, C \rangle$ denote the standard $\mathcal{H}_\infty$ synthesis interconnection depicted in Fig. 2. From Fig. 2 and after some algebra,

$$\begin{bmatrix} z \\ y \\ u \end{bmatrix} = \left(\begin{bmatrix} H_{11} & 0 & H_{12} \\ H_{21} & 0 & H_{22} \\ 0 & 0 & I \end{bmatrix} + \begin{bmatrix} H_{12} \\ H_{22} \\ I \end{bmatrix} C(I - H_{22}C)^{-1} \begin{bmatrix} H_{21} & -I & H_{22} \end{bmatrix} \right) \begin{bmatrix} w \\ v_1 \\ v_2 \end{bmatrix}. \tag{2}$$

$$\begin{bmatrix} G^* \\ \tilde{G} \end{bmatrix} \begin{bmatrix} G & \tilde{G}^* \end{bmatrix} = \begin{bmatrix} G & \tilde{G}^* \end{bmatrix} \begin{bmatrix} G^* \\ \tilde{G} \end{bmatrix} = I.$$

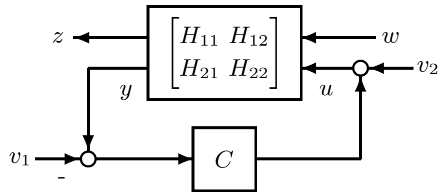

Fig. 2. The standard $\mathcal{H}_\infty$ synthesis interconnection.

Definition 2. *Given a generalized plant $H \in \mathcal{R}$ and a controller $C \in \mathcal{R}$. $\langle H, C\rangle$ is said to be well-posed if the transfer function in* (2) *belongs to $\mathcal{R}$, that is, if $(I - H_{22}C)^{-1} \in \mathcal{R}$. Also, $\langle H, C\rangle$ is said to be internally stable if, in addition, the transfer function in* (2) *belongs to $\mathcal{RH}_\infty$. Finally, H is said to be stabilizable if there exists a C such that $\langle H, C\rangle$ is internally stable.* [6]

It will be shown in the sequel, that if $[H_{22}, C]$ is stable, then the larger the size of $\|\mathcal{F}_l\,(H, C)\|_\infty^{-1}$, the larger the size of the set of plants that C is guaranteed to stabilize. Therefore, it can be argued that $\|\mathcal{F}_l\,(H, C)\|_\infty^{-1}$ is a measure of the robust stability of a feedback system. Also, the nominal performance of a feedback system, that is the performance with no plant uncertainty, can be related to H and $\|\mathcal{F}_l\,(H, C)\|_\infty^{-1}$ (by performance it is typically meant reference tracking, disturbance rejection and noise rejection). This is because typically the size of any closed-loop transfer function can be bounded above in terms of the size of any weighting functions absorbed in H and $\|\mathcal{F}_l\,(H, C)\|_\infty^{-1}$. [7] Therefore, $\|\mathcal{F}_l\,(H, C)\|_\infty^{-1}$ is a very important quantity that summarizes the robustness and performance of a feedback system.

Definition 3. *Given a plant $P \in \mathcal{R}^{p\times q}$, a generalized plant $H \in \mathcal{R}$ with $H_{22} = P$, and a controller $C \in \mathcal{R}^{q\times p}$. Define the stability margin $b^H(P, C)$ of the feedback interconnection $\langle H, C\rangle$ by:*

$$b^H(P, C) := \begin{cases} \|\mathcal{F}_l\,(H, C)\|_\infty^{-1}, & \text{if } \mathcal{F}_l\,(H, C) \in \mathcal{RL}_\infty \text{ and } [P, C] \text{ is internally stable} \\ 0, & \text{otherwise.} \end{cases}$$

The choice of generalized plant H and the required size of the stability margin are application specific. Now, we define a distance measure between a nominal plant and a perturbed plant.

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[6] If $\left[\begin{array}{c|c} A & B_2 \\ \hline C_2 & D_{22} \end{array}\right]$ is the state-space realization of H_{22} inherited from a stabilizable and detectable state-space realization of H, then this definition of "H is stabilizable" is equivalent to (A, B_2) being stabilizable and (C_2, A) being detectable [12].

[7] For an example of how nominal performance can be related to H and $\|\mathcal{F}_l\,(H, C)\|_\infty^{-1}$, see [13, p. 493, Theorem 18.11].

Definition 4. *Given a plant $P \in \mathcal{R}^{p\times q}$, a generalized plant $H \in \mathcal{R}$ with $H_{22} = P$, and a perturbed plant $P_\Delta \in \mathcal{R}^{p\times q}$. Let the set of all admissible uncertainties that satisfy consistency of equations be given by:*

$$\boldsymbol{\Delta} = \left\{\Delta \in \mathcal{RL}_\infty : (I - H_{11}\Delta)^{-1} \in \mathcal{R}, P_\Delta = \mathcal{F}_u\left(H, \Delta\right)\right\}.$$

Define the distance measure $d^H(P, P_\Delta)$ between plants P and P_Δ for the uncertainty structure $\mathcal{F}_u\left(H, \Delta\right)$ by: [8]

$$d^H(P, P_\Delta) := \begin{cases} \min_{\Delta\in\boldsymbol{\Delta}} \|\Delta\|_\infty, & \text{if } \boldsymbol{\Delta} \neq \emptyset \\ \infty, & \text{otherwise.} \end{cases}$$

Note that $d^H(P, P) = 0$ since $P = \mathcal{F}_u\left(H, 0\right)$. It will be shown in the sequel, that the smaller the size of $d^H(P, P_\Delta)$, the smaller the worst-case degradation of the stability margin due to a plant perturbation. Therefore, we will say that $d^H(P, P_\Delta)$ can be interpreted as the distance between two plants from a feedback perspective. [9]

Definition 5. *Given a plant $P \in \mathcal{R}^{p\times q}$, a generalized plant $H \in \mathcal{R}$ with $H_{22} = P$, and a perturbed plant $P_\Delta \in \mathcal{R}^{p\times q}$. Define*

$$\boldsymbol{\Delta}^{\min} := \begin{cases} \{\Delta \in \boldsymbol{\Delta} : \|\Delta\|_\infty = d^H(P, P_\Delta)\}, & \text{if } \boldsymbol{\Delta} \neq \emptyset \\ \emptyset, & \text{otherwise.} \end{cases}$$

The set $\boldsymbol{\Delta}^{\min}$ is the set of the smallest-sized $\Delta \in \mathcal{RL}_\infty$ that satisfy the consistency equation $P_\Delta = \mathcal{F}_u\left(H, \Delta\right)$. Note that $\boldsymbol{\Delta}^{\min} \subset \boldsymbol{\Delta}$ and that $\boldsymbol{\Delta} \neq \emptyset$ if, and only if, $\boldsymbol{\Delta}^{\min} \neq \emptyset$.

2.3 Robust stability

The following theorem gives a necessary and sufficient condition for the stability of a perturbed feedback system given a bound on the distance between the nominal and perturbed plants.

Theorem 1 (Robust Stability). *Given a plant $P \in \mathcal{R}^{p\times q}$, a stabilizable generalized plant $H \in \mathcal{R}$ with $H_{22} = P$, a perturbed plant $P_\Delta \in \mathcal{R}^{p\times q}$ and a controller $C \in \mathcal{R}^{q\times p}$ such that $d^H(P, P_\Delta) < b^H(P, C)$, then*

$$[P_\Delta, C] \text{ is internally stable} \Leftrightarrow$$
$$\exists \Delta \in \boldsymbol{\Delta}^{\min} : \eta(P_\Delta) = \eta(P) + \text{wno}\det(I - H_{11}\Delta), \quad (3)$$

[8] We are using min instead of inf in the definition of $d^H(P, P_\Delta)$ as we know that the minimizer belongs to the set $\boldsymbol{\Delta}$ for our use in Section 3. In general, one should use inf and adjust a few technical issues that arise.

[9] We have not yet explored whether $d^H(\cdot, \cdot)$ is a metric on $\mathcal{R}^{p\times q}$ or some subset of $\mathcal{R}$ for any uncertainty structure H, though it can be shown that in some cases it is a metric.

where the winding number is evaluated on a contour indented to the right around any imaginary axis poles of P and P_Δ.

Proof. If $d^H(P,P_\Delta) < b^H(P,C)$, then $d^H(P,P_\Delta) < \infty$ and $b^H(P,C) > 0$, and therefore by definition, $\boldsymbol{\Delta} \neq \emptyset$ and $[P,C]$ is internally stable. $\boldsymbol{\Delta} \neq \emptyset$ implies that $\boldsymbol{\Delta}^{\min} \neq \emptyset$.

$$\begin{aligned} & d^H(P,P_\Delta) < b^H(P,C) \\ \Rightarrow \quad & \|\Delta\|_\infty < \|\mathcal{F}_l\,(H,C)\|_\infty^{-1}, \quad \forall \Delta \in \boldsymbol{\Delta}^{\min} \\ \Rightarrow \quad & \|\Delta\,\mathcal{F}_l\,(H,C)\|_\infty < 1, \quad \forall \Delta \in \boldsymbol{\Delta}^{\min}. \end{aligned}$$

Therefore, for all $\Delta \in \boldsymbol{\Delta}^{\min}$

- $\Delta \in \mathcal{RL}_\infty$;
- $\|\Delta\,\mathcal{F}_l\,(H,C)\|_\infty < 1$;
- $(I - H_{11}\Delta)^{-1} \in \mathcal{R}$; and
- $P_\Delta = \mathcal{F}_u\,(H,\Delta)$.

The proof follows from [6, p. 45, Lemma 1.22]. ($\Rightarrow$) Since H is stabilizable and $[P,C]$ is internally stable, from Lemma 1.22 in [6]

$$\begin{aligned} & [P_\Delta, C] \text{ is internally stable} \Rightarrow \\ & \qquad \eta(P_\Delta) = \eta(P) + \operatorname{wno}\det(I - H_{11}\Delta), \quad \forall \Delta \in \boldsymbol{\Delta}^{\min}, \end{aligned}$$

where the winding number is evaluated on a contour indented to the right around any imaginary axis poles of P and P_Δ. ($\Leftarrow$) Also from Lemma 1.22 in [6],

$$\begin{aligned} & [P_\Delta, C] \text{ is internally stable} \Leftarrow \\ & \qquad \exists \Delta \in \boldsymbol{\Delta}^{\min} : \eta(P_\Delta) = \eta(P) + \operatorname{wno}\det(I - H_{11}\Delta), \end{aligned}$$

where the winding number is evaluated on the aforementioned contour.

The robust stability theorem also holds if, instead of supposing that $d^H(P,P_\Delta) < b^H(P,C)$, we suppose that $\mathcal{F}_l\,(H,C) \in \mathcal{RL}_\infty$, that $[P,C]$ is internally stable and that for all $\Delta \in \boldsymbol{\Delta}^{\min}$

$$\overline{\sigma}(\Delta(j\omega)) < \frac{1}{\overline{\sigma}(\mathcal{F}_l\,(H,C))(j\omega)}, \quad \forall \omega.$$

If $b^H(P,C) > \epsilon$ (≥ 0) and H is stabilizable, then $[P_\Delta, C]$ is internally stable for all P_Δ that belong to the set

$$\begin{aligned} \mathcal{P} = \big\{ & P_\Delta = \mathcal{F}_u\,(H,\Delta) : d^H(P,P_\Delta) \leq \epsilon,\ \Delta \in \mathcal{RL}_\infty, \\ & (I - H_{11}\Delta)^{-1} \in \mathcal{R},\ \eta(P_\Delta) = \eta(P) + \operatorname{wno}\det(I - H_{11}\Delta) \big\}. \end{aligned}$$

Therefore, a controller C that achieves a stability margin greater than ϵ with a nominal plant P, is guaranteed to stabilize all the plants in the set $\mathcal{P}$.

The robust stability theorem will be of practical use if, given a nominal plant P, a generalized plant H and a perturbed plant P_Δ, we can readily compute $d^H(P, P_\Delta)$ and rewrite the winding number condition in terms of only P, H and P_Δ, that is, independent of the perturbation $\Delta \in \boldsymbol{\Delta}^{\min}$.

What follows is a step-by-step procedure for computing $d^H(P, P_\Delta)$ and rewriting the winding number condition. Given $P \in \mathcal{R}^{p\times q}$, $H \in \mathcal{R}$ with $H_{22} = P$, and $P_\Delta \in \mathcal{R}^{p\times q}$:

1. Simplify $P_\Delta = \mathcal{F}_u(H, \Delta)$ assuming that $(I - H_{11}\Delta)$ is invertible in $\mathcal{R}$;
2. Parameterize the set $\boldsymbol{\Delta}$;
3. Compute $d^H(P, P_\Delta)$; and
4. Parameterize the set $\boldsymbol{\Delta}^{\min}$ and rewrite the winding number condition independent of $\Delta \in \boldsymbol{\Delta}^{\min}$.

Our experience from working with the generalized plants described in the next section suggests, that steps 1 and 3 are fairly straightforward but steps 2 and 4 can be much harder. [10] In Section 3, we will apply our procedure for a particular choice H.

2.4 Generalized plants considered

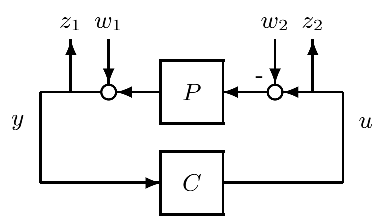

Fig. 3. The left 4-block interconnection.

Given a plant $P \in \mathcal{R}$ and a controller $C \in \mathcal{R}$. Consider the feedback interconnection illustrated in Fig. 3 that will be referred to as the left 4-block interconnection. From Fig. 3,

$$\begin{bmatrix} z_1 \\ z_2 \\ \hline y \end{bmatrix} = \left[\begin{array}{cc|c} I & -P & P \\ 0 & 0 & I \\ \hline I & -P & P \end{array}\right] \begin{bmatrix} w_1 \\ w_2 \\ \hline u \end{bmatrix}.$$

In order to pin down the technical machinery, in the remainder of this paper, we will only consider generalized plants that can be derived from Fig. 3. Let

[10] There is evidence that $d^H(P, P_\Delta)$ can also be computed if $\boldsymbol{\Delta}$ is a structured uncertainty set. This will be reported in a subsequent paper.

$$z = S_z \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}, \quad \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = S_w \, w,$$

where $S_w, S_z \in \mathcal{R}$ select and possibly filter the inputs and outputs of interest,

$$H = \left[\begin{array}{c|c} S_z & \\ \hline & I \end{array}\right] \left[\begin{array}{cc|c} I & -P & P \\ 0 & 0 & I \\ \hline I & -P & P \end{array}\right] \left[\begin{array}{c|c} S_w & \\ \hline & I \end{array}\right] \tag{4}$$

and

$$w = \Delta z,$$

where $\Delta \in \mathcal{R}$.

If $\left(I - S_z \begin{bmatrix} I & -P \\ 0 & 0 \end{bmatrix} S_w \Delta \right)^{-1} \in \mathcal{R}$, then $P_\Delta \in \mathcal{R}$ and

$$\begin{aligned}
& P_\Delta = \mathcal{F}_u (H, \Delta) \\
\Leftrightarrow \quad & P_\Delta = P + \begin{bmatrix} I & -P \end{bmatrix} S_w \Delta \left(I - S_z \begin{bmatrix} I & -P \\ 0 & 0 \end{bmatrix} S_w \Delta \right)^{-1} S_z \begin{bmatrix} P \\ I \end{bmatrix} \\
\Leftrightarrow \quad & P_\Delta - P = \begin{bmatrix} I & -P \end{bmatrix} S_w \Delta \left(I - S_z \begin{bmatrix} I \\ 0 \end{bmatrix} \begin{bmatrix} I & -P \end{bmatrix} S_w \Delta \right)^{-1} S_z \begin{bmatrix} P \\ I \end{bmatrix} \\
\Leftrightarrow \quad & P_\Delta - P = \left(I - \begin{bmatrix} I & -P \end{bmatrix} S_w \Delta \, S_z \begin{bmatrix} I \\ 0 \end{bmatrix} \right)^{-1} \begin{bmatrix} I & -P \end{bmatrix} S_w \Delta S_z \begin{bmatrix} P \\ I \end{bmatrix} \\
\Leftrightarrow \quad & \left(I - \begin{bmatrix} I & -P \end{bmatrix} S_w \Delta \, S_z \begin{bmatrix} I \\ 0 \end{bmatrix} \right) (P_\Delta - P) = \begin{bmatrix} I & -P \end{bmatrix} S_w \Delta S_z \begin{bmatrix} P \\ I \end{bmatrix} \\
\Leftrightarrow \quad & P_\Delta - P = \begin{bmatrix} I & -P \end{bmatrix} S_w \Delta S_z \left(\begin{bmatrix} P \\ I \end{bmatrix} + \begin{bmatrix} P_\Delta - P \\ 0 \end{bmatrix} \right) \\
\Leftrightarrow \quad & P_\Delta - P = \begin{bmatrix} I & -P \end{bmatrix} S_w \Delta \, S_z \begin{bmatrix} P_\Delta \\ I \end{bmatrix}.
\end{aligned} \tag{5}$$

Also, if $(I - PC)^{-1} \in \mathcal{R}$, then $\mathcal{F}_l (H, C) \in \mathcal{R}$ and

$$
\begin{aligned}
&\mathcal{F}_l(H, C) \\
&= S_z \begin{bmatrix} I & -P \\ 0 & 0 \end{bmatrix} S_w + S_z \begin{bmatrix} P \\ I \end{bmatrix} C(I - PC)^{-1} \begin{bmatrix} I & -P \end{bmatrix} S_w \\
&= S_z \begin{bmatrix} I \\ 0 \end{bmatrix} \begin{bmatrix} I & -P \end{bmatrix} S_w + S_z \begin{bmatrix} P \\ I \end{bmatrix} C(I - PC)^{-1} \begin{bmatrix} I & -P \end{bmatrix} S_w \\
&= S_z \left(\begin{bmatrix} I \\ 0 \end{bmatrix} + \begin{bmatrix} P \\ I \end{bmatrix} C(I - PC)^{-1} \right) \begin{bmatrix} I & -P \end{bmatrix} S_w \\
&= S_z \left(\begin{bmatrix} I - PC \\ 0 \end{bmatrix} + \begin{bmatrix} PC \\ C \end{bmatrix} \right) (I - PC)^{-1} \begin{bmatrix} I & -P \end{bmatrix} S_w \\
&= S_z \begin{bmatrix} I \\ C \end{bmatrix} (I - PC)^{-1} \begin{bmatrix} I & -P \end{bmatrix} S_w. \qquad (6)
\end{aligned}
$$

Uncertainty structure	S_w	S_z
Output inverse multiplicative	$\begin{bmatrix} I \\ 0 \end{bmatrix}$	$\begin{bmatrix} I & 0 \end{bmatrix}$
Additive	$\begin{bmatrix} I \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 & I \end{bmatrix}$
Inverse additive	$\begin{bmatrix} 0 \\ I \end{bmatrix}$	$\begin{bmatrix} I & 0 \end{bmatrix}$
Input multiplicative	$\begin{bmatrix} 0 \\ I \end{bmatrix}$	$\begin{bmatrix} 0 & I \end{bmatrix}$
Output inverse multiplicative and additive	$\begin{bmatrix} I \\ 0 \end{bmatrix}$	I
Inverse additive and input multiplicative	$\begin{bmatrix} 0 \\ I \end{bmatrix}$	I
Output inverse multiplicative and inverse additive	I	$\begin{bmatrix} I & 0 \end{bmatrix}$
Additive and input multiplicative	I	$\begin{bmatrix} 0 & I \end{bmatrix}$
Left 4-block	I	I
Left coprime	$\begin{bmatrix} (R\tilde{M})^{-1} \\ 0 \end{bmatrix}$	I

Table 1. Uncertainty structures considered. $\{\tilde{N}, \tilde{M}\}$ is a normalized left coprime factorization of P and $R \in \mathfrak{G}\mathcal{H}_\infty$.

Table 1 lists the uncertainty structures that will be considered. The theory and tools developed in this paper readily apply to *any* uncertainty structure that can be derived from the left and right 4-block interconnections. [11]

2.5 Robust performance

The robust stability theorem gives a necessary and sufficient condition for the stability of a perturbed feedback system. In addition to checking whether $[P_\Delta, C]$ is stable, an engineer typically wishes to know the residual stability margin $b^{H_\Delta}(P_\Delta, C)$, where H_Δ is the generalized plant that corresponds to the perturbed plant P_Δ. We will now derive an upper and a lower bound for the stability margin of a perturbed feedback system, and an upper bound for $\|\mathcal{F}_l(H_\Delta, C) - \mathcal{F}_l(H, C)\|_\infty$ which is a measure of the difference in performance between the nominal and perturbed feedback systems.

To derive the bounds, we will need the following lemma.

Lemma 1. *Given a nominal plant $P \in \mathcal{R}^{p\times q}$, a generalized plant*

$$H = \left[\begin{array}{c|c} S_z & \\ \hline & I \end{array}\right] \left[\begin{array}{cc|c} I & -P & P \\ 0 & 0 & I \\ \hline I & -P & P \end{array}\right] \left[\begin{array}{c|c} S_w & \\ \hline & I \end{array}\right],$$

where $S_w, S_z \in \mathcal{R}$, a controller $C \in \mathcal{R}^{q\times p}$ and a perturbation $\Delta \in \mathcal{R}$ such that

- $(I - PC)^{-1} \in \mathcal{R}$;
- $(I - \mathcal{F}_l(H, C)\,\Delta)^{-1} \in \mathcal{R}$; *and*
- $(I - H_{11}\Delta)^{-1} \in \mathcal{R}$.

Let $P_\Delta = \mathcal{F}_u(H, \Delta)$, $S_{w_\Delta} = S_w(I - k\Delta S_z S_w)^{-1} \in \mathcal{R}$ for a given $k \in \{0, 1\}$,

$$H_\Delta = \left[\begin{array}{c|c} S_z & \\ \hline & I \end{array}\right] \left[\begin{array}{cc|c} I & -P_\Delta & P_\Delta \\ 0 & 0 & I \\ \hline I & -P_\Delta & P_\Delta \end{array}\right] \left[\begin{array}{c|c} S_{w_\Delta} & \\ \hline & I \end{array}\right]$$

and $S = (1-k)S_z S_w$, then

$$\mathcal{F}_l(H_\Delta, C) = \mathcal{F}_l(H, C) + \mathcal{F}_l(H, C)\,\Delta(\mathcal{F}_l(H_\Delta, C) - S).$$

[11] The generalized plant of the right 4-block interconnection is equal to $\left[\begin{array}{cc|c} 0 & P & P \\ 0 & I & I \\ \hline -I & P & P \end{array}\right]$ and can be derived from Fig. 1 with $\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} y \\ u \end{bmatrix}$. Hence, it is a dual problem and can be treated easily using similar techniques.

Proof. First consider the case $k = 0$. We will only prove the case when $S_w = I$ and $S_z = I$ (that is, left 4-block uncertainty) for simplicity, as the proof in the general case is identical (with S_w and S_z carried on either side to obtain the required result).

$$\begin{aligned}
H_\Delta &= \begin{bmatrix} I & -P_\Delta & P_\Delta \\ 0 & 0 & I \\ I & -P_\Delta & P_\Delta \end{bmatrix} = \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & I \\ I & 0 & 0 \end{bmatrix} + \begin{bmatrix} I \\ 0 \\ I \end{bmatrix} P_\Delta \begin{bmatrix} 0 & -I & I \end{bmatrix} \\
&= \mathcal{F}_u \left(\left[\begin{array}{c|ccc} 0 & 0 & -I & I \\ \hline I & I & 0 & 0 \\ 0 & 0 & 0 & I \\ I & I & 0 & 0 \end{array}\right], \mathcal{F}_u(H, \Delta) \right) = \mathcal{F}_u \left(H \star \left[\begin{array}{c|ccc} 0 & 0 & -I & I \\ \hline I & I & 0 & 0 \\ 0 & 0 & 0 & I \\ I & I & 0 & 0 \end{array}\right], \Delta \right) \\
&= \mathcal{F}_u \left(\underbrace{\left[\begin{array}{cc|ccc} I & -P & 0 & -P & P \\ 0 & 0 & 0 & -I & I \\ \hline I & -P & I & -P & P \\ 0 & 0 & 0 & 0 & I \\ I & -P & I & -P & P \end{array}\right]}_{J}, \Delta \right).
\end{aligned}$$

By expanding $\left(I - H \begin{bmatrix} \Delta & 0 \\ 0 & C \end{bmatrix} \right)$ in terms of its Schur complements,

$$\begin{aligned}
\det \left(I - H \begin{bmatrix} \Delta & 0 \\ 0 & C \end{bmatrix} \right) &= \det(I - H_{11}\Delta) \det(I - P_\Delta C) \\
&= \det(I - PC) \det(I - \mathcal{F}_l(H, C)\Delta).
\end{aligned}$$

Since $(I - PC)$, $(I - \mathcal{F}_l(H, C)\Delta)$ and $(I - H_{11}\Delta)$ are invertible in $\mathcal{R}$, it follows that $\det(I - P_\Delta C)(\infty) \neq 0$ and thus $(I - P_\Delta C)$ is also invertible in $\mathcal{R}$.

$$\begin{aligned}
\mathcal{F}_l(H_\Delta, C) &= \mathcal{F}_l(\mathcal{F}_u(J,\Delta), C) = \mathcal{F}_u(\mathcal{F}_l(J,C),\Delta) \\
&= \mathcal{F}_u\left(\mathcal{F}_l\left(\left[\begin{array}{cc|c} 0 & -I & I \\ 0 & 0 & I \\ \hline I & I & 0 \end{array}\right] \star H, C\right), \Delta\right) \\
&= \mathcal{F}_l\left(\mathcal{F}_u\left(\left[\begin{array}{c|cc} 0 & -I & I \\ \hline 0 & 0 & I \\ I & I & 0 \end{array}\right], \Delta\right), \mathcal{F}_l(H,C)\right) \\
&= \mathcal{F}_l\left(\left[\begin{array}{c|c} 0 & I \\ \hline I-\Delta & \Delta \end{array}\right], \mathcal{F}_l(H,C)\right) \\
&= \mathcal{F}_l(H,C)(I-\Delta\mathcal{F}_l(H,C))^{-1}(I-\Delta) \\
&= (I-\mathcal{F}_l(H,C)\Delta)^{-1}\mathcal{F}_l(H,C)(I-\Delta).
\end{aligned}$$

Therefore, $\mathcal{F}_l(H_\Delta, C) \in \mathcal{R}$ and

$$\begin{aligned}
& \mathcal{F}_l(H_\Delta, C) = (I-\mathcal{F}_l(H,C)\Delta)^{-1}\mathcal{F}_l(H,C)(I-\Delta) \\
\Leftrightarrow \quad & (I-\mathcal{F}_l(H,C)\Delta)\mathcal{F}_l(H_\Delta, C) = \mathcal{F}_l(H,C)(I-\Delta) \\
\Leftrightarrow \quad & \mathcal{F}_l(H_\Delta, C) = \mathcal{F}_l(H,C) + \mathcal{F}_l(H,C)\Delta(\mathcal{F}_l(H_\Delta, C) - I).
\end{aligned}$$

This concludes the proof for the first case.

Now consider the case $k = 1$. From the first case,

$$S_z\mathcal{F}_l\left(\left[\begin{array}{cc|c} I & -P_\Delta & P_\Delta \\ 0 & 0 & I \\ \hline I & -P_\Delta & P_\Delta \end{array}\right], C\right) S_w = \mathcal{F}_l(H,C) + \mathcal{F}_l(H,C)\Delta\left(S_z\mathcal{F}_l\left(\left[\begin{array}{cc|c} I & -P_\Delta & P_\Delta \\ 0 & 0 & I \\ \hline I & -P_\Delta & P_\Delta \end{array}\right], C\right) S_w - S_z S_w\right)$$

which can be rearranged as

$$S_z\mathcal{F}_l\left(\left[\begin{array}{cc|c} I & -P_\Delta & P_\Delta \\ 0 & 0 & I \\ \hline I & -P_\Delta & P_\Delta \end{array}\right], C\right) S_w = \mathcal{F}_l(H,C)(I-\Delta S_z S_w) + \mathcal{F}_l(H,C)\Delta S_z\mathcal{F}_l\left(\left[\begin{array}{cc|c} I & -P_\Delta & P_\Delta \\ 0 & 0 & I \\ \hline I & -P_\Delta & P_\Delta \end{array}\right], C\right) S_w.$$

The result follows by multiplying both sides of the above equation by $(I - \Delta S_z S_w)^{-1}$ and noting that in this case

$$\mathcal{F}_l(H_\Delta, C) = S_z\mathcal{F}_l\left(\left[\begin{array}{cc|c} I & -P_\Delta & P_\Delta \\ 0 & 0 & I \\ \hline I & -P_\Delta & P_\Delta \end{array}\right], C\right) S_w (I-\Delta S_z S_w)^{-1}.$$

The following theorem bounds the stability margin of a perturbed feedback system and the performance difference between the nominal and perturbed feedback systems.

Theorem 2. *[Robust Performance] Given a nominal plant $P \in \mathcal{R}^{p\times q}$, a stabilizable generalized plant*

$$H = \left[\begin{array}{c|c} S_z & \\ \hline & I \end{array}\right] \left[\begin{array}{cc|c} I & -P & P \\ 0 & 0 & I \\ \hline I & -P & P \end{array}\right] \left[\begin{array}{c|c} S_w & \\ \hline & I \end{array}\right],$$

where $S_w, S_z \in \mathcal{R}$, a perturbed plant $P_\Delta \in \mathcal{R}^{p\times q}$ and a controller $C \in \mathcal{R}^{q\times p}$ such that $d^H(P, P_\Delta) < b^H(P, C)$. Assume furthermore that there exists a $\Delta \in \boldsymbol{\Delta}^{\min}$ that satisfies

$$\eta(P_\Delta) = \eta(P) + \text{wno}\det(I - H_{11}\Delta),$$

where the winding number is evaluated on a contour indented to the right around any imaginary axis poles of P and P_Δ. Let $S_{w_\Delta} = S_w(I - k\Delta S_z S_w)^{-1} \in \mathcal{R}$ for a given $k \in \{0, 1\}$,

$$H_\Delta = \left[\begin{array}{c|c} S_z & \\ \hline & I \end{array}\right] \left[\begin{array}{cc|c} I & -P_\Delta & P_\Delta \\ 0 & 0 & I \\ \hline I & -P_\Delta & P_\Delta \end{array}\right] \left[\begin{array}{c|c} S_{w_\Delta} & \\ \hline & I \end{array}\right]$$

and $S = (1-k)S_z S_w$.

Then the following results hold when $S \in \mathcal{RL}_\infty$:

1. $\mathcal{F}_l(H_\Delta, C) \in \mathcal{RL}_\infty$ and $[P_\Delta, C]$ is internally stable

2.

$$\left|b^{H_\Delta}(P_\Delta, C) - b^H(P, C)\right| \le \|\mathcal{F}_l(H_\Delta, C) - S\|_\infty\, b^{H_\Delta}(P_\Delta, C)\, d^H(P, P_\Delta)$$

3.

$$\|\mathcal{F}_l(H_\Delta, C) - \mathcal{F}_l(H, C)\|_\infty \le \frac{\|\mathcal{F}_l(H_\Delta, C) - S\|_\infty\, d^H(P, P_\Delta)}{b^H(P, C)}.$$

Proof. We will only consider the case $k = 0$, as the proof for the other case is identical.

If $d^H(P, P_\Delta) < b^H(P, C)$, then $d^H(P, P_\Delta) < \infty$ and $b^H(P, C) > 0$, and therefore by definition, $\boldsymbol{\Delta} \neq \emptyset$, $\mathcal{F}_l(H, C) \in \mathcal{RL}_\infty$ and $[P, C]$ is internally stable. $\boldsymbol{\Delta} \neq \emptyset$ implies that $\boldsymbol{\Delta}^{\min} \neq \emptyset$.

$$\begin{aligned}
& d^H(P, P_\Delta) < b^H(P, C) \\
\Rightarrow\quad & \|\Delta\mathcal{F}_l(H, C)\|_\infty < 1, \quad \forall\Delta \in \boldsymbol{\Delta}^{\min} \\
\Leftrightarrow\quad & 0 < 1 - \overline{\sigma}(\Delta\mathcal{F}_l(H, C))(j\omega), \quad \forall\omega,\ \forall\Delta \in \boldsymbol{\Delta}^{\min} \\
\Rightarrow\quad & 0 < \underline{\sigma}(I - \Delta\mathcal{F}_l(H, C))(j\omega), \quad \forall\omega,\ \forall\Delta \in \boldsymbol{\Delta}^{\min} \\
\Leftrightarrow\quad & \det(I - \Delta\mathcal{F}_l(H, C))(j\omega) \neq 0, \quad \forall\omega,\ \forall\Delta \in \boldsymbol{\Delta}^{\min}.
\end{aligned}$$

Thus, $(I - \Delta\,\mathcal{F}_l\,(H,C))^{-1} \in \mathcal{RL}_\infty$ for all $\Delta \in \boldsymbol{\Delta}^{\min}$. Since internal stability of $[P,C]$ implies that $(I-PC)^{-1} \in \mathcal{RH}_\infty$ and $\Delta \in \boldsymbol{\Delta}^{\min}$ implies that $(I - H_{11}\Delta)^{-1} \in \mathcal{R}$, from Lemma 1 and for all $\Delta \in \boldsymbol{\Delta}^{\min}$,

$$\mathcal{F}_l\,(H_\Delta,C) = \mathcal{F}_l\,(H,C) + \mathcal{F}_l\,(H,C)\,\Delta(\mathcal{F}_l\,(H_\Delta,C) - S_zS_w) \tag{7}$$
$$\Leftrightarrow \quad (I - \mathcal{F}_l\,(H,C)\,\Delta)\mathcal{F}_l\,(H_\Delta,C) = \mathcal{F}_l\,(H,C)\,(I - \Delta S_zS_w)$$
$$\Leftrightarrow \quad \mathcal{F}_l\,(H_\Delta,C) = \mathcal{F}_l\,(H,C)\,(I - \Delta\mathcal{F}_l\,(H,C))^{-1}(I - \Delta\,S_zS_w). \tag{8}$$

Since $\Delta, (I - \Delta\,\mathcal{F}_l\,(H,C))^{-1} \in \mathcal{RL}_\infty$ for all $\Delta \in \boldsymbol{\Delta}^{\min}$ and $b^H(P,C) > 0$ implies that $\mathcal{F}_l\,(H,C) \in \mathcal{RL}_\infty$, if $S_zS_w \in \mathcal{RL}_\infty$, then from (8), $\mathcal{F}_l\,(H_\Delta,C) \in \mathcal{RL}_\infty$ for all $\Delta \in \boldsymbol{\Delta}^{\min}$.

From Theorem 1 and because by supposition there exists a $\Delta \in \boldsymbol{\Delta}^{\min}$ that satisfies the winding number condition, $[P_\Delta, C]$ is internally stable. This concludes the proof of part (a).

Inequality (b) follows from the properties of the $\mathcal{L}_\infty$-norm and (7).

$$\begin{aligned}
& \|\mathcal{F}_l\,(H_\Delta,C)\|_\infty \le \|\mathcal{F}_l\,(H,C)\|_\infty + \\
& \quad \|\mathcal{F}_l\,(H,C)\|_\infty \|\Delta\|_\infty \|\mathcal{F}_l\,(H_\Delta,C) - S_zS_w\|_\infty \\
\Leftrightarrow \quad & \|\mathcal{F}_l\,(H,C)\|_\infty^{-1} \le \|\mathcal{F}_l\,(H_\Delta,C)\|_\infty^{-1} + \\
& \|\mathcal{F}_l\,(H_\Delta,C) - S_zS_w\|_\infty \|\Delta\|_\infty \|\mathcal{F}_l\,(H_\Delta,C)\|_\infty^{-1} \\
& \text{(dividing both sides by } \|\mathcal{F}_l\,(H,C)\|_\infty \|\mathcal{F}_l\,(H_\Delta,C)\|_\infty) \\
\Leftrightarrow \quad & -\|\mathcal{F}_l\,(H_\Delta,C) - S_zS_w\|_\infty \|\Delta\|_\infty \|\mathcal{F}_l\,(H_\Delta,C)\|_\infty^{-1} \le \\
& \|\mathcal{F}_l\,(H_\Delta,C)\|_\infty^{-1} - \|\mathcal{F}_l\,(H,C)\|_\infty^{-1}.
\end{aligned}$$

Similarly,

$$\|\mathcal{F}_l\,(H_\Delta,C)\|_\infty^{-1} - \|\mathcal{F}_l\,(H,C)\|_\infty^{-1} \le \|\mathcal{F}_l\,(H_\Delta,C) - S_zS_w\|_\infty \|\Delta\|_\infty \|\mathcal{F}_l\,(H_\Delta,C)\|_\infty^{-1},$$

and therefore,

$$\left|\|\mathcal{F}_l\,(H_\Delta,C)\|_\infty^{-1} - \|\mathcal{F}_l\,(H,C)\|_\infty^{-1}\right| \le \|\mathcal{F}_l\,(H_\Delta,C) - S_zS_w\|_\infty \|\Delta\|_\infty \|\mathcal{F}_l\,(H_\Delta,C)\|_\infty^{-1}.$$

Since $b^H(P,C) > 0$, $\|\mathcal{F}_l\,(H,C)\|_\infty = 1/b^H(P,C)$ and since $\Delta \in \boldsymbol{\Delta}^{\min}$, $\|\Delta\|_\infty = d^H(P,P_\Delta)$. The inequality follows by noting that since $\mathcal{F}_l\,(H_\Delta,C) \in \mathcal{RL}_\infty$ and $[P_\Delta,C]$ is internally stable, $\|\mathcal{F}_l\,(H_\Delta,C)\|_\infty = 1/b^{H_\Delta}(P_\Delta,C)$.

We will now prove inequality (c). Again from the properties of the $\mathcal{L}_\infty$-norm and (7),

$$\|\mathcal{F}_l\,(H_\Delta,C) - \mathcal{F}_l\,(H,C)\|_\infty \le \|\mathcal{F}_l\,(H,C)\|_\infty \|\Delta\|_\infty \|\mathcal{F}_l\,(H_\Delta,C) - S_zS_w\|_\infty.$$

The result then follows.

Note that it is straightforward to derive analogues of both inequalities that are valid at each frequency.

If $\|\mathcal{F}_l(H_\Delta, C) - S\|_\infty b^{H_\Delta}(P_\Delta, C) \leq 1$, as is the case for additive, inverse additive, left 4-block and coprime factor uncertainties, we can state the following corollary which is a more concise version of the robust performance theorem.

Corollary 1. *For any nominal plant $P \in \mathcal{R}^{p\times q}$ with a stabilizable and detectable state-space realization, perturbed plant $P_\Delta \in \mathcal{R}^{p\times q}$ and controller $C \in \mathcal{R}^{q\times p}$. Let*

- $S_w = \begin{bmatrix} W_1 \\ 0 \end{bmatrix}$, $S_z = \begin{bmatrix} 0 & W_2 \end{bmatrix}$ *and* $S_{w_\Delta} = S_w$, *where* $W_1, W_2 \in \mathcal{RL}_\infty$ *(that is, additive uncertainty); or*
- $S_w = \begin{bmatrix} 0 \\ W_1 \end{bmatrix}$, $S_z = \begin{bmatrix} W_2 & 0 \end{bmatrix}$ *and* $S_{w_\Delta} = S_w$, *where* $W_1, W_2 \in \mathcal{RL}_\infty$ *(that is, inverse additive uncertainty); or*
- $S_w = I$, $S_z = I$ *and* $S_{w_\Delta} = S_w$ *(that is, left 4-block uncertainty); or*
- $S_w = \begin{bmatrix} (R\tilde{M})^{-1} \\ 0 \end{bmatrix}$, $S_z = I$ *and* $S_{w_\Delta} = S_w(I - \Delta S_z S_w)^{-1}$ *(that is, coprime factor uncertainty),*

$$H = \left[\begin{array}{c|c} S_z & \\ \hline & I \end{array}\right] \left[\begin{array}{cc|c} I & -P & P \\ 0 & 0 & I \\ \hline I & -P & P \end{array}\right] \left[\begin{array}{c|c} S_w & \\ \hline & I \end{array}\right]$$

and

$$H_\Delta = \left[\begin{array}{c|c} S_z & \\ \hline & I \end{array}\right] \left[\begin{array}{cc|c} I & -P_\Delta & P_\Delta \\ 0 & 0 & I \\ \hline I & -P_\Delta & P_\Delta \end{array}\right] \left[\begin{array}{c|c} S_{w_\Delta} & \\ \hline & I \end{array}\right].$$

If

$$\exists \Delta \in \boldsymbol{\Delta}^{\min} : \eta(P_\Delta) = \eta(P) + \operatorname{wno}\det(I - H_{11}\Delta),$$

where the winding number is evaluated on a contour indented to the right around any imaginary axis poles of P and P_Δ, then

$$b^{H_\Delta}(P_\Delta, C) \geq b^H(P, C) - d^H(P, P_\Delta). \tag{9}$$

Furthermore, when $b^H(P, C) > d^H(P, P_\Delta)$,

$$\|\mathcal{F}_l(H_\Delta, C) - \mathcal{F}_l(H, C)\|_\infty \leq \frac{d^H(P, P_\Delta)}{b^H(P, C) b^{H_\Delta}(P_\Delta, C)}. \tag{10}$$

Proof. First suppose that $b^H(P, C) \leq d^H(P, P_\Delta)$. Then the right-hand-side of inequality (9) is ≤ 0. But from Definition 3, $b^{H_\Delta}(P_\Delta, C) \geq 0$ and hence (9) is automatically true.

Now suppose that $b^H(P,C) > d^H(P,P_\Delta)$. The proof follows from Theorem 2, noting that if P has a stabilizable and detectable state-space realization, then H is stabilizable, and that in the case of coprime factor uncertainty

$$S_{w_\Delta} = S_w(I - \Delta S_z S_w)^{-1} = \begin{bmatrix}(R\tilde{M})^{-1} \\ 0\end{bmatrix}\left(I - \Delta\begin{bmatrix}(R\tilde{M})^{-1} \\ 0\end{bmatrix}\right)^{-1}$$

belongs to $\mathcal{R}$ for all $\Delta \in \boldsymbol{\Delta}^{\min}$ because by definition

$$(I - H_{11}\Delta)^{-1} = \left(I - \begin{bmatrix}(R\tilde{M})^{-1} \\ 0\end{bmatrix}\Delta\right)^{-1}$$

belongs to $\mathcal{R}$ for all $\Delta \in \boldsymbol{\Delta}^{\min}$. Inequalities (9) and (10) immediately follow from conditions (b) and (c) of Theorem 2 on noting that for each of the cases considered $\|\mathcal{F}_l(H_\Delta, C) - S\|_\infty b^{H_\Delta}(P_\Delta, C) \le 1$.

3 Left 4-block uncertainty characterization with a left 4-block performance measure

In this section, we illustrate the novel generic procedure described in Section 2 on a left 4-block uncertainty characterization with a left 4-block performance measure and derive results specific for this case. The results for this particular (and only this particular) case have also been derived via other technical machinery in [5, 6]. Consequently, in this section, we are illustrating that the generic procedure of Section 2 reproduces existing results in a simple systematic manner. We shall indicate in Section 4 that new results, generated by our novel procedure of Section 2 on different uncertainty structures and performance measures, are also possible.

Towards this end, consider Figure 3 again and note that for this particular case $S_w = I$ and $S_z = I$, as also seen in Table 1.

3.1 Define the robust stability margin $b(P,C)$

Plugging in $S_w = I$ and $S_z = I$ in equation (4) gives

$$H = \left[\begin{array}{cc|c} I & -P & P \\ 0 & 0 & I \\ \hline I & -P & P \end{array}\right]$$

and hence

$$\mathcal{F}_l(H,C) = \begin{bmatrix} I \\ C \end{bmatrix}(I - PC)^{-1}\begin{bmatrix} I & -P \end{bmatrix}.$$

Consequently, straight from Definition 3, we see that the stability margin $b(P,C)$ for left 4-block uncertainty characterizations is given by:

$$b(P,C) := \begin{cases} \left\| \begin{bmatrix} I \\ C \end{bmatrix} (I - PC)^{-1} \begin{bmatrix} I & -P \end{bmatrix} \right\|_\infty^{-1} & \text{if } [P,C] \text{ is internally stable,} \\ 0 & \text{otherwise.} \end{cases} \tag{11}$$

In this specific case, $\mathcal{F}_l(H,C) \in \mathcal{RL}_\infty$ is dropped in (11) since $[P,C]$ is internally stable automatically guarantees this.

3.2 When is the given data consistent and what are all solutions?

Given a nominal plant P and a perturbed plant P_Δ, we would now like to derive necessary and sufficient conditions for there to exist a $\Delta \in \mathcal{RL}_\infty$ satisfying

$$P_\Delta = \mathcal{F}_u(H,\Delta).$$

That is, we would like to know when the given P, P_Δ data is consistent with the uncertainty characterization considered, expressed via generalized plant H. Furthermore, provided these necessary and sufficient conditions are satisfied, we would like to parameterize all solutions Δ that fit the given P, P_Δ data.

For this specific case, plugging in $S_w = I$ and $S_z = I$ in equation (5) gives

$$\begin{aligned} & P_\Delta = \mathcal{F}_u(H,\Delta) \\ \Leftrightarrow\quad & P_\Delta - P = \begin{bmatrix} I & -P \end{bmatrix} \Delta \begin{bmatrix} P_\Delta \\ I \end{bmatrix} \end{aligned}$$

which can then be equivalently rearranged into

$$\begin{aligned} \Leftrightarrow\quad & P - P_\Delta = \begin{bmatrix} -I & P \end{bmatrix} \Delta \begin{bmatrix} P_\Delta \\ I \end{bmatrix} \\ \Leftrightarrow\quad & \tilde{M}(P - P_\Delta)M_\Delta = \tilde{G}\Delta G_\Delta \\ \Leftrightarrow\quad & \tilde{G}G_\Delta = \tilde{G}\Delta G_\Delta \qquad (12) \\ \Leftrightarrow\quad & \tilde{G}G_\Delta = \tilde{G}\begin{bmatrix} \tilde{G}^* & G \end{bmatrix} \begin{bmatrix} \tilde{G} \\ G^* \end{bmatrix} \Delta \begin{bmatrix} \tilde{G}_\Delta^* & G_\Delta \end{bmatrix} \begin{bmatrix} \tilde{G}_\Delta \\ G_\Delta^* \end{bmatrix} G_\Delta \\ \Leftrightarrow\quad & \tilde{G}G_\Delta = \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} \tilde{G} \\ G^* \end{bmatrix} \Delta \begin{bmatrix} \tilde{G}_\Delta^* & G_\Delta \end{bmatrix} \begin{bmatrix} 0 \\ I \end{bmatrix} \\ \Leftrightarrow\quad & \Delta = \begin{bmatrix} \tilde{G}^* & G \end{bmatrix} \begin{bmatrix} Q_1 & \tilde{G}G_\Delta \\ Q_2 & Q_3 \end{bmatrix} \begin{bmatrix} \tilde{G}_\Delta \\ G_\Delta^* \end{bmatrix} \qquad (13) \end{aligned}$$

where $Q_i \in \mathcal{RL}_\infty$ are arbitrary.

Consequently, for this specific case, the given P, P_Δ data is *always consistent* with the uncertainty characterization considered, as there always exists at least one solution for Δ (assuming that $(I-H_{11}\Delta)^{-1} \in \mathcal{R}$). Also, given any P, P_Δ pair, there in fact always exist multiple solutions for $\Delta \in \mathcal{RL}_\infty$, and the non-uniqueness in the solution Δ is parameterized above by the arbitrary objects $Q_i \in \mathcal{RL}_\infty$.

3.3 Define the solution set $\boldsymbol{\Delta}$

Since $P_\Delta = \mathcal{F}_u(H, \Delta)$ is well-posed when $\det(I - H_{11}\Delta)(\infty) \neq 0$, we need to express $\det(I - H_{11}\Delta)(\infty)$ independently of Δ. To this end, observe that:

$$
\begin{aligned}
&\det(I - H_{11}\Delta) \\
&= \det\left(I - \begin{bmatrix} I & -P \\ 0 & 0 \end{bmatrix} \Delta\right) \\
&= \det\left(I + \begin{bmatrix} I \\ 0 \end{bmatrix} \begin{bmatrix} -I & P \end{bmatrix} \Delta\right) \\
&= \det\left(I + \begin{bmatrix} \tilde{M}^{-1} \\ 0 \end{bmatrix} \tilde{G} \begin{bmatrix} \tilde{G}^* & G \end{bmatrix} \begin{bmatrix} Q_1 & \tilde{G}G_\Delta \\ Q_2 & Q_3 \end{bmatrix} \begin{bmatrix} \tilde{G}_\Delta \\ G_\Delta^* \end{bmatrix}\right) \\
&= \det\left(I + \begin{bmatrix} \tilde{M}^{-1} \\ 0 \end{bmatrix} \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} Q_1 & \tilde{G}G_\Delta \\ Q_2 & Q_3 \end{bmatrix} \begin{bmatrix} \tilde{G}_\Delta \\ G_\Delta^* \end{bmatrix}\right) \\
&= \det\left(I + \begin{bmatrix} \tilde{G}_\Delta \\ G_\Delta^* \end{bmatrix} \begin{bmatrix} \tilde{M}^{-1} \\ 0 \end{bmatrix} \begin{bmatrix} Q_1 & \tilde{G}G_\Delta \end{bmatrix}\right) \qquad (14) \\
&= \det\left(I + \begin{bmatrix} \tilde{G}_\Delta \\ G_\Delta^* \end{bmatrix} \begin{bmatrix} \begin{pmatrix} \tilde{M}^{-1}Q_1 \\ 0 \end{pmatrix} & (GM^{-1}M_\Delta - G_\Delta) \end{bmatrix}\right) \\
&= \det\left(\begin{bmatrix} \tilde{G}_\Delta \\ G_\Delta^* \end{bmatrix} \begin{bmatrix} \tilde{G}_\Delta^* + \begin{pmatrix} \tilde{M}^{-1}Q_1 \\ 0 \end{pmatrix} & GM^{-1}M_\Delta \end{bmatrix}\right). \qquad (15)
\end{aligned}
$$

Consequently,

$$
\det(I - H_{11}\Delta)(\infty) \neq 0 \quad \Leftrightarrow \quad \det\left(\begin{bmatrix} \tilde{G}_\Delta^* + \begin{pmatrix} \tilde{M}^{-1}Q_1 \\ 0 \end{pmatrix} & G \end{bmatrix}\right)(\infty) \neq 0.
$$

We are now in a position to define the solution set $\boldsymbol{\Delta}$, which contains all feasible solutions Δ satisfying $P_\Delta = \mathcal{F}_u(H, \Delta)$, for left 4-block uncertainty characterizations. It is clear from Definition 4 and the preceeding derivations that set $\boldsymbol{\Delta}$ reduces to

$$
\boldsymbol{\Delta} = \left\{ \Delta = \begin{bmatrix} \tilde{G}^* & G \end{bmatrix} \begin{bmatrix} Q_1 & \tilde{G}G_\Delta \\ Q_2 & Q_3 \end{bmatrix} \begin{bmatrix} \tilde{G}_\Delta \\ G_\Delta^* \end{bmatrix} \ : \ Q_i \in \mathcal{RL}_\infty, \right.
$$
$$
\left. \det\left(\begin{bmatrix} \tilde{G}_\Delta^* + \begin{pmatrix} \tilde{M}^{-1}Q_1 \\ 0 \end{pmatrix} & G \end{bmatrix}\right)(\infty) \neq 0 \right\}
$$

in this specific case. Note also that $\boldsymbol{\Delta}^{\min}$ is never empty in this specific case for $\overline{\sigma}(\tilde{G}G_\Delta)(\infty) < 1$.

3.4 Define the distance measure $d(P, P_\Delta)$

Since $\begin{bmatrix}\tilde{G}^* \; G\end{bmatrix}$ and $\begin{bmatrix}\tilde{G}_\Delta \\ G_\Delta^*\end{bmatrix}$ are all-pass, and $\boldsymbol{\Delta}^{\min} \neq \emptyset$, it easily follows straight from Definition 4 that the distance measure $d(P, P_\Delta)$ for left 4-block uncertainty characterizations is given by:

$$\begin{aligned} d(P, P_\Delta) &= \min_{\Delta \in \boldsymbol{\Delta}} \|\Delta\|_\infty \\ &= \min_{Q_i \in \mathcal{RL}_\infty} \left\| \begin{bmatrix}\tilde{G}^* \; G\end{bmatrix} \begin{bmatrix} Q_1 & \tilde{G}G_\Delta \\ Q_2 & Q_3 \end{bmatrix} \begin{bmatrix}\tilde{G}_\Delta \\ G_\Delta^*\end{bmatrix} \right\|_\infty \\ &= \min_{Q_i \in \mathcal{RL}_\infty} \left\| \begin{bmatrix} Q_1 & \tilde{G}G_\Delta \\ Q_2 & Q_3 \end{bmatrix} \right\|_\infty \\ &= \left\| \tilde{G}G_\Delta \right\|_\infty . \end{aligned} \tag{16}$$

Note that the minimizer belongs to the set $\boldsymbol{\Delta}$ because we have assumed that $\overline{\sigma}(\tilde{G}G_\Delta)(\infty) < 1$.

3.5 Write the winding number condition independent of $\boldsymbol{\Delta}$

Now, we need to rewrite winding number condition (3) independently of Δ. This will give us a necessary and sufficient condition for robust stability as presented in Theorem 1 that is calculable from the given data as it is independent of Δ.

Towards this end, note that for left 4-block uncertainty characterizations, winding number condition (3) reduces to

$$\eta(P_\Delta) - \eta(P) = \operatorname{wno}\det\left(\begin{bmatrix}\tilde{G}_\Delta \\ G_\Delta^*\end{bmatrix} \begin{bmatrix} \tilde{G}_\Delta^* + \begin{pmatrix} \tilde{M}^{-1}Q_1 \\ 0 \end{pmatrix} & GM^{-1}M_\Delta \end{bmatrix} \right) \tag{17}$$

via equation (15). Therefore,

$$\begin{aligned} 0 &= \operatorname{wno}\det\left(\begin{bmatrix}\tilde{G}_\Delta \\ G_\Delta^*\end{bmatrix} \begin{bmatrix} \tilde{G}_\Delta^* + \begin{pmatrix} \tilde{M}^{-1}Q_1 \\ 0 \end{pmatrix} & G \end{bmatrix} \right) \\ &= \operatorname{wno}\det \begin{bmatrix} I - \tilde{M}_\Delta \tilde{M}^{-1}Q_1 & \tilde{G}_\Delta G \\ N_\Delta^* \tilde{M}^{-1} Q_1 & G_\Delta^* G \end{bmatrix} . \end{aligned} \tag{18}$$

Before we proceed in massaging this winding number condition further, note that [14] proved that in 4-block uncertainty characterizations (as the case considered here),

$$\left\| \begin{bmatrix} I \\ C \end{bmatrix} (I - PC)^{-1} \begin{bmatrix} I & -P \end{bmatrix} \right\|_\infty \geq 1.$$

Hence $b(P, C) \leq 1$. Consequently, in this specific case, $d(P, P_\Delta) < 1$ whenever the supposition $d(P, P_\Delta) < b(P, C)$ (see Theorem 1) is enforced. Then, since

$d(P, P_\Delta) = \left\| \tilde{G} G_\Delta \right\|_\infty < 1$ and $\underline{\sigma}\left(G_\Delta^* G\right)^2 = 1 - \overline{\sigma}\left(\tilde{G} G_\Delta\right)^2$ on the $j\omega$-axis [6, p. 121], it follows that $(G_\Delta^* G)^{-1} \in \mathcal{RL}_\infty$. Exploiting this fact, we can use Schur complements to rewrite equation (18) as

$$0 = \operatorname{wno}\det \begin{bmatrix} I - \tilde{M}_\Delta \tilde{M}^{-1} Q_1 - \tilde{G}_\Delta G (G_\Delta^* G)^{-1} N_\Delta^* \tilde{M}^{-1} Q_1 & 0 \\ 0 & G_\Delta^* G \end{bmatrix}$$

which is equivalent to

$$\begin{aligned}
&\operatorname{wno}\det(G_\Delta^* G) \\
&= -\operatorname{wno}\det[I - (\tilde{M}_\Delta + \tilde{G}_\Delta G (G_\Delta^* G)^{-1} N_\Delta^*) \tilde{M}^{-1} Q_1] \\
&= -\operatorname{wno}\det[I - \tilde{M}_\Delta (I + (P_\Delta - P) M (G_\Delta^* G)^{-1} N_\Delta^*) \tilde{M}^{-1} Q_1] \\
&= -\operatorname{wno}\det[I - \tilde{M}_\Delta (I + (P_\Delta - P)(I + P_\Delta^* P)^{-1} P_\Delta^*) \tilde{M}^{-1} Q_1] \\
&= -\operatorname{wno}\det[I - \tilde{M}_\Delta (I + P_\Delta P_\Delta^*)(I + P P_\Delta^*)^{-1} \tilde{M}^{-1} Q_1] \\
&= -\operatorname{wno}\det[I - \tilde{M}_\Delta^{-*} (I + P P_\Delta^*)^{-1} \tilde{M}^{-1} Q_1] \\
&= -\operatorname{wno}\det[I - (\tilde{G} \tilde{G}_\Delta^*)^{-1} Q_1] \\
&= -\operatorname{wno}\det[(\tilde{G} \tilde{G}_\Delta^*)^{-1} (\tilde{G} \tilde{G}_\Delta^* - Q_1)] \\
&= \operatorname{wno}\det[\tilde{G} \tilde{G}_\Delta^*] - \operatorname{wno}\det[\tilde{G} \tilde{G}_\Delta^* - Q_1]. \qquad (19)
\end{aligned}$$

Hence, equation (19) is a formulation of the desired winding number condition written independently of Δ so that it can be computed as a necessary and sufficient condition for robust stability as presented in Theorem 1. However, for this to be checked, one needs to check the condition over all $Q_i \in \mathcal{RL}_\infty$ that yield a $\Delta \in \boldsymbol{\Delta}^{\min}$ (i.e. satisfy $\left\| \begin{bmatrix} Q_1 & \tilde{G} G_\Delta \\ Q_2 & Q_3 \end{bmatrix} \right\|_\infty = \left\| \tilde{G} G_\Delta \right\|_\infty$ and $\det\left(\begin{bmatrix} \tilde{G}_\Delta^* + \begin{pmatrix} \tilde{M}^{-1} Q_1 \\ 0 \end{pmatrix} & G \end{bmatrix} \right)(\infty) \neq 0$).

Non-uniqueness in $\boldsymbol{\Delta}^{\min}$ provides no extra freedom

The following lemma states that winding number condition (19) can be simplified further and can be made also independent of Q_1 whenever the supposition $d(P, P_\Delta) < b(P, C)$ (see Theorem 1) is enforced.[12] This makes computation of the winding number test considerably easier.

Lemma 2. *Given normalized graph symbols $\tilde{G}, G_\Delta, \tilde{G}_\Delta$ defined in Section 1.1 and $\|\tilde{G} G_\Delta\|_\infty < 1$. Then*

$$\operatorname{wno}\det[\tilde{G} \tilde{G}_\Delta^*] = \operatorname{wno}\det[\tilde{G} \tilde{G}_\Delta^* - Q_1]$$

for all $Q_i \in \mathcal{RL}_\infty$ that satisfy

[12] The supposition $d(P, P_\Delta) < b(P, C)$ implies $\|\tilde{G} G_\Delta\|_\infty < 1$ via [14].

$$\det\left(\left[\tilde{G}_\Delta^* + \begin{pmatrix}\tilde{M}^{-1}Q_1\\ 0\end{pmatrix} \quad G\right]\right)(\infty) \neq 0 \quad \textit{and} \quad \left\|\begin{bmatrix}Q_1 & \tilde{G}G_\Delta\\ Q_2 & Q_3\end{bmatrix}\right\|_\infty = \left\|\tilde{G}G_\Delta\right\|_\infty.$$

Proof. Since

$$\left\|\left[Q_1\ \tilde{G}G_\Delta\right]\right\|_\infty \leq \left\|\begin{bmatrix}Q_1 & \tilde{G}G_\Delta\\ Q_2 & Q_3\end{bmatrix}\right\|_\infty = \left\|\tilde{G}G_\Delta\right\|_\infty,$$

it follows that

$$\begin{aligned}
& Q_1Q_1^* \leq \|\tilde{G}G_\Delta\|_\infty^2\, I - \tilde{G}G_\Delta G_\Delta^*\tilde{G}^* \quad \forall\omega \in \mathbb{R}\cup\{\infty\}\\
\Rightarrow\quad & \overline{\sigma}(Q_1)^2 \leq \overline{\lambda}\big(\|\tilde{G}G_\Delta\|_\infty^2\, I - \tilde{G}G_\Delta G_\Delta^*\tilde{G}^*\big) \quad \forall\omega \in \mathbb{R}\cup\{\infty\}\\
& (\text{since } AA^* \leq B \Rightarrow \overline{\sigma}(A)^2 \leq \overline{\lambda}(B))\\
\Leftrightarrow\quad & \overline{\sigma}(Q_1)^2 \leq \|\tilde{G}G_\Delta\|_\infty^2 - \overline{\sigma}\big(\tilde{G}G_\Delta\big)^2 \quad \forall\omega \in \mathbb{R}\cup\{\infty\}\\
\Rightarrow\quad & \overline{\sigma}(Q_1)^2 < 1 - \overline{\sigma}\big(\tilde{G}G_\Delta\big)^2 \quad \forall\omega \in \mathbb{R}\cup\{\infty\}\\
& (\text{since } \|\tilde{G}G_\Delta\|_\infty < 1 \text{ was assumed})\\
\Leftrightarrow\quad & \overline{\sigma}(Q_1) < \underline{\sigma}\big(\tilde{G}\tilde{G}_\Delta^*\big) \quad \forall\omega \in \mathbb{R}\cup\{\infty\}\\
& (\text{since } \underline{\sigma}\big(\tilde{G}\tilde{G}_\Delta^*\big)^2 = 1 - \overline{\sigma}\big(\tilde{G}G_\Delta\big)^2 \text{ via [6, p. 121]})\\
\Rightarrow\quad & \operatorname{wno}\det(\tilde{G}\tilde{G}_\Delta^* - Q_1) = \operatorname{wno}\det(\tilde{G}\tilde{G}_\Delta^*)\\
& (\text{via [6, p. 16]}).
\end{aligned}$$

Consequently, whenever the supposition $d(P, P_\Delta) < b(P, C)$ (see Theorem 1) is enforced[12], it follows that equation (19) reduces to

$$\operatorname{wno}\det(G_\Delta^* G) = 0. \tag{20}$$

That is, Q_1 in equation (19) does not yield any extra freedom on top of Vinnicombe's winding number test [6, Definition 3.1, p. 119]. Hence, equation (20) is the desired winding number condition written independently of Δ so that it can be computed as a necessary and sufficient condition for robust stability as presented in Theorem 1. It is also easily computable.

3.6 State robust stability and robust performance theorems

In this subsection, we rewrite the generic Robust Stability Theorem 1 and the generic Robust Performance Theorem 2 for this specific design case. That is, we reduce Robust Stability Theorem 1 and Robust Performance Theorem 2 specifically for left 4-block uncertainty characterizations with left 4-block performance measures.

Theorem 3 (Robust Stability - Four block). *Given a plant $P \in \mathcal{R}^{p\times q}$, a perturbed plant $P_\Delta \in \mathcal{R}^{p\times q}$ and a controller $C \in \mathcal{R}^{q\times p}$. Define normalized* *graph symbols* $G, \tilde{G}, G_\Delta, \tilde{G}_\Delta$ *as in Section 1.1, a stability margin* $b(P,C)$ *as in* (11) *and a distance measure* $d(P, P_\Delta)$ *as in* (16).

Furthermore, suppose $d(P, P_\Delta) < b(P,C)$*. Then*

$$[P_\Delta, C] \text{ is internally stable} \Leftrightarrow \operatorname{wno}\det(G_\Delta^* G) = 0,$$

where the winding number is evaluated on a contour indented to the right around any imaginary axis poles of P *and* P_Δ*.*

Proof. Trivial reduction of Theorem 1. Note that the supposition that "H is stabilizable" is automatically fulfilled in this specific design case, and hence does not need to be independently enforced, since H has the following special form (i.e. P repeated in a special way):

$$H = \left[\begin{array}{cc|c} I & -P & P \\ 0 & 0 & I \\ \hline I & -P & P \end{array}\right].$$

Similarly, we now reduce Robust Performance Theorem 2 specifically for left 4-block uncertainty characterizations with left 4-block performance measures.

Theorem 4 (Robust Performance - Four block). *Given the suppositions of Theorem 3 and furthermore assuming* $d(P, P_\Delta) < b(P,C)$ *and* $\operatorname{wno}\det(G_\Delta^* G) = 0$*, where the winding number is evaluated on a contour indented to the right around any imaginary axis poles of* P *and* P_Δ*.*

Then

$$|b(P_\Delta, C) - b(P,C)| \le d(P, P_\Delta) \tag{21}$$

and

$$\|\mathcal{F}_l(H_\Delta, C) - \mathcal{F}_l(H, C)\|_\infty \le \frac{d(P, P_\Delta)}{b(P,C)b(P_\Delta, C)}, \tag{22}$$

where

$$H = \left[\begin{array}{cc|c} I & -P & P \\ 0 & 0 & I \\ \hline I & -P & P \end{array}\right] \quad \text{and} \quad H_\Delta = \left[\begin{array}{cc|c} I & -P_\Delta & P_\Delta \\ 0 & 0 & I \\ \hline I & -P_\Delta & P_\Delta \end{array}\right].$$

Proof. Trivial reduction of Theorem 2 on noting that

$$\begin{aligned} \|\mathcal{F}_l(H_\Delta, C) - S_z S_w\|_\infty &= \|\mathcal{F}_l(H_\Delta, C) - I\|_\infty \\ &= \|\mathcal{F}_l(H_\Delta, C)\|_\infty \\ &= \frac{1}{b(P_\Delta, C)}. \end{aligned}$$

The second equality follows from [13, Lemma 18.6, p. 484] since $\mathcal{F}_l(H_\Delta, C)$ is idempotent in this specific case. The third equality follows from definition (11) of $b(P_\Delta, C)$ on noting that $[P_\Delta, C]$ is internally stable via Theorem 3.

A tighter robust performance result exploiting case-specific geometry

In this specific case (i.e. left 4-block uncertainty characterization with a left 4-block performance measure), it is possible to derive a slightly tighter result by exploiting case-specific geometry.

To do this, let K (resp. $\tilde{K}$) denote a normalized right (resp. left) inverse graph symbol for controller C and let $G, \tilde{G}, G_\Delta, \tilde{G}_\Delta$ be defined as in Section 1.1. Then, since $S_w = I$ and $S_z = I$, it is easy to see that

$$\begin{aligned}\mathcal{F}_l\left(H, C\right) &= \begin{bmatrix} I \\ C \end{bmatrix} (I - PC)^{-1} \begin{bmatrix} I & -P \end{bmatrix} \quad \text{via (6)} \\ &= K(\tilde{G}K)^{-1}\tilde{G} \\ &= I - G(\tilde{K}G)^{-1}\tilde{K}\end{aligned}$$

and similarly

$$\begin{aligned}\mathcal{F}_l\left(H_\Delta, C\right) &= K(\tilde{G}_\Delta K)^{-1}\tilde{G}_\Delta \\ &= I - G_\Delta(\tilde{K}G_\Delta)^{-1}\tilde{K}.\end{aligned}$$

Consequently, using the result of Lemma 1 for this specific case (i.e. with $S_w = I$ and $S_z = I$), we get

$$\begin{aligned}
& \mathcal{F}_l\left(H_\Delta, C\right) = \mathcal{F}_l\left(H, C\right) + \mathcal{F}_l\left(H, C\right) \Delta(\mathcal{F}_l\left(H_\Delta, C\right) - I) \\
\Leftrightarrow \quad & K(\tilde{G}_\Delta K)^{-1}\tilde{G}_\Delta = K(\tilde{G}K)^{-1}\tilde{G} \\
& \quad - K(\tilde{G}K)^{-1}\tilde{G}\Delta G_\Delta(\tilde{K}G_\Delta)^{-1}\tilde{K} \\
\Rightarrow \quad & (\tilde{G}_\Delta K)^{-1}\tilde{G}_\Delta\tilde{K}^* = (\tilde{G}K)^{-1}\tilde{G}\tilde{K}^* - (\tilde{G}K)^{-1}\tilde{G}G_\Delta(\tilde{K}G_\Delta)^{-1} \\
& \quad \text{(pre-multiply by } K^*\text{, postmultiply by } \tilde{K}^*\text{, substitute} \\
& \quad \tilde{G}G_\Delta = \tilde{G}\Delta G_\Delta \text{ via (12))} \\
\Rightarrow \quad & \overline{\sigma}\left[(\tilde{G}_\Delta K)^{-1}\tilde{G}_\Delta\tilde{K}^*\right] \leq \frac{\overline{\sigma}(\tilde{G}\tilde{K}^*)}{\underline{\sigma}(\tilde{G}K)} + \frac{\overline{\sigma}(\tilde{G}G_\Delta)}{\underline{\sigma}(\tilde{G}K)\,\underline{\sigma}(\tilde{K}G_\Delta)} \\
& \quad \text{(via singular value inequalities)} \\
\Leftrightarrow \quad & \frac{\sqrt{1 - \underline{\sigma}(\tilde{G}_\Delta K)^2}}{\underline{\sigma}(\tilde{G}_\Delta K)} \leq \frac{\sqrt{1 - \underline{\sigma}(\tilde{G}K)^2}}{\underline{\sigma}(\tilde{G}K)} + \frac{\overline{\sigma}(\tilde{G}G_\Delta)}{\underline{\sigma}(\tilde{G}K)\,\underline{\sigma}(\tilde{K}G_\Delta)} \\
& \quad \text{(via [6, Lemma 2.2(ii), p. 71] using } Z_\Delta = \tilde{G}_\Delta \begin{bmatrix} K & \tilde{K}^* \end{bmatrix} \\
& \quad \text{satisfying } Z_\Delta Z_\Delta^* = I \text{ and } Z = \tilde{G}\begin{bmatrix} K & \tilde{K}^* \end{bmatrix} \text{ satisfying } ZZ^* = I) \\
\Leftrightarrow \quad & \beta\sqrt{1 - \alpha^2} - \alpha\sqrt{1 - \beta^2} \leq \gamma \\
& \quad (\alpha = \underline{\sigma}(\tilde{G}_\Delta K) = \underline{\sigma}(\tilde{K}G_\Delta), \beta = \underline{\sigma}(\tilde{G}K), \gamma = \overline{\sigma}(\tilde{G}G_\Delta))
\end{aligned}$$

$$\begin{aligned} \Leftrightarrow\quad & \sin(\hat{\beta}-\hat{\alpha}) \le \sin\hat{\gamma} \\ & \quad (\hat{\alpha} = \arcsin\alpha, \hat{\beta} = \arcsin\beta, \hat{\gamma} = \arcsin\gamma \text{ such that} \\ & \quad \hat{\alpha}, \hat{\beta}, \hat{\gamma} \in [0,\pi] \text{ since } \alpha,\beta,\gamma \in [0,1] \text{ — see [6])} \\ \Leftrightarrow\quad & \arcsin\beta - \arcsin\alpha \le \arcsin\gamma. \end{aligned}$$

It can be similarly shown that "$\arcsin\alpha - \arcsin\beta \le \arcsin\gamma$", thus giving us

$$|\arcsin\alpha - \arcsin\beta| \le \arcsin\gamma.$$

Then since

$$\alpha = \underline{\sigma}(\tilde{G}_\Delta K) = \frac{1}{\overline{\sigma}(\mathcal{F}_l\,(H_\Delta, C))},\ \beta = \underline{\sigma}(\tilde{G}K) = \frac{1}{\overline{\sigma}(\mathcal{F}_l\,(H, C))},\ \gamma = \overline{\sigma}(\tilde{G}G_\Delta),$$

it follows that

$$|\arcsin b(P_\Delta, C) - \arcsin b(P, C)| \le \arcsin d(P, P_\Delta) \tag{23}$$

on noting that $[P, C]$ and $[P_\Delta, C]$ are internally stable in Theorem 2.

4 Other structures

The systematic procedure proposed in Section 2 was used in Section 3 on the specific problem of left 4-block uncertainty characterizations with left 4-block performance measures to derive, in a different way, results that already existed in the literature [5, 6]. In doing so, we have shown that the proposed systematic procedure captures completely existing results in the literature.

This systematic procedure of Section 2 can also be used on several other uncertainty structures and several other performance measures to derive brand new robust stability and robust performance results. It allows us to define stability margins and distance measures for each specific uncertainty structure and performance measure considered. This then allows us to give a necessary and sufficient winding number condition for stability of a perturbed system, and residual performance guarantees, when a nominal system is replaced by a perturbed system for each specific uncertainty structure and performance measure considered.

We have derived such results (using the outline systematic procedure) for a number of different uncertainty structures and performance measures, including to mention a few: Additive, Inverse Additive, Multiplicative, Inverse Multiplicative, Coprime Factor and 4-block uncertainty characterizations with Additive, Inverse Additive, Multiplicative, Inverse Multiplicative, Coprime Factor and 4-block performance measures, and various mixes of the above as also outlined in Table 1.

These results are not given here due to space constraints and will be published shortly elsewhere.

5 Conclusions

Given a nominal plant model, an uncertainty structure, weights and a perturbed plant model we have defined a measure of distance between the nominal and perturbed plant models that is essentially a measure of difference between the two plants from a feedback perspective.

We have shown how to compute the distance measure using model validation ideas.

We have also derived a necessary and sufficient condition for the stability of the perturbed feedback system given a bound on the distance between the two plants. Furthermore, we have derived upper and lower bounds for the residual stability margin and an upper bound for a measure of the difference in performance. These bounds justify our claim that the distance measure is a measure of difference from a feedback perspective.

All the above results are generic and can be applied to a large number of uncertainty structures. To show that our procedure works we have applied it to left 4-block uncertainty with left 4-block performance, a case that has already been studied by Vinnicombe using different techniques and we systematically reproduce his results.

We have also applied our procedure to other uncertainty structures not discussed in this paper.

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