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On multipliers for bounded and monotone nonlinearities[☆]

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ABSTRACT

Recent results in equivalence between classes of multipliers for slope-restricted nonlinearities are extended to multipliers for bounded and monotone nonlinearities. This extension requires a slightly modified version of the Zames–Falb theorem and a more general definition of phase-substitution. The results in this paper resolve apparent contradictions in the literature on classes of multipliers for bounded and monotone nonlinearities.

1. Introduction

Different classes of multipliers can be used for analysing the stability of a Lur'e system (see Fig. 1) where the nonlinearity is bounded and monotone. A loop transformation allows us to analyse slope-restricted nonlinearities with the same classes of multipliers [1]. Apparently contradictory results can be found in the literature with respect to which class provides better results. On the one hand, it is stated that a complete search over the class of Zames–Falb multipliers will provide the best result that can be achieved [2,3]. On the other hand, searches over a subclass of Zames–Falb multipliers [4,5] have been improved by adding a Popov multiplier [6–8].

The class of Zames–Falb multipliers is formally given in the celebrated paper [1]. Two main results are given: Theorem 1 in [1] presents the Zames–Falb multipliers for bounded and monotone nonlinearities; Corollary 2 in [1] applies the Zames–Falb multipliers to slope-restricted nonlinearities via a loop transformation. We have formally shown in [9] that the class of Zames–Falb multipliers for slope-restricted nonlinearities, i.e. using Corollary 2 in [1], should provide the best result in comparison with any other class of multipliers available in the literature. The result relies on the fact that only biproper plants need to be considered in the search for a Zames–Falb multiplier, since the original plant becomes biproper after the loop transformation in Fig. 2 [1,10].

However, for bounded and monotone nonlinearities, biproperness of the LTI system G cannot be assumed without loss of generality. But the conditions of Theorem 1 in [1] cannot hold when the plant is strictly proper. An example has been proposed in [11] where the addition of a Popov multiplier to the Zames–Falb multiplier is essential to guarantee the stability of the Lur'e system. This prompts the natural question: is the addition of a Popov multiplier an improvement over the class of Zames–Falb multipliers for bounded and monotone nonlinearities? In fact, we show that this restriction of the conditions of Theorem 1 in [1] leads to more fundamental contradictions.

This paper proposes a slightly modified version of Theorem 1 in [1] in such a way that strictly proper plants can be analysed. Then, generalizations of phase-substitution and phase-containment defined in [9] are given in order to show the relationship between classes of multipliers. As a result, we show that a search over the class of Zames–Falb multipliers is also sufficient for bounded and monotone nonlinearities, i.e. if there is no suitable Zames–Falb multiplier then there is no suitable multiplier within any other class of multipliers. This paper resolves some apparent paradoxes, providing consistency to results in the literature.

The structure of the paper is as follows. Section 2 gives preliminary results; in particular, the equivalence results in [9] are stated and the differences between the cases of slope-restricted and bounded and monotone nonlinearities are highlighted. Section 3 provides the relationships between classes for the case of bounded and monotone nonlinearities. Section 4 analyses the example given in [11], showing that there exists a Zames–Falb multiplier that provides the stability result under our modification

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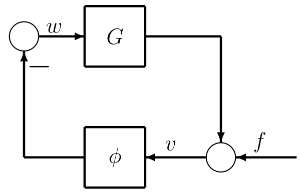

Fig. 1. Lur'e system.

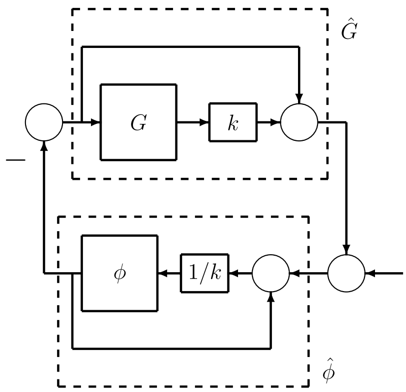

Fig. 2. Loop shifting transforms a slope restricted nonlinearity ϕ into a monotone nonlinearity $\hat{\phi}$. Simultaneously, a new linear system $\hat{G}$ is generated. In [9], we have shown that when generated via loop shifting $\hat{G}$ can be assumed biproper without loss of generality from the necessity of the Kalman conjecture (for further discussion, see Section 2.3 in [9]), but such an assumption cannot be made when there is no loop shifting.

of Theorem 1 in [1]. Finally, the conclusions of this paper are given in Section 5.

2. Notation and preliminary results

Let $\mathcal{L}_2^m[0,\infty)$ be the Hilbert space of all square integrable and Lebesgue measurable functions $f : [0,\infty) \to \mathbb{R}^m$. Similarly, $\mathcal{L}_2^m(-\infty,\infty)$ can be defined for $f : (-\infty,\infty) \to \mathbb{R}^m$. Given $T \in \mathbb{R}$, a truncation of the function f at T is given by $f_T(t) = f(t)\,\forall t \le T$ and $f_T(t) = 0\,\forall t > T$. The function f belongs to the extended space $\mathcal{L}_{2e}^m[0,\infty)$ if $f_T \in \mathcal{L}_2^m[0,\infty)$ for all $T > 0$. In addition, $\mathcal{L}_1(-\infty,\infty)$ (henceforth $\mathcal{L}_1$) is the space of all absolute integrable functions; given a function $h : \mathbb{R} \to \mathbb{R}$ such that $h \in \mathcal{L}_1$, its $\mathcal{L}_1$-norm is given by

$$\|h\|_1 = \int_{-\infty}^{\infty} |h(t)|\,\mathrm{d}t. \tag{1}$$

A nonlinearity $\phi : \mathcal{L}_{2e}[0,\infty) \to \mathcal{L}_{2e}[0,\infty)$ is said to be memoryless if there exists $N : \mathbb{R} \to \mathbb{R}$ such $(\phi v)(t) = N(v(t))$ for all $t \in \mathbb{R}$. Henceforward we assume that $N(0) = 0$. A memoryless nonlinearity ϕ is said to be bounded if there exists a positive constant C such that $|N(x)| < C|x|$ for all $x \in \mathbb{R}$. The nonlinearity ϕ is said to be monotone if for any two real numbers x_1 and x_2 we have

$$0 \le \frac{N(x_1) - N(x_2)}{x_1 - x_2}. \tag{2}$$

The nonlinearity ϕ is said to be odd if $N(x) = -N(-x)$ for all $x \in \mathbb{R}$.

This paper focuses the stability of the feedback interconnection of a proper stable LTI system G and a bounded and monotone nonlinearity ϕ, represented in Fig. 1 and given by

$$\begin{cases} v = f + Gw, \\ w = -\phi v. \end{cases} \tag{3}$$

Since G is a stable LTI system, the exogenous input in this part of the loop can be taken as the zero signal without loss of generality. It is well-posed if the map $(v, w) \mapsto (0, f)$ has a causal inverse on $\mathcal{L}_{2e}^2[0,\infty)$; this interconnection is $\mathcal{L}_2$-stable if for any $f \in \mathcal{L}_2[0,\infty)$, then $Gw \in \mathcal{L}_2[0,\infty)$ and $\phi v \in \mathcal{L}_2[0,\infty)$, and it is absolutely stable if it is $\mathcal{L}_2$-stable for all ϕ within the class of nonlinearities. In addition, $G(j\omega)$ means the transfer function of the LTI system G. Finally, given an operator M, then M^* means its $\mathcal{L}_2$-adjoint (see [12] for a definition). For LTI systems, $M^*(s) = M^\top(-s)$, where $^\top$ means transpose.

The standard notation $\mathbf{L}_\infty$ ($\mathbf{RL}_\infty$) is used for the space of all (proper real rational) transfer functions bounded on the imaginary axis and infinity; $\mathbf{RH}_\infty$ ($\mathbf{RH}_2$) is used for the space of all (strictly) proper real rational transfer functions such that all their poles have strictly negative real parts; and $\mathbf{RH}_\infty^-$ is used for the space of all proper real rational transfer functions such that all their poles have strictly positive real parts. Moreover, the subset of $\mathbf{RH}_2$ with positive DC gain is referred to as $\mathbf{RH}_2^+$. The H_∞-norm of a SISO transfer function G is defined as

$$\|G\|_\infty = \sup_{\omega\in\mathbb{R}}(|G(j\omega)|). \tag{4}$$

With some acceptable abuse of notation, given a rational strictly proper transfer function $H(s)$ bounded on the imaginary axis, $\|H\|_1$ means the $\mathcal{L}_1$-norm of the impulse response of $H(s)$.

2.1. Zames–Falb theorem and multipliers

The original Theorem 1 in [1] can be stated as follows:

Theorem 2.1 (*[1]*). *Consider the feedback system in Fig. 1 with $G \in \mathbf{RH}_\infty$, and a bounded and monotone nonlinearity ϕ. Assume that the feedback interconnection is well-posed. Then suppose that there exists a convolution operator $M : \mathcal{L}_2(-\infty,\infty) \to \mathcal{L}_2(-\infty,\infty)$ whose impulse response is of the form*

$$m(t) = \delta(t) - \sum_{i=0}^{\infty} z_i\delta(t - t_i) - z_a(t), \tag{5}$$

where δ is the Dirac delta function and

$$\sum_{i=0}^{\infty} |z_i| < \infty, \quad z_a \in \mathcal{L}_1, \text{ and } t_i \in \mathbb{R}\,\forall i \in \mathbb{N}. \tag{6}$$

Assume that:

(i)
$$\|z_a\|_1 + \sum_{i=0}^{\infty} |z_i| < 1, \tag{7}$$

(ii) *either ϕ is odd or $z_a(t) > 0$ for all $t \in \mathbb{R}$ and $z_i > 0$ for all $i \in \mathbb{N}$, and*

(iii) *there exists $\delta > 0$ such that*
$$\operatorname{Re}\{M(j\omega)G(j\omega)\} \ge \delta \quad \forall\omega \in \mathbb{R}. \tag{8}$$

Then the feedback interconnection (3) is $\mathcal{L}_2$-stable. ■

Eqs. (5)–(7) in Theorem 2.1 provide the class of Zames–Falb multipliers. It is a subset of $\mathbf{L}_\infty$, i.e. it is not limited to rational transfer functions. However, for the remainder of this paper we restrict our attention to such rational multipliers, i.e. we set $z_i = 0$ for all $i \in \mathbb{N}$.

Definition 2.2. The class of SISO rational Zames–Falb multipliers $\mathcal{M}$ contains all SISO rational transfer functions $M \in \mathbf{RL}_\infty$ such that $M(s) = 1 - Z(s)$, where $Z(s)$ is a rational strictly proper transfer function and $\|Z\|_1 < 1$.

Lemma 2.3 (*[5]*)**.** *Let $M \in \mathbf{RL}_\infty$ be a rational transfer function with $M(s) = M(\infty) + \widehat{M}(s)$, where $\widehat{M}(s)$ denotes its associated strictly proper transfer function. Then, $M(s)/M(\infty)$ is a Zames–Falb multiplier if and only if $\|\widehat{M}\|_1 < M(\infty)$.* ■

If $M \in \mathbf{RH}_\infty$, the multiplier is said to be causal. If $M \in \mathbf{RH}_\infty^-$, the multiplier is said to be anticausal. Otherwise, the multiplier is noncausal (see [8] for further details).

2.2. List of classes of multipliers for monotone and bounded nonlinearities

The first class of multipliers we consider for bounded nonlinearities is the class of Popov multipliers:

Definition 2.4. The class of Popov multipliers is given by

$$M(s) = 1 + qs, \tag{9}$$

where $q \in \mathbb{R}$.

Following [11,6], two extensions of the class of Zames–Falb multipliers by combination with the Popov multipliers have been proposed:

Definition 2.5. The class of Popov-extended Zames–Falb multipliers is given by

$$M_{\text{PZF}}(s) = qs + M(s) \tag{10}$$

where $q \in \mathbb{R}$ and where $M(s)$ belongs to the class of Zames–Falb multipliers.

Definition 2.6. The class of Popov plus Zames–Falb multipliers is given by

$$M_{\text{P+ZF}}(s) = \vartheta(1 + qs) + M(s) \tag{11}$$

where $q \in \mathbb{R}$, where $\vartheta > 0$ and where $M(s)$ belongs to the class of Zames–Falb multipliers.

Another important class of multipliers is generated by including both a Popov multiplier and a quadratic term. It was given originally by Yakubovich [13], and an LMI search over this set has been proposed in [14]:

Definition 2.7. The class of Park's multipliers is given by

$$M_P(s) = 1 + \frac{bs}{-s^2 + a^2} \tag{12}$$

where a and b are real numbers.

Following [15], an extension of the class of Zames–Falb multipliers with this quadratic term can be proposed:

Definition 2.8. The class of Yakubovich–Zames–Falb multipliers is given by

$$M_{\text{YZF}}(s) = -\kappa^2 s^2 + M(s), \quad \kappa \in \mathbb{R}, \tag{13}$$

where $\kappa \in \mathbb{R}$ and $M(s)$ is a Zames–Falb multiplier.

2.3. Previous equivalence results

In [9], Theorem 1 in [1] is considered but restricted to a particular set of biproper plants $\hat{G}(s)$, as a result of a previous loop transformation (see Fig. 2). Under such a restriction, a search over the class of Zames–Falb multipliers should be sufficient to obtain the best possible result using any other class in the literature.

Definition 2.9. The subset $\mathcal{SR} \subset \mathbf{RH}_\infty$ is defined as follows

$$\mathcal{SR} = \{G \in \mathbf{RH}_\infty : G^{-1} \in \mathbf{RH}_\infty \text{ and } G(\infty) > 0\}. \tag{14}$$

This characterization of $\mathcal{SR}$ plays a key role to show that Popov multipliers are "limiting cases" of Zames–Falb multipliers and is also essential for the extension using the Popov multipliers. With this aim, some definitions are mathematically formalized in [9]. For instance, a definition of phase-substitution is proposed with respect to $\mathcal{SR}$:

Definition 2.10 (*[9]*)**.** Let M_a and M_b be two multipliers and $G \in \mathcal{SR}$. The multiplier M_b is a *phase-substitute* of the multiplier M_a when

$$\text{Re}\,\{M_a(j\omega)G(j\omega)\} \geq \delta_1 \quad \forall \omega \in \mathbb{R} \tag{15}$$

for some $\delta_1 > 0$ implies

$$\text{Re}\,\{M_b(j\omega)G(j\omega)\} \geq \delta_2 \quad \forall \omega \in \mathbb{R} \tag{16}$$

for some $\delta_2 > 0$.

Using Definition 2.10 for phase-substitution, the relationship between two classes can be given as follows:

Definition 2.11 (*[9]*)**.** Let $\mathcal{M}_A$ and $\mathcal{M}_B$ be two classes of multipliers. The class $\mathcal{M}_A$ is *phase-contained* within the class $\mathcal{M}_B$ if given a multiplier $M_a \in \mathcal{M}_A$, then there exists $M_b \in \mathcal{M}_B$ such that it is a phase-substitute of M_a.

Result 2.12 (*[9]*)**.** Under the assumption $G(s) \in \mathcal{SR}$, the classes of multipliers given in Section 2.2 are phase-contained within the class of Zames–Falb multipliers. ■

A graphical interpretation of Result 2.12 is given in Fig. 3.

In this paper, we focus on extending Result 2.12 to monotone and bounded nonlinearities. For this, strictly proper plants must be included in the set of interest. Then Result 2.12 is no longer valid in general since Popov multipliers are only phase-contained under Definition 2.11 within the class of Zames–Falb multipliers if $G \in \mathcal{SR}$.

All constant gains K are included in the class of bounded and monotone nonlinearities. Trivially, a necessary condition for absolute stability is that the feedback interconnection of G and a constant gain K must be $\mathcal{L}_2$-stable for any value of K. Thus if G is biproper, then G must belong to $\mathcal{SR}$ as commented in [9] and Result 2.12 can be applied. Therefore we can restrict our attention to strictly proper plants without loss of generality. Further, we only consider strictly proper plants with positive DC gain $\mathbf{RH}_2^+$, i.e. $G(\infty) = 0$ and $G(0) > 0$. It is straightforward to show that if $G(0) < 0$, then the feedback interconnection of G and $K = -\frac{1}{G(0)}$ is not $\mathcal{L}_2$-stable.

2.4. Counterexample

Let us consider the plant given by

$$G(s) = \frac{b}{s + a} \tag{17}$$

where $a, b > 0$. If the nonlinearity is bounded and monotone, then Theorem 2.1 is not able to demonstrate the absolute stability of this system since, given $\delta > 0$, there exists no Zames–Falb multiplier M satisfying

$$\text{Re}\left\{M(j\omega)\frac{b}{j\omega + a}\right\} \geq \delta \quad \forall \omega \in \mathbb{R}, \tag{18}$$

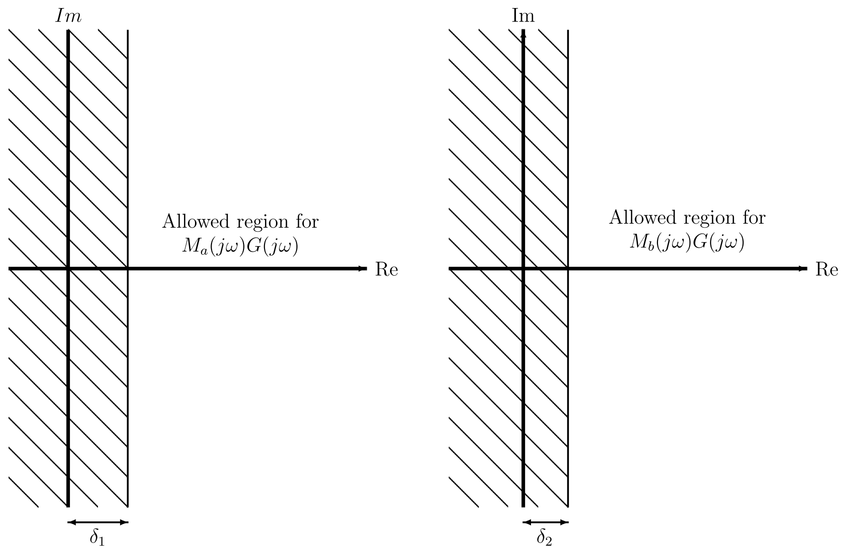

Fig. 3. For any $G \in \mathcal{SR}$, when there exist a multiplier M_a within the classes of multipliers in Section 2.2, and $\delta_1 > 0$ such that the Nyquist plot of $M_a(j\omega)G(j\omega)$ is within the allowed region, then there exist a Zames–Falb multiplier M_b and $\delta_2 > 0$ such that the Nyquist plot of $M_b(j\omega)G(j\omega)$ is within the allowed region (see [16] and c.f. Fig. 4).

for some $\delta > 0$, since $\lim_{\omega\to\infty} M(j\omega) = M(\infty) > 0$ for any Zames–Falb multiplier M, thus $\lim_{\omega\to\infty} M(j\omega)\frac{b}{j\omega+a} = 0$.

However, it is possible to find $\delta > 0$ and a Popov-extended Zames–Falb multiplier (Definition 2.5) such that:

$$\mathrm{Re}\left\{M_{\mathrm{PZF}}(j\omega)\frac{b}{j\omega+a}\right\} \geq \delta \quad \forall\omega \in \mathbb{R}, \tag{19}$$

since the transfer function on the left side is now biproper. So, the use of a Popov-extended Zames–Falb multiplier seems to outperform the original class of Zames–Falb multipliers. A similar example is discussed in [11] and a similar conclusion is drawn.

But we are immediately led into a more fundamental paradox. For any nonlinearity bounded with a finite constant $C > 0$, the Circle Criterion [10] states that the feedback in Fig. 1 is absolutely stable if $1 + CG(s)$ is strictly positive real (SPR). It is straightforward that $1+CG(s)$ is SPR for any finite constant C. Using the same argument, we conclude that a constant multiplier outperforms the class of Zames–Falb multipliers. Nevertheless, the class of constant multipliers is included within the Zames–Falb multipliers.

The difficulties arise because the original version of the Zames–Falb theorem is not adequate for strictly proper plants. If we use this version of the Theorem to compare multipliers, we must conclude not only that the Popov-extended Zames–Falb multipliers are superior to the original class of Zames–Falb multipliers (as argued in [11]), but also that the class of constant multipliers can outperform the original class of Zames–Falb multipliers. This is clearly paradoxical, and in the following section we address the paradox by modifying the Zames–Falb theorem itself.

3. Main results

In the following, we state a modification of the original Zames–Falb theorem which is able to cope with strictly proper plants. Then, a more general definition of phase-substitution is given. Finally, we will show that the class of Popov and Popov-extended Zames–Falb multipliers are "phase-contained" within the original class of Zames–Falb multipliers under our more general definition.

3.1. Modification of the Zames–Falb theorem

We have seen that if G is strictly proper then no multiplier within the class of Zames–Falb multipliers satisfies (8). However, this conservatism can be avoided by exploiting the boundedness of the nonlinearity. In the IQC framework [17], it is straightforward to combine the positivity constraint and boundedness constraint of the nonlinearity. Applying Corollary 1 in [17], we can propose an alternative version of the Zames–Falb theorem.

Corollary 3.1. *Consider the feedback system in Fig. 1 with $G \in \mathbf{RH}_\infty$ and any bounded and monotone nonlinearity ϕ. Assume that the feedback interconnection is well-posed. If there exists a Zames–Falb multiplier such that*

$$\mathrm{Re}\{M(j\omega)G(j\omega)\} \geq \epsilon G^*(j\omega)G(j\omega) \quad \forall\omega \in \mathbb{R} \tag{20}$$

for some $\epsilon > 0$, then the feedback interconnection (3) is $\mathcal{L}_2$-stable. ■

Remark 3.2. Note that the homotopy conditions imposed by the IQC theorem are trivially satisfied for these classes of nonlinearities.

Remark 3.3. The extension of results in [18] in order to show the equivalence between IQC and classical passivity theory for Corollary 3.1 is possible by using classical results in factorization [19].

3.2. General definition of phase-substitution

The modification of the Zames–Falb theorem shows that Definition 3.1 in [9] is not general. A general definition of phase-substitution should allow different properties of the multiplier to hold as they arise either in different stability theorems or in different versions of the same stability theorem. We will use the classical concept of quadratic constraint [20,17].

Definition 3.4. The plant G and multiplier M satisfy the frequency quadratic constraint QC(ϵ, δ) if

$$\begin{bmatrix} G(j\omega) \\ 1 \end{bmatrix}^* \begin{bmatrix} -2\epsilon & M^*(j\omega) \\ M(j\omega) & -2\delta \end{bmatrix} \begin{bmatrix} G(j\omega) \\ 1 \end{bmatrix} \geq 0 \quad \forall\omega \in \mathbb{R}. \tag{21}$$

Loosely speaking, a multiplier M_a can be phase-substituted by a multiplier M_b if M_b is able to show the same stability properties as M_a. As different versions of stability theorems can use different quadratic constraints, a generalized definition of phase-substitution is given as follows:

Definition 3.5. Let M_a and M_b be two multipliers and let $\mathcal{G}$ be a set of plants. The multiplier M_b is a $\mathrm{QC}(\epsilon_a, \delta_a)$–$\mathrm{QC}(\epsilon_b, \delta_b)$ *phase-substitute with respect to* $\mathcal{G}$ of the multiplier M_a if whenever the pair $\{M_a, G\}$ satisfies the frequency quadratic constraint $\mathrm{QC}(\epsilon_a, \delta_a)$ for G within a set $\mathcal{G}$, then the pair $\{M_b, G\}$ also satisfies the frequency quadratic constraint $\mathrm{QC}(\epsilon_b, \delta_b)$.

Remark 3.6. Definition 2.10 is a particular case of Definition 3.5 where $\mathrm{QC}(\epsilon_a, \delta_a) = \mathrm{QC}(0, \delta_1)$ and $\mathrm{QC}(\epsilon_b, \delta_b) = \mathrm{QC}(0, \delta_2)$.

With this generalization, different classes of multipliers can be analysed under different quadratic constraints. For example, Corollary 3.1 avoids the conservatism of Theorem 2.1 when it is applied for monotone nonlinearities. Thus the following counterpart of Definition 3.1 in [9] is appropriate here.

Definition 3.7. Let M_a and M_b be two multipliers and $G \in \mathbf{RH}_2^+$. The multiplier M_b is a $\mathrm{QC}(0, \delta)$–$\mathrm{QC}(\epsilon, 0)$ *phase-substitute* with respect to $\mathbf{RH}_2^+$ of the multiplier M_a when

$$\mathrm{Re}\,\{M_a(j\omega)G(j\omega)\} \geq \delta \quad \forall \omega \in \mathbb{R} \tag{22}$$

for some $\delta > 0$ implies

$$\mathrm{Re}\,\{M_b(j\omega)G(j\omega)\} \geq \epsilon G^*(j\omega)G(j\omega) \quad \forall \omega \in \mathbb{R} \tag{23}$$

for some $\epsilon > 0$.

This relationship between multipliers can be straightforwardly extended to classes of multipliers:

Definition 3.8. Let $\mathcal{M}_A$ and $\mathcal{M}_B$ be two classes of multipliers. The class $\mathcal{M}_A$ is $\mathrm{QC}(0, \delta)$–$\mathrm{QC}(\epsilon, 0)$ *phase-contained with respect to* $\mathbf{RH}_2^+$ within the class $\mathcal{M}_B$ if given a multiplier $M_a \in \mathcal{M}_A$, then there exists $M_b \in \mathcal{M}_B$ such that it is a $\mathrm{QC}(0, \delta)$–$\mathrm{QC}(\epsilon, 0)$ *phase-substitute* with respect to $\mathbf{RH}_2^+$ of M_a.

Henceforth we will use the terminology "*phase-contained in the sense of* Definition 3.8" to mean "$\mathrm{QC}(0, \delta)$–$\mathrm{QC}(\epsilon, 0)$ *phase-contained with respect to* $\mathbf{RH}_2^+$".

3.3. Popov multipliers

In this section, we state that the class of Popov multipliers, the class of Popov-extended Zames–Falb multipliers, and the class of Popov plus Zames–Falb multipliers are phase-contained within the class of Zames–Falb multipliers for bounded and monotone nonlinearities.

Lemma 3.9. *The class of Popov multipliers with positive constant q is phase-contained in the sense of Definition 3.8 within the class of causal first order Zames–Falb multipliers.*

Proof. Assume that M_P is a Popov multiplier with $q > 0$ such that

$$\mathrm{Re}\,\{M_P(j\omega)G(j\omega)\} \geq \delta \quad \forall \omega \in \mathbb{R} \tag{24}$$

for some $\delta > 0$ and $G \in \mathbf{RH}_2^+$. Note that if (24) holds, then G is strictly proper with relative degree one. In the following, we show that M_P can be phase-substituted in the sense of Definition 3.8 by the following Zames–Falb multiplier:

$$M(j\omega) = \frac{(1 + qj\omega)}{1 + \kappa j\omega} \tag{25}$$

where κ is appropriately small.

The phase of $(1+qj\omega)G(j\omega)$ is as close as desired to the phase of $\frac{(1+qj\omega)}{1+\kappa j\omega}G(j\omega)$ at low frequency by choosing $\kappa > 0$ sufficiently small. However, the high frequency range must be carefully considered. Since G satisfies (24), M_PG must be biproper, hence G must have relative degree -1. As a result, at high frequency, the plant can be described as:

$$G(j\omega) = \frac{G_r}{\omega^2} - j\frac{G_i}{\omega},$$

where $G_i > 0$ and $G_r \in \mathbb{R}$. The real part of the above product between the Zames–Falb multiplier and G is given by

$$\begin{aligned} &\mathrm{Re}\left\{\frac{(q\kappa\omega^2 + 1) + j\omega(q - \kappa)}{\kappa^2\omega^2 + 1}\left(\frac{G_r}{\omega^2} - j\frac{G_i}{\omega}\right)\right\} \\ &= \left\{\frac{(q\kappa\omega^2 + 1)G_r}{\omega^2(\kappa^2\omega^2 + 1)} + \frac{G_i(q - \kappa)}{\kappa^2\omega^2 + 1}\right\}. \end{aligned} \tag{26}$$

Considering only the terms in ω^{-2}, then it follows

$$\mathrm{Re}\left\{\frac{(1 + qj\omega)}{1 + \kappa j\omega}G(j\omega)\right\} = \left(\frac{q - \kappa}{\kappa^2}G_i + \frac{q}{\kappa}G_r\right)\frac{1}{\omega^2} \tag{27}$$

when $\omega \to \infty$. For any value of G_r and a sufficient small value of κ, the real part in (27) approaches zero as ω^{-2}. Finally, as $G^*(j\omega)G(j\omega)$ also approaches zero as ω^{-2} at high frequencies, once κ has been chosen, taking

$$\epsilon = \frac{\left(\frac{q-\kappa}{\kappa^2}G_i + \frac{q}{\kappa}G_r\right)}{2G_i^2} > 0$$

then

$$\begin{aligned} &\mathrm{Re}\left\{\frac{1 + qs}{1 + \kappa s}G(j\omega)\right\} - \epsilon G^*(j\omega)G(j\omega) \\ &= \frac{1}{2}\left(\frac{q - \kappa}{\kappa^2}G_i + \frac{q}{\kappa}G_r\right)\frac{1}{\omega^2} > 0 \end{aligned} \tag{28}$$

when $\omega \to \infty$.

Note that low frequency and high frequency constraints require small values of κ. Hence, choosing the minimum κ for satisfying both conditions, the result is obtained.

In summary, if the Popov multiplier M_P and $G \in \mathbf{RH}_2^+$ satisfy the constraint $\mathrm{QC}(0, \delta)$ for some $\delta > 0$, then there exist a Zames–Falb multiplier M and positive constant $\epsilon > 0$ such that M and G satisfy the constraint $\mathrm{QC}(\epsilon, 0)$. ■

Lemma 3.10. *The class of Popov multipliers with negative constant q is phase-contained in the sense of Definition 3.8 within the class of anticausal first order Zames–Falb multipliers.*

Proof. Similar to Lemma 3.9 but with $\kappa < 0$. ■

As a result, we can conclude that any Popov multiplier can be phase-substituted by a Zames–Falb multiplier in the sense of Definition 3.8.

Lemma 3.11. *The class of Popov-extended Zames–Falb multipliers is phase-contained in the sense of Definition 3.8 within the class of Zames–Falb multipliers.*

Proof. Following the same idea as in [9], given a multiplier $M \in \mathcal{M}$ then $M(s) = 1 + H(s)$ for some strictly proper transfer function $H(s)$ with $\|H(s)\|_1 < 1$. Then, there exists $\rho > 0$ such that $\|H(s)\|_1 < 1 - \rho$. Thus,

$$M(s) = \rho + ((1 - \rho) + H(s)) = \rho + M'(s) \tag{29}$$

where $M'(s)$ is a Zames–Falb multiplier. Hence, (11) can be rewritten as follows

$$M_{\mathrm{PZF}}(s) = \rho\left(1 + \frac{q}{\rho}s\right) + (M(s) - \rho) \tag{30}$$

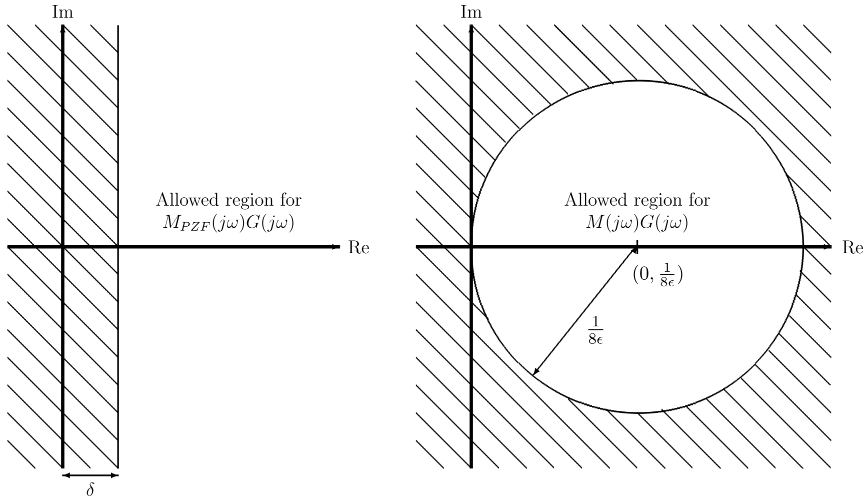

Fig. 4. For any $G \in \mathcal{RH}_2^+$, when there exist a Popov-extended Zames–Falb multiplier M_{PZF} and a $\delta > 0$ such that the Nyquist plot of $M_{PZF}(j\omega)G(j\omega)$ is within the allowed region, then there exist a Zames–Falb multiplier M and an $\epsilon > 0$ such that the Nyquist plot of $M(j\omega)G(j\omega)$ is within the allowed region. We have used the inequality $M^*(j\omega)M(j\omega) < 2^2$ (see Fig. 3 in [5]), hence $\mathrm{Re}(M(j\omega)G(j\omega)) > 8\epsilon|M(j\omega)G(j\omega)|^2$ for all $\omega \in \mathbb{R}$. See [16] for further details on the classical frequency description of SISO output strictly passive systems.

and choosing κ as small as desired, it holds that

$$M(s) = \rho\left(\frac{1+\frac{q}{\rho}s}{1+\kappa s}\right) + (M(s) - \rho) \qquad (31)$$

is a Zames–Falb multiplier. Thus applying the same procedure as in the proof of Lemma 3.9 leads to the result: if the Popov-extended multiplier M_{PZF} and $G \in \mathbf{RH}_2^+$ satisfy the constraint QC$(0, \delta)$ for some $\delta > 0$, then there exist a Zames–Falb multiplier M and positive constant $\epsilon > 0$ such that M and G satisfy the constraint QC$(\epsilon, 0)$. ■

A graphical interpretation of Lemma 3.11 is given in Fig. 4.

Corollary 3.12. *The class of Popov plus Zames–Falb multipliers is phase-contained in the sense of Definition 3.8 within the class of Zames–Falb multipliers.*

Proof. This result is part of the above proof. ■

3.4. Popov multiplier for boundedness condition

In many cases the properties of the nonlinearity may differ from the conditions of Theorem 2.1. A subtle distinction arises for nonlinearities that are monotone and with known finite bound C.

Although Theorem 2.1 may be used, there is an inherent conservativeness as the value of the bound is not exploited. The additional sector bound allows a less conservative stability criterion than Theorem 2.1. Loosely speaking, the feedback interconnection is stable provided there exists some Zames–Falb multiplier $M(s)$, some Popov multiplier $(1 + qs)$ and some $\lambda > 0$ such that for all ω

$$\mathrm{Re}\,\{M(j\omega)G(\omega j) + \lambda(1 + qs)[1 + CG(j\omega)]\} > 0. \qquad (32)$$

Then a Popov multiplier can be more appropriate than a Zames–Falb multiplier if C is small.

A similar observation has been stated for the case of slope-restricted nonlinearities with a sector condition smaller than its slope condition [6].

4. Example

Let us consider the example given by [11], where it is suggested that the class of Popov-extended Zames–Falb multipliers is wider than the class of Zames–Falb multipliers. As commented in [11], a search over the set of Zames–Falb multipliers is not able to find the stability of this example if Theorem 2.1 is used. This is trivial since the plant

$$G(s) = \frac{(2s^2 + s + 2)(s + 100)}{(s + 10)^2(s^2 + 5s + 20)} \qquad (33)$$

is strictly proper (a factor −1 has been included to consider negative feedback). On the other hand, [11] shows a Popov-extended Zames–Falb multiplier M_{PZF} such that

$$\mathrm{Re}\,\{M_{PZF}(j\omega)G(j\omega)\} > \delta \quad \forall\omega \in \mathbb{R}. \qquad (34)$$

Hence, the stability of the feedback interconnection is guaranteed. However, the use of Corollary 3.1 allows us to replace the Popov-extended Zames–Falb multiplier, and the stability of the feedback interconnection can also be ensured.

The multiplier proposed in [11] is

$$M_{PZF}(s) = 0.04s + 1 + \frac{0.92}{s - 1} = 0.04\frac{s^2 + 24s - 2}{s - 1}. \qquad (35)$$

Considering that the phase of G reaches a constant value at approximately 10^3 rad/s, a phase-substitute Zames–Falb multiplier of that in (35) can be constructed as follows

$$M(s) = 0.01\frac{4s + 1}{0.001s + 1} + \left(0.99 + \frac{0.92}{s - 1}\right). \qquad (36)$$

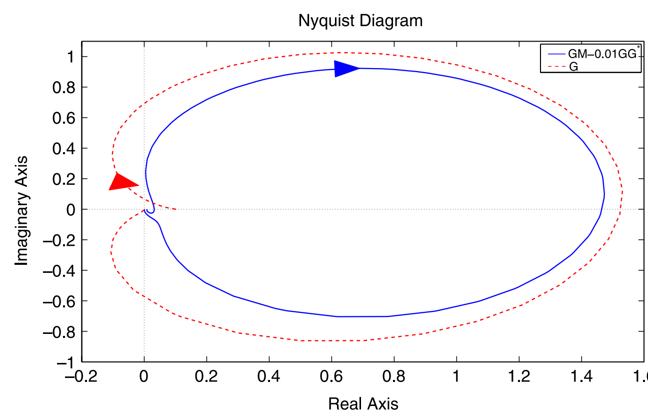

Fig. 5. Nyquist plot of $G(j\omega)$ and $G(j\omega)M(j\omega) - 0.01G^*(j\omega)G(j\omega)$.

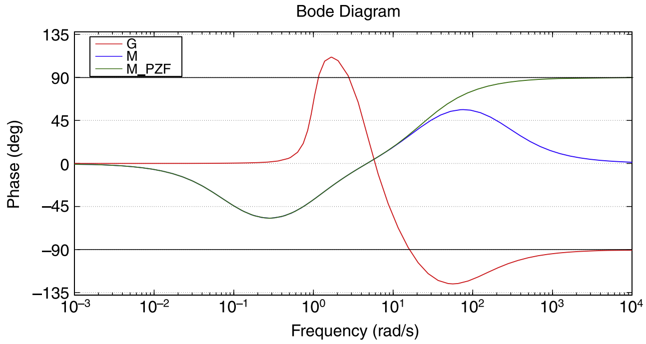

Fig. 6. Phase of $M(j\omega)$, $M_{PZF}(j\omega)$, and $G(j\omega)$. The extra pole in $M(j\omega)$ is included at high frequency when the phase of the plant is near to $-90°$, so that the addition of the phases of $M(j\omega)$ and $G(j\omega)$ is above $-90°$. At low frequency, $M(j\omega)$ and $M_{PZF}(j\omega)$ have approximately the same phase. It is worth noting that the pole can be included at a frequency as high as desired.

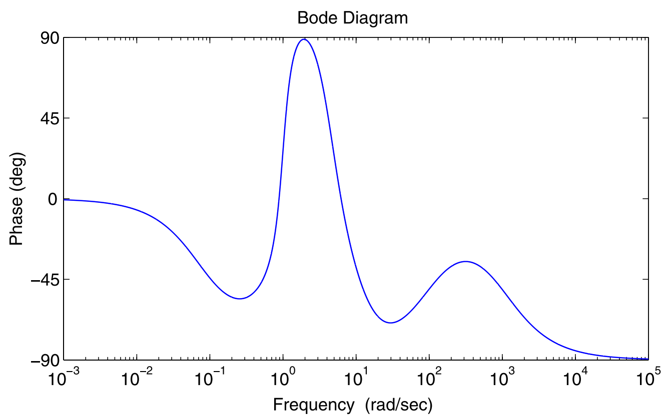

Fig. 7. Phase of $G(j\omega)M(j\omega) - 0.01G^*(j\omega)G(j\omega)$. The phase lies between -90 and 90 indicating that the real part is always positive.

The phase of both multipliers are shown in Fig. 6. We find $\mathrm{Re}\{G(j\omega)M(j\omega) - 0.01G^*(j\omega)G(j\omega)\} > 0$ for all frequencies (see Figs. 5 and 7).

5. Conclusions

This paper has analysed the apparent contradiction between different results in the literature for bounded and monotone nonlinearities. The original version of the Zames–Falb theorem has an inherent conservatism for strictly proper plants. This conservatism has been exploited in the literature to suggest that the class of Popov-extended Zames–Falb multipliers is a wider class of multipliers. However, a slightly modified version of the Zames–Falb theorem allows us to extend the equivalence result presented in [9] for the case of slope-restricted nonlinearities to the case of bounded and monotone nonlinearities.

As a conclusion, the Zames–Falb multipliers is also the widest available class of multipliers for bounded and monotone nonlinearities. The example given by Jönsson [11] is used for demonstrating our results.

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