On Local Input-Output Stability of Nonlinear Feedback Systems via Local Graph Separation

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***Abstract*— A new type of local input-output stability for nonlinear systems is defined, called M-local boundedness, which can be viewed as a local version of established definitions of global boundedness. This definition states that the system is bounded if the input Lebesgue signal has a norm smaller than M. Using graph separation concepts and a novel topological argument, which partitions the output space of the system into feasible and infeasible regions based on the restriction of the system input, sufficient conditions for M-local boundedness of a nonlinear feedback system are derived. Using this theorem, a new local nonlinear small gain condition is found for a closed-loop system with additive inputs. This small gain condition is then used in a numerical example, in which a differential equation with a quadratic element was partitioned into a feedback system and bounds on the norm of the input were found which ensured the system was M-locally stable.**

***Index Terms*— Robust control, stability of nonlinear systems.**

I. Introduction

WHEN working with feedback systems, it is crucial to ensure the stability of the closed-loop system. Conditions on the two separate systems are desired such that, when met, stability of the feedback loop is assured. Such conditions were presented by Zames [13], [12], using functional analysis to determine that if the open loop elements can be partitioned into two separate conic sectors, then the closed loop was stable. Both the small gain and passivity theorems were shown to partition the function space into conic sectors.

Years later, Safonov [7] proved that Zames' conic sectors were a special case of the more general graph separation principle. Safonov showed that, if the graphs (that is, the space of all possible input-output pairs) of the two systems can be proven to be separate except at the origin and the distance between the graphs increases monotonically, then the feedback loop must be bounded. The conic sectors presented by Zames were then a tool for "topologically separating" the input-output space. Additionally, Safonov's theory allowed inputs to enter the loop arbitrarily rather than restricting them to additive-only inputs as in Zames' work. Teel [9] in turn expanded upon Safonov's work, providing a simpler version of Safonov's main stability theorem. Teel also combined new nonlinear small gain results [5] that leveraged monotone gain functions with Safonov's improved conic sectors, thus further increasing the scope of the theory.

All of this work dealt with global stability of feedback systems. Definitions of local input-output stability, and its relationship with local Lyapunov stability, have been proposed and proven by Vidyasagar and Vannelli [10] as well as Choi [2]. However, this work deals only with individual systems, not feedback loops. In both [6] and [1], attempts are made to prove theorems of local input-output stability for nonlinear feedback systems, but both works impose restrictions on the systems and system spaces not found in the global input-output stability literature: [6] deals with unique dual-normed spaces, gives only a linear local small gain theorem, and requires disturbances to be additive; and [1] requires signals to belong to Sobolev spaces, and again only gives a type of small gain condition and restricts the system to additive disturbances. It is only in [3] that local conditions are related to system graphs, but in this case it is local robustness to system perturbations that is investigated; local stability is already assumed.

This paper aims to provide a more general and widely applicable local input-output stability theorem by returning to the graph separation ideas of Safonov and Teel. The mild assumption that system outputs belong to an extended Lebesgue space is used to prove the connectedness of the space of system outputs. Using this property, a novel topological line of reasoning is employed to prove the local graph separation result. This theorem relaxes the restriction that the graphs are globally separated and adds the additional constraint that the inputs are restricted to a certain size. Using this result, a local nonlinear small gain condition is derived in a manner similar to that in [9], which is then applied to a feedback system for which stability cannot be proven via global methods. The paper is organised as follows: Section II defines the notation used throughout. Section III introduces established definitions in the field of topology and gives some important preliminary lemmas required for the main result. In Section IV system boundedness is described as per [9], and norm-restricted subsets of function spaces and M-local boundedness are defined. The main result of this paper, a local input-output stability theorem for feedback systems, is given in Section V, which is used to derive a local nonlinear small gain condition in Section VI for which a worked example is given.

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II. Notation

Notation is standard throughout unless stated otherwise.

The Euclidean norm of a vector in $\mathbb{R}^n$ is denoted $|\cdot|$. $\mathcal{L}_p^n$ is the space of Lebesgue measurable functions mapping $[0,\infty)$ to $\mathbb{R}^n$ where the Euclidean norm of the function output raised to the p^{th} power has a finite integral. For $f \in \mathcal{L}_p^n$, $\|f\|^p = \int_{[0,\infty)} |f(t)|^p \mathrm{d}\mu$ where μ is the Lebesgue measure. The truncation operator π_τ is such that $\pi_\tau f(t) = f(t)$ if $\tau \geq t$ and zero otherwise. We often denote $\pi_\tau f$ as f_τ.

The extended Lebesgue space $\mathcal{L}_{pe}^n$ permits functions f when $f_\tau \in \mathcal{L}_p^n$ for all $\tau \in [0,\infty)$.

Systems are described as relations between two (extended) vector spaces. $\Sigma \subset \mathcal{U}_e \times \mathcal{Y}_e$ means that the system Σ is a relation between the spaces $\mathcal{U}_e$ and $\mathcal{Y}_e$. By convention, the space(s) of inputs come(s) before the space(s) of outputs. $(u,y) \in \Sigma$ states that (u,y) is a permissible input-output pair of Σ. The range of Σ is denoted by $\mathrm{rng}(\Sigma)$, such that $(u,y) \in \Sigma \subset \mathcal{U}_e \times \mathcal{Y}_e \Rightarrow y \in \mathrm{rng}(\Sigma) \subset \mathcal{Y}_e$.

III. Topology definitions & lemmas

The main result of this paper is achieved by treating the elements of the closed-loop as mappings between spaces rather than as time-domain based operators. Hence, before the main theory can be presented some machinery from the mathematical field of topology is required. Definitions 1 and 2 given in this section are well established and can be found in [11], for example.

Definition 1 (Path): A path from point x to y in a topological space (respectively subspace, subset) $\mathcal{X}$ is a continuous function of the form $f : [0,1] \to \mathcal{X}$ where $f(0) = x$ and $f(1) = y$.

Definition 2 (Path-connected spaces): A topological space (respectively subspace, subset) $\mathcal{X}$ is path-connected if there exists a path between any two points in $\mathcal{X}$.

Definition 3: Define the set of a Lebesgue function y and all its truncations as T_y, such that

$$T_y = \{y | y \in \mathcal{L}_{pe}^n\} \cup \{y_\tau | y \in \mathcal{L}_{pe}^n, \tau \in [0,\infty)\}. \quad (1)$$

The following lemma proves that T_y is path-connected. This is crucial to the local graph separation result, as it shows that the space of the truncations of a dynamical system's outputs must also be path-connected.

Lemma 1: Let $y \in \mathcal{L}_{pe}^n$. Then T_y is path-connected and contains the origin.

Proof: To prove that there is a path between all truncated functions in T_y, define the function f as

$$f : [0,1] \to T_y, \;\; f(r) = y_{(1-r)\tau_1 + r\tau_2} \quad (2)$$

for any $\tau_1, \tau_2 \in [0,\infty)$. If f is continuous over all r, it is a path between y_{τ_1} and y_{τ_2} and therefore T_y is path-connected.

To prove continuity, we prove that for all $r \in [0,1]$ we have

$$\lim_{\delta r \to 0} \|f(r+\delta r) - f(r)\| = 0. \quad (3)$$

From the definition of the Lebesgue norm we have

$$\|f(r+\delta r) - f(r)\|^p \quad (4)$$
$$= \|y_{(1-r-\delta r)\tau_1 + (r+\delta r)\tau_2} - y_{(1-r)\tau_1 + r\tau_2}\|^p \quad (5)$$
$$= \int_{[0,\infty)} |y_{(1-r-\delta r)\tau_1 + (r+\delta r)\tau_2}(t) - y_{(1-r)\tau_1 + r\tau_2}(t)|^p \mathrm{d}\mu \quad (6)$$
$$= \int_{[(1-r)\tau_1 + r\tau_2, (1-r-\delta r)\tau_1 + (r+\delta r)\tau_2]} |y(t)|^p \mathrm{d}\mu. \quad (7)$$

As $\delta r \to 0$, the limits of integration coincide such that the integration is over a point. By definition, a point has zero Lebesgue measure, and so for all $r \in [0,1]$, all $y \in \mathcal{L}_{pe}^n$ and all $\tau_1, \tau_2 \in [0,\infty)$, we have $\lim_{\delta r \to 0} \|f(r+\delta r) - f(r)\| = 0$, proving that f is continuous. Therefore, there is a path between all truncations of y in T_y.

It can be proven similarly that there is a path in T_y from all truncations y_τ to y by taking the limit as $\tau_2 \to \infty$, as $\lim_{\tau \to \infty} y_\tau = y$. Hence all pairs of points in T_y are joined by a path in T_y, and so T_y is path-connected.

The proof that T_y includes the origin for all $y \in \mathcal{L}_{pe}^n$ follows a similar argument. Investigating the norm of y_0 gives:

$$\|y_0\|^p = \int_{[0,\infty)} |y_0(t)|^p \mathrm{d}\mu \quad (8)$$
$$= \int_{[0,0]} |y(t)|^p \mathrm{d}\mu = 0. \quad (9)$$

This result follows as, once again, the Lebesgue measure of a point is zero. Hence, y_0 is equal to zero almost everywhere; equivalently, y_0 is at the origin of $\mathcal{L}_{pe}^n$ for all $y \in \mathcal{L}_{pe}^n$. By definition $y_0 \in T_y$, and hence T_y must include the origin. ■

Definition 4: Let $(u,y) \in \Sigma \subset \mathcal{U}_e \times \mathcal{Y}_e$, where $\mathcal{U}_e$, $\mathcal{Y}_e$ are extended Lebesgue spaces. Define the output space of Σ, the space which includes all possible outputs and truncations of outputs of Σ, as $\mathcal{Y}_\Sigma$ such that

$$\mathcal{Y}_\Sigma = \bigcup_{y \in \mathrm{rng}(\Sigma)} T_y. \quad (10)$$

Lemma 2: Let $(u,y) \in \Sigma \subset \mathcal{U}_e \times \mathcal{Y}_e$, where $\mathcal{U}_e$, $\mathcal{Y}_e$ are extended Lebesgue spaces. Then $\mathcal{Y}_\Sigma$ is path-connected and contains the origin.

Proof: By Lemma 1, T_y is path-connected and contains the origin. Therefore, the union of many such T_y will contain the origin and will be path-connected as the origin is a common point for every T_y. ■

IV. Boundedness, norm-restricted subsets and M-local boundedness

Having introduced the required topological material which will be necessary for the proof of the main theorem, the concepts of system boundedness, subsets of signals with bounded norms and the original concept of M-local boundedness will be covered in this section. These are the final tools required before the local graph separation theorem can be presented. The definitions regarding boundedness are taken from [9].

Definition 5 (Class-$\mathcal{K}$ functions [9]): Let a function γ be the mapping $\gamma : [0,\infty) \to [0,\infty)$. Then:

- γ is in class $\mathcal{K}_+$ (denoted $\gamma \in \mathcal{K}_+$) if it is continuous and non-decreasing;
- γ is in class $\mathcal{K}$ (denoted $\gamma \in \mathcal{K}$) if it is continuous, non-decreasing and zero at zero;
- γ is in class $\mathcal{K}_l$ (denoted $\gamma \in \mathcal{K}_l$) if it is linear, i.e. $\exists\, k > 0$ such that $\gamma(s) = ks$.

Definition 6 (System boundedness [9]): A dynamical system Σ is

- bounded if there exists a function $\gamma \in \mathcal{K}_+$
- stable if there exists a function $\gamma \in \mathcal{K}$
- finite-gain stable if there exists a function $\gamma \in \mathcal{K}_l$

such that, for all $(u, y) \in \Sigma \subset \mathcal{U}_e \times \mathcal{Y}_e$ where $\mathcal{U}_e$ and $\mathcal{Y}_e$ are extended Lebesgue spaces, and for all $\tau \in [0, \infty)$, we have $\|y_\tau\| \leq \gamma(\|u_\tau\|)$. As $\mathcal{K}_l \in \mathcal{K} \in \mathcal{K}_+$, we will use 'bounded with gain γ' as a generic phrase, where the stability ($\gamma \in \mathcal{K}$) or finite-gain stability ($\gamma \in \mathcal{K}_l$) of the system can be determined by the properties of the gain function γ.

As stated in the introduction, the local graph separation theorem requires the feedback system inputs to be in some sense bounded. The following definition formalises this notion.

Definition 7 (Norm-restricted subsets of function spaces): Let $x \in \mathcal{X}_e$ where $\mathcal{X}_e$ is an extended Lebesgue space. Define the subset $\mathcal{X}_e^{\leq M} \subset \mathcal{X}_e$ as the set of all x which have Lebesgue norm less than or equal to M, i.e.

$$\mathcal{X}_e^{\leq M} = \{x \in \mathcal{X}_e \mid \|x\| \leq M \text{ where } M > 0\}, \tag{11}$$

where $\|\cdot\|$ is the Lebesgue norm with which $\mathcal{X}_e$ is equipped.

Remark 1: By definition, $\mathcal{X}_e^{\leq M}$ is a sphere of radius M centred at the origin, and $\mathcal{X}_e^{\leq M} \subset \mathcal{X}_e^{\leq N}$ for all $N > M$.

Definition 8 (System with norm-restricted input): Let the dynamical system Σ be such that $(u, y) \in \Sigma \subset \mathcal{U}_e \times \mathcal{Y}_e$. Additionally, let the input of Σ be restricted such that $u \in \mathcal{U}_e^{\leq M}$. We then denote the system Σ with norm restricted input as $\Sigma^{\leq M}$.

Remark 2: Note that the output of $\Sigma^{\leq M}$, y, is not norm restricted in the same way as u. Instead, the bound on y is determined by M and the gain of $\Sigma^{\leq M}$ as will be seen in Definition 9.

We now give our definition of local stability. Note that this is subtly different to other local stability definitions given in the literature (e.g. [10], [2], [6], [1]) as the maximum input size is given directly in the definition. This is practically useful as it informs the engineer of which inputs will and will not render the system stable. Furthermore, it also permits a nonlinear gain function γ.

Definition 9 (M-local boundedness): Consider a dynamical system with norm-restricted input given by $(u, y) \in \Sigma^{\leq M}$, where $\Sigma \subset \mathcal{U}_e \times \mathcal{Y}_e$ and $\mathcal{U}_e$, $\mathcal{Y}_e$ are extended Lebesgue spaces. Then, $\Sigma^{\leq M}$ is M-locally bounded with gain γ if there exists $\gamma \in \mathcal{K}_+$ such that, for all $(u, y) \in \Sigma^{\leq M}$ and all $\tau \in [0, \infty)$, we have $\|y_\tau\| \leq \gamma(\|u_\tau\|) \leq \gamma(M)$.

Remark 3: The definition of M-local boundedness can be seen as an extension of system boundedness. The restriction in Definition 6 that $\|y\|_\tau \leq \gamma(\|u_\tau\|)$ holds for all input-output pairs of Σ is relaxed, instead only requiring that it holds for all $(u, y) \in \Sigma$ when $u \in \mathcal{U}_e^{\leq M}$, in return for only guaranteeing boundedness when $u \in \mathcal{U}_e^{\leq M}$ instead of guaranteeing global boundedness. If $M \to \infty$, $\Sigma^{\leq M} \to \Sigma$ and the definition of M-local boundedness becomes that of system boundedness.

V. Canonical Feedback System and Local Graph Separation Theorem

This section defines the canonical feedback system and presents the main result of this paper: the local graph separation theorem which guarantees M-local boundedness of said system. This result is reached by application of a novel topological argument which separates the feasible region of the closed-loop output space from the infeasible region, thus allowing the space to be partitioned and requiring stability conditions to be checked only within the feasible region. Without such topological separation, the stability conditions would have to be met within the entire output space, as the feasible region of the output space would be unknown.

The canonical feedback system given below is of the general form used by both Teel and Safonov ([9], [7]). Unlike traditional feedback representations, it allows the inputs to enter each feedback element arbitrarily rather than restricting them to be additive. The signals u_1 and u_2 could be the initial states of $\Sigma^{(1)}$ and $\Sigma^{(2)}$, for example. This allows for a more general and therefore more powerful M-local boundedness theorem, which can always be specialised to fit a given feedback system.

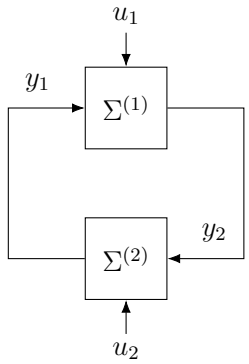

Fig. 1. Canonical feedback system.

Consider the system shown in Figure 1. We define $u = (u_1, u_2)$, $y = (y_1, y_2)$ and $z = (u, y)$. Let $\mathcal{U}_{1e}$, $\mathcal{U}_{2e}$, $\mathcal{Y}_{1e}$, $\mathcal{Y}_{2e}$, $\mathcal{U}_e$, $\mathcal{Y}_e$ and $\mathcal{Z}_e$ be extended Lebesgue spaces such that $y = (y_1, y_2) \in \mathcal{Y}_{1e} \times \mathcal{Y}_{2e} = \mathcal{Y}_e$, $u = (u_1, u_2) \in \mathcal{U}_{1e} \times \mathcal{U}_{2e} = \mathcal{U}_e$ and $z = (u, y) \in \mathcal{U}_e \times \mathcal{Y}_e = \mathcal{Z}_e$. Finally, define the entire system as $\underline{\Sigma}$, so $(u, y) \in \underline{\Sigma} \subset \mathcal{U}_e \times \mathcal{Y}_e = \mathcal{Z}_e$.

Definition 10 (Graph of $\Sigma^{(i)}$ [9]): For each $\Sigma^{(i)}$, $(i \in \{1, 2\})$ in Figure 1, we define the graph of $\Sigma^{(i)}$ at input u_i as the set of points $(y_1, y_2) \in \mathcal{Y}_{1e} \times \mathcal{Y}_{2e}$ satisfying $(u_i, y_1, y_2) \in \Sigma^{(i)}$, i.e.

$$\mathcal{G}^{(i)}(u_i) = \{(y_1, y_2) \in \mathcal{Y}_{1e} \times \mathcal{Y}_{2e} \mid (u_i, y_1, y_2) \in \Sigma^{(i)}\}. \tag{12}$$

Remark 4: Via Definition 10, the system in Figure 1 can be described by $y \in \mathcal{G}^{(1)}(u_1) \cap \mathcal{G}^{(2)}(u_2)$.

The following theorem, the main result of this paper, is modelled on Proposition 5.1 of [9]. It states that if the conditions given in Proposition 5.1 hold only for bounded u and y, not for all u and y, then the system is M-locally bounded. These conditions must hold in the region where $\|y_\tau\| \leq \gamma(M) + \epsilon$, where γ is the gain of the closed-loop, M is the bound on u and ϵ is some positive real number. This is to ensure that all outputs for which $\|y_\tau\| \leq \gamma(\|u_\tau\|)$ is true can reside within the bounded region of $\mathcal{Y}_e$ where the conditions which ensure boundedness hold.

Theorem 1 (M-local boundedness of Figure 1): Consider the system in Figure 1. Suppose there exists some functionals

$\lambda^{(1)},\ \lambda^{(2)} : \mathcal{Z} \to \mathbb{R}$, some $\gamma \in \mathcal{K}$, and some $M, \epsilon > 0$ such that, for all $\tau \in [0, \infty)$ and all Lebesgue functions

$$z = (u, y) \in \mathcal{U}_e^{\leq M} \times \mathcal{Y}_e^{\leq(\gamma(M)+\epsilon)}, \tag{13}$$

we have

$$y \in \mathcal{G}^{(1)}(u_1) \Rightarrow \lambda^{(1)}(z_\tau) \leq 0 \tag{14a}$$

$$y \in \mathcal{G}^{(2)}(u_2) \Rightarrow \lambda^{(2)}(z_\tau) \leq 0 \tag{14b}$$

$$\max\{\lambda^{(1)}(z_\tau), \lambda^{(2)}(z_\tau)\} \leq 0 \Rightarrow \|y_\tau\| \leq \gamma(\|u_\tau\|). \tag{14c}$$

Then the system is M-locally bounded with gain γ.

Proof: If the conditions given in (14) held for all $(u, y) \in \mathcal{U}_e \times \mathcal{Y}_e$, then the system would be globally bounded with gain γ as per Proposition 5.1 in [9].

However, these conditions only hold for $(u, y) \in \mathcal{U}_e^{\leq M} \times \mathcal{Y}_e^{\leq(\gamma(M)+\epsilon)}$. From Lemma 2, we have that

$$(u, y) \in \underline{\Sigma}^{\leq M} \Rightarrow y \in \mathcal{Y}_{\underline{\Sigma}^{\leq M}} \text{ and } y_\tau \in \mathcal{Y}_{\underline{\Sigma}^{\leq M}} \forall \tau \in [0, \infty)$$

where $\mathcal{Y}_{\underline{\Sigma}^{\leq M}}$ is path-connected and includes the origin. To prove M-local boundedness, we must prove that

$$\mathcal{Y}_{\underline{\Sigma}^{\leq M}} \subset \mathcal{Y}_e^{\leq(\gamma(M)+\epsilon)}.$$

If this is true, u and y both reside within the required norm restricted subsets for conditions (14) to hold, and hence M-local boundedness of the system in Figure 1 follows.

We will find $\mathcal{Y}_{\underline{\Sigma}^{\leq M}}$ via contradiction. Consider the spherical hyper-surface, centred on the origin, defined by the set

$$Y^* = \{y \in \mathcal{Y} \mid \|y\| = \gamma(M) + \epsilon^* \text{ with } \epsilon^* > 0\}. \tag{15}$$

Assume that $y \in Y^* \cap \mathcal{G}^{(1)}(u_1) \cap \mathcal{G}^{(2)}(u_2)$, i.e. some point $y \in Y^*$ is a solution to the system, and that $u \in \mathcal{U}_e^{\leq M}$. Furthermore, assume that $\epsilon^* = \epsilon$. Then, by definition of Y^*, $y \in \mathcal{Y}_e^{\leq(\gamma(M)+\epsilon)}$ and therefore conditions (14a) and (14b) are satisfied, which in turn satisfy (14c) giving $\|y_\tau\| \leq \gamma(\|u_\tau\|)$ for all $\tau \in [0, \infty)$. However, by definition $\|u_\tau\| \leq M$, and $\|y_\tau\| \leq \gamma(M)$ for all $\tau \in [0, \infty)$ implies that $\|y_\tau\| < \gamma(M)+\epsilon$ for all $\tau \in [0, \infty)$ for any $\epsilon > 0$. This contradicts our assumption that $y \in Y^*$, as

$$\|y\| = \sup_{\tau \in [0,\infty)} \|y_\tau\| = \lim_{\tau \to \infty} \|y_\tau\|, \tag{16}$$

because $\|y_\tau\|$ is monotonically increasing with τ. Hence, if $u \in \mathcal{U}_e^{\leq M}$ and $\epsilon^* = \epsilon$, conditions (14) imply that $Y^* \cap \mathcal{G}^{(1)}(u_1) \cap \mathcal{G}^{(2)}(u_2) = \emptyset$. This means that Y^* is an infeasible set for all $y \in \mathcal{G}^{(1)}(u_1) \cap \mathcal{G}^{(2)}(u_2)$ when $(u, y) \in \underline{\Sigma}^{\leq M}$. In (15) we defined Y^* as a spherical hyper-surface centred on the origin. By Lemma 2, $\mathcal{Y}_{\underline{\Sigma}^{\leq M}}$ is path-connected, therefore it cannot exist both in the space enclosed by Y^* and in the space outside of Y^*. Additionally, the same lemma states that $\mathcal{Y}_{\underline{\Sigma}^{\leq M}}$ must contain the origin, so $\mathcal{Y}_{\underline{\Sigma}^{\leq M}}$ must be entirely within the space enclosed by Y^*. This proves that $\mathcal{Y}_{\underline{\Sigma}^{\leq M}} \subset \mathcal{Y}_e^{\leq(\gamma(M)+\epsilon)}$, as Y^* is exactly the boundary of $\mathcal{Y}_e^{\leq(\gamma(M)+\epsilon)}$, and hence the system is M-locally bounded with gain γ. ∎

In the above theorem, as $\epsilon > 0$, we can replace $\gamma(M) + \epsilon$ everywhere with some $N > 0$, given N is chosen such that $N \geq \gamma(M) + \epsilon > \gamma(M)$. This gives the following corollary to Theorem 1, which simplifies the application of the theorem greatly as it allows N to be chosen a priori without knowledge of γ, and instead we check that $N > \gamma(M)$ once γ has been found.

Corollary 1: Consider the system in Figure 1. Suppose there exists some functionals $\lambda^{(1)},\ \lambda^{(2)} : \mathcal{Z} \to \mathbb{R}$, some $\gamma \in \mathcal{K}$ and some $M, N \in \mathbb{R}^+$, such that for all $\tau \in \mathbb{R}^+$ and all Lebesgue functions $z = (u, y) \in \mathcal{U}_e^M \times \mathcal{Y}_e^N$, we have

$$y \in \mathcal{G}^{(1)}(u_1) \Rightarrow \lambda^{(1)}(z_\tau) \leq 0 \tag{17a}$$

$$y \in \mathcal{G}^{(2)}(u_2) \Rightarrow \lambda^{(2)}(z_\tau) \leq 0 \tag{17b}$$

$$\max\{\lambda^{(1)}(z_\tau), \lambda^{(2)}(z_\tau)\} \leq 0 \Rightarrow \|y_\tau\| \leq \gamma(\|u_\tau\|). \tag{17c}$$

Then, if $N > \gamma(M)$, the system is M-locally bounded with gain γ.

Remark 5: Analogous to the explanation given in [9], Theorem 1 and Corollary 1 can be seen as local graph separation conditions by viewing the functionals $\lambda^{(1)}$, $\lambda^{(2)}$ as defining the boundaries of $\mathcal{G}^{(1)}$ and $\mathcal{G}^{(2)}$. The first two conditions ensure that the graphs of each system are the points at which $\lambda^{(1)}(z_\tau) \leq 0$ and $\lambda^{(2)}(z_\tau) \leq 0$ respectively, and the third condition ensures that the output y is related to the input u via the gain γ wherever these graphs intersect. The added requirement that these conditions need only hold when the norms of y and u are less than a certain threshold differentiates this graph separation condition as local rather than global.

It is suggested that Theorem 1 and Corollary 1 are applied using slightly different methods; the method for applying Corollary 1 is iterative and the method for applying Theorem 1 is not. These methods are described below.

To apply Corollary 1, do the following:

1) Choose values for M and N a priori.
2) Find functionals $\lambda^{(1)}, \lambda^{(2)}$ such that (17) hold for the chosen M and N.
3) If $N > \gamma(M)$ using the gain function γ found via (17c), the system is M-locally bounded. If not, return to step 1 and choose a smaller M or a larger N. Repeat these steps until we have $N > \gamma(M)$.

The method for applying Theorem 1 is direct but only possible if relatively simple functionals $\lambda^{(1)}, \lambda^{(2)}$ can be found. It is as follows:

1) Find functionals $\lambda^{(1)}, \lambda^{(2)}$ for which (17a), (17b) hold globally.
2) Find bounds on M and $\gamma(M) + \epsilon$ such that (17c) is satisfied when M and $\gamma(M)+\epsilon$ are within these bounds.
3) Using these bounds, find the maximum value of M such that the bound on M and the bound on $\gamma(M) + \epsilon$ are both satisfied for a chosen value of ϵ.

The benefit of applying Corollary 1 as above is that it is far less computationally intensive and is applicable to more complex systems. All that is required is an M small enough and an N large enough for the conditions to hold, which is checked at the final step.

The benefit of applying Theorem 1 is that it produces a maximum possible M for a given ϵ and γ, which is not true of the method for applying Corollary 1. However, note that conservatism is often introduced both by the choice of functionals and the method in which γ is found, so this result is only optimal given those choices.

On the subject of conservatism, when applying either of these methods great care must be taken when finding a suitable γ. Unlike for global boundedness proofs, where only the fact that γ belonged to $\mathcal{K}_+$ was of importance, now the value which $\gamma(M)$ evaluates to is also important. Therefore, the less conservatism that can be introduced when finding γ, such as by application of inequality relations like the triangle inequality or various signal norm inequalities, the smaller $\gamma(M)$ will be and hence the larger M is permitted to be.

VI. Nonlinear local small gain example

A. Derivation of nonlinear local small gain condition

As an example of an application of Corollary 1, we will derive a nonlinear local small gain condition for a system with additive disturbances as shown in Figure 2. This example is structurally similar to the nonlinear small gain condition derived by Teel in [9, Section V-B].

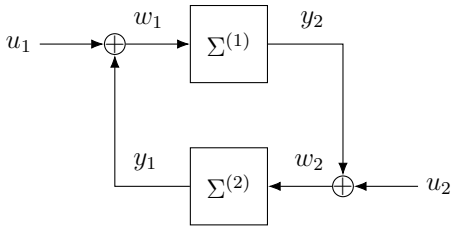

Fig. 2. Feedback system with additive disturbances.

Theorem 2 (Nonlinear local small gain condition): Define $\|u\| = \max\{\|u_1\|, \|u_2\|\}$ and $\|y\| = \max\{\|y_1\|, \|y_2\|\}$. Suppose there exists some $\gamma^{(1)}, \gamma^{(2)} \in \mathcal{K}$ such that for $(i,j) \in \{(1,2),(2,1)\}$

$$\gamma^{(i)} \circ \gamma^{(j)}(s) < s \quad \forall\, s \in [0, M+N] \tag{18}$$

where $M, N > 0$, and that

$$y \in \mathcal{G}^{(1)}(u_1) \Rightarrow \|y_{2,\tau}\| \leq \gamma^{\star(1)}(\|u_{1,\tau} + y_{1,\tau}\|), \tag{19}$$

$$y \in \mathcal{G}^{(2)}(u_2) \Rightarrow \|y_{1,\tau}\| \leq \gamma^{\star(2)}(\|u_{2,\tau} + y_{2,\tau}\|) \tag{20}$$

for all $\tau \in [0,\infty)$, where $\gamma^{\star(\cdot)} = \frac{1}{2}\gamma^{(\cdot)}$.

Define $\gamma \in \mathcal{K}$ as

$$\gamma(s) = \max\{s, \gamma^{\star(1)}(2s), \gamma^{\star(2)}(2s)\}. \tag{21}$$

Then, by Corollary 1, the closed-loop system of $\Sigma^{(1)}$ and $\Sigma^{(2)}$ given in Fig. 2 is M-locally bounded with gain γ if $N > \gamma(M)$.

Proof: Define the functionals

$$\lambda^{(1)}(z_\tau) = \|y_{2,\tau}\| - \gamma^{\star(1)}(\|y_{1,\tau} + u_{1,\tau}\|), \tag{22}$$

$$\lambda^{(2)}(z_\tau) = \|y_{1,\tau}\| - \gamma^{\star(2)}(\|y_{2,\tau} + u_{2,\tau}\|). \tag{23}$$

From (19) and (20), this gives $y \in \mathcal{G}^{(1)}(u_1) \Rightarrow \lambda^{(1)}(z_\tau) \leq 0$ and $y \in \mathcal{G}^{(2)}(u_2) \Rightarrow \lambda^{(2)}(z_\tau) \leq 0$.

Next, we must prove that $\max\{\lambda^{(1)}(z_\tau), \lambda^{(2)}(z_\tau)\} \leq 0 \Rightarrow \|y_\tau\| \leq \gamma(\|u_\tau\|)$ for some $\gamma \in \mathcal{K}_+$ and for all $\tau \in [0,\infty)$. First, we have

$$\max\{\lambda^{(1)}(z_\tau), \lambda^{(2)}(z_\tau)\} \leq 0 \Rightarrow$$
$$\|y_\tau\| \leq \max\{\gamma^{\star(1)}(\|y_{1,\tau} + u_{1,\tau}\|),$$
$$\gamma^{\star(2)}(\|y_{2,\tau} + u_{2,\tau}\|)\} \tag{24}$$

For $i \in \{1,2\}$, by monotonicity of $\gamma^{\star(i)}$ and the triangle inequality, we have

$$\gamma^{\star(i)}(\|y_{i,\tau} + u_{i,\tau}\|) \leq$$
$$\max\{\gamma^{\star(i)}(2\|y_{i,\tau}\|), \gamma^{\star(i)}(2\|u_{i,\tau}\|)\} \tag{25}$$

Then for $(i,j) \in \{(1,2),(2,1)\}$, via (19), (20), (18) and the triangle inequality, for all $\|u_\tau\| \leq M$ and $\|y_\tau\| \leq N$ we have

$$\gamma^{\star(i)}(2\|y_{i,\tau}\|) \leq \gamma^{\star(i)} \circ \left[2\gamma^{\star(j)}(\|y_{j,\tau} + u_{j,\tau}\|)\right] \tag{26}$$

$$\leq \frac{1}{2}\gamma^{(i)} \circ \gamma^{(j)}(\|y_{j,\tau} + u_{j,\tau}\|) \tag{27}$$

$$< \frac{1}{2}\|y_{j,\tau} + u_{j,\tau}\| \tag{28}$$

$$< \max\{\|y_{j,\tau}\|, \|u_{j,\tau}\|\}. \tag{29}$$

Substituting (25) and (29) into (24), for all $\|u_\tau\| \leq M$ and $\|y_\tau\| \leq N$ we have

$$\|y_\tau\| \leq \max\{\|y_{1,\tau}\|, \|y_{2,\tau}\|, \|u_{1,\tau}\|, \|u_{2,\tau}\|,$$
$$\gamma^{\star(1)}(2\|u_{1,\tau}\|), \gamma^{\star(2)}(2\|u_{2,\tau}\|)\}. \tag{30}$$

Now assume that $\|u_\tau\| \leq M$, $\|y_\tau\| \leq N$ and

$$\|y_\tau\| > \max\{\|u_{1,\tau}\|, \|u_{2,\tau}\|,$$
$$\gamma^{\star(1)}(2\|u_{1,\tau}\|), \gamma^{\star(2)}(2\|u_{2,\tau}\|)\}. \tag{31}$$

By (29) and (30), this must give $\|y_\tau\| < \max\{\|y_{1,\tau}\|, \|y_{2,\tau}\|\}$. However, this contradicts the definition of $\|y\|$, which means we must have

$$\|y_\tau\| \leq \max\{\|u_{1,\tau}\|, \|u_{2,\tau}\|,$$
$$\gamma^{\star(1)}(2\|u_{1,\tau}\|), \gamma^{\star(2)}(2\|u_{2,\tau}\|)\}$$
$$\leq \gamma(\|u_\tau\|) \tag{32}$$

when $\|u_\tau\| \leq M$ and $\|y_\tau\| \leq N$, for all $\tau \in [0,\infty)$ and where γ is given by (21). Hence, if $N > \gamma(M)$, by Corollary 1 the closed loop system is M-locally bounded with gain γ. ■

Remark 6: Note that while local stability via small gain theory for such an interconnection is intuitive if the input signals are small enough, the result given in Theorem 2 provides quantitative bounds on the size of input signals the closed-loop can permit before becoming unstable, and the possible range of output signals these inputs could produce.

B. Application of nonlinear local small gain with quadratic nonlinearity

It will be illustrated how this nonlinear local small gain condition can be applied to a real-world system. The following derivation applies Theorem 2 to a feedback loop where one element is an LTI system and another is a static nonlinearity the gain of which can be upper-bounded by a quadratic function.

Suppose that, for the systems depicted in Figure 2, the system $\Sigma^{(2)}$ is a linear time invariant system G and the system $\Sigma^{(1)}$ is a static nonlinearity Φ with magnitude that can be bounded by a quadratic such that

$$\|y_{1,\tau}\| \leq \|G\|\|w_{2,\tau}\| \text{ and} \tag{33}$$

$$\|y_{2,\tau}\| \leq k\|w_{1,\tau}\|^2 \tag{34}$$

where the Lebesgue norm $\|\cdot\|$ is the $\mathcal{L}_\infty$ norm given by $\|\cdot\| = \sup_t |\cdot(t)|$, $\|G\|$ is the induced $\mathcal{L}_\infty$ to $\mathcal{L}_\infty$ gain of the LTI system G, and $k > 0$.

Comparing to Theorem 2, this gives $\gamma^{\star(1)}(s) = ks^2$ and $\gamma^{\star(2)}(s) = \|G\|s$. By Theorem 1, Theorem 2 holds with N replaced everywhere with $\gamma(M) + \epsilon$ for any $\epsilon < 0$. Hence, for stability, we require for $(i, j) \in \{(1, 2), (2, 1)\}$ that

$$\gamma^{(i)} \circ \gamma^{(j)}(s) < s \quad \forall s \in [0, M + \gamma(M) + \epsilon]. \tag{35}$$

First, for $(i, j) = (1, 2)$ we require

$$2\gamma^{\star(1)} \circ \left[2\gamma^{\star(2)}(s)\right] < s \Leftrightarrow \tag{36}$$

$$2k(2\|G\|s)^2 = 8k\|G\|^2 s^2 < s \Leftrightarrow s < (8k\|G\|^2)^{-1}, \tag{37}$$

and for $(i, j) = (2, 1)$ we require

$$2\gamma^{\star(2)} \circ \left[2\gamma^{\star(1)}(s)\right] < s \Leftrightarrow \tag{38}$$

$$2\|G\|(2ks^2) < s \Leftrightarrow s < (4k\|G\|)^{-1}. \tag{39}$$

Putting together (37) and (39) gives the requirement that

$$s < (4k\|G\|)^{-1} \times \min\{(2\|G\|)^{-1}, 1\} = \delta \tag{40}$$

Hence, the system is M-locally stable (as $\gamma \in \mathcal{K}$) for all M where $M + \gamma(M) + \epsilon < \delta$ for some $\epsilon > 0$ where γ and δ are given by (21) and (40) respectively.

Substituting $\gamma^{\star(1)}$ and $\gamma^{\star(2)}$ into (21) gives

$$M + \gamma(M) = M \times \max\{\alpha, M\beta\} \tag{41}$$

where

$$\alpha = \max\{2, 2\|G\| + 1\} \text{ and } \beta = 1 + 4k. \tag{42}$$

Therefore, (35) is satisfied if both

$$M\alpha < \delta - \epsilon \text{ and } M^2\beta < \delta - \epsilon. \tag{43}$$

Combining both equations in (43) proves that the system is M-locally stable with gain γ for all M such that

$$M < \min\{(\delta - \epsilon)/\alpha, [(\delta - \epsilon)/\beta]^{\frac{1}{2}}\} \tag{44}$$

where α, β and δ are given by (42) and (40) respectively.

C. Numerical example

Finally, we will work through a numerical example using this result. Consider the forced nonlinear differential equation with quadratic nonlinearity with state x and input u, given by

$$\ddot{x} + \dot{x} + 100x + 0.88x^2 = 3.53u. \tag{45}$$

Stability of such a system can be analysed directly via well known techniques (see [4], [8] for examples). However, it can also be analysed via the result given in Section VI-B by re-framing it as the feedback interconnection of an LTI system and a static quadratic nonlinearity.

Such a re-framing gives a quadratic nonlinearity Φ with gain $k = 0.25$ for $\Sigma^{(1)}$ and a second-order LTI system G with $\|G\| = 0.5$ for $\Sigma^{(2)}$. Substituting these values into (44) and choosing a sufficiently small ϵ yields that this system is M-locally stable with gain γ for all $M < 1$. Note that the quadratic term in (45) ensures the system is not globally stable, rendering global graph separation approaches such as the Circle or Popov criteria incompatible.

Remark 7: This result is quite conservative, as is common with small gain conditions. However, local versions of global stability tools which are in general less conservative than small gain, such as the IQC framework and Zames-Falb multipliers, should give less conservative results than those derived here. This is instead included as an example of how Theorem 1 and Corollary 1 can be applied.

VII. Conclusion

In this paper the concept of M-local boundedness was introduced, which is more general than previous local stability conditions which have been defined in the literature. By applying graph separation principles from [7], [9] (which give sufficient conditions for the global boundedness of feedback systems) in a local manner and using properties of the Lebesgue integral alongside a novel topological argument, a theorem guaranteeing M-local boundedness of a feedback system with Lebesgue outputs was proven. This theorem was then used to find a local nonlinear small gain condition for a system with additive disturbances, and this condition was applied to a system described by a nonlinear differential equation as a numerical example to give bounds on the inputs which guarantee that the system is M-locally stable.

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