Negative Imaginary Theory for a Class of Linear Time-Varying Systems

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***Abstract*—This letter introduces the notion of linear time-varying (LTV) negative imaginary systems. LTV negative imaginary systems are defined using a time-domain dissipative supply rate $w(u, \dot{y})$ that depends on input to the system (u), time-derivative of the system's output ($\dot{y}$) and an index $\delta \geq 0$. For $\delta > 0$, it gives rise to a strict subclass within the LTV negative imaginary systems, termed as LTV output strictly negative imaginary systems. For characterizing the proposed class of systems, a set of linear differential matrix inequality conditions is derived based on the given state-space realization. Subsequently, LTV negative imaginary theory is specialized to linear parameter-varying (LPV) cases for which, the differential matrix inequality conditions can easily be avoided by considering the rate of variation of the uncertain parameters as independent LMI variables. Finally, a set of sufficient conditions is derived which ensures that the origin is a globally asymptotically stable equilibrium point of an unforced positive feedback interconnection of two uniformly asymptotically stable LTV negative imaginary systems.**

***Index Terms*—LTV negative imaginary systems, LPV negative imaginary systems, non autonomous systems, global uniform asymptotic stability.**

I. Introduction

NEGATIVE imaginary (NI) systems theory was introduced in [1] and was inspired by the positive position feedback control of lightly-damped flexible structures with colocated position sensors and force actuators. In the SISO setting, NI property translates into a class of systems having negative imaginary frequency response, that is, the Nyquist plot of such systems reside in the union of third and fourth quadrants of the complex plane for all $\omega \in (0, \infty)$. NI theory finds potential applications in vibration control of lightly-damped flexible structures [1], cantilever beams [2], large space structures [3] and robotic manipulators [3], in control of nano-positioning systems, etc. NI theory has drawn significant interest of the control theorists and practising engineers over the past twelve years mainly due to its simple internal stability condition that depends only on the DC loop gain. Of late, NI theory has been extended to improper and non-rational systems [4], [5] and also to discrete-time LTI systems [6].

Although negative imaginary literature has witnessed rapid progress in both theory and applications over the past decade, the area of linear time-varying NI systems is not yet well explored. From the system theory perspective, the time-domain formulation of NI systems closely resembles (but are not identical to) the notions of counterclockwise dynamics [7] and input-output Hamiltonian systems [8]. In [9], NI theory has been extended to SISO LTI systems with parametric uncertainty that varies in known intervals. However, to the best of the authors' knowledge, the notion of LTV NI systems has not been addressed so far in the NI literature.

This letter lays the theoretical foundation of LTV NI systems theory. Such systems are described via a time-domain input-output approach exploiting the concept of classical dissipativity theory. A strict subset of the LTV NI class, called the LTV Output Strictly Negative Imaginary (OSNI) systems, is also proposed. State-space characterizations are provided for both LTV NI and OSNI systems which rely on linear differential matrix inequalities (LDMI). Subsequently, LTV NI systems have been specialized to LPV NI systems to be able to use the LMIs instead of the LDMIs. Finally, a closed-loop stability result is also established for a positive feedback interconnection of LTV NI and OSNI systems.

Notation: The space of all real-valued, absolutely continuous time-domain functions $f : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$ is denoted by $AC(\mathbb{R}_{\geq 0})$. If $f(t) \in AC(\mathbb{R}_{\geq 0})$, then $\dot{f}(t) = \frac{\mathrm{d}}{\mathrm{d}t} f(t)$ exists forward in time as a measurable function that remains bounded almost everywhere [10]. The space of all locally square integrable, absolutely continuous time-domain functions is defined as $\mathbb{U}^m = \{f : \mathbb{R} \to \mathbb{R}^m : f(t) = 0 \text{ when } t < 0, \int_0^T f(t)^T f(t)\,\mathrm{d}t < \infty\ \forall T \in [0, \infty) \text{ and } f(t) \in AC(\mathbb{R}_{\geq 0})\}$.

II. Linear Time-Varying NI Systems Theory

A. Definition and Properties

Consider a class of finite-dimensional, square, LTV dynamical systems described by the state-space equations

$$\Sigma : \begin{cases} \dot{x}(t) = A(t)x(t) + B(t)u(t), \quad x(0) = x_0; \\ y(t) = Cx(t), \end{cases} \tag{1}$$

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where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$ and $y(t) \in \mathbb{R}^m$ $\forall t \in \mathbb{R}_{\geq 0}$ and the matrices $A(t)$ and $B(t)$ are assumed to be continuous and bounded $\forall t \in \mathbb{R}_{\geq 0}$. Note that in this section, the admissible inputs u are considered to be in the space $\mathbb{U}^m$ along with sufficient smoothness properties such that unique solution of the state trajectory $x(t)$ exists forward in time $\forall t \in \mathbb{R}_{\geq 0}$ and also $x \in \mathscr{L}_{2e}^n$. Hence $\dot{y}(t) = C\dot{x}(t) = CA(t)x(t) + CB(t)u(t)$ also exists forward in time and $\dot{y} \in \mathscr{L}_{2e}^m$.

Definition 1 (LTV NI Systems): Let Σ be a finite-dimensional, square, LTV system as described in (1). Then Σ is said to be an LTV NI system if there exists a constant $\beta \in \mathbb{R}$ such that

$$\int_0^T \dot{y}(t)^T u(t)\, \mathrm{d}t \geq \beta \tag{2}$$

for any admissible $u \in \mathbb{U}^m$, any initial condition $x_0 \in \mathbb{R}^n$ and all $T \in [0, \infty)$.

Remark 1: In the SISO LTI case, the name 'negative imaginary' is motivated by the Nyquist plot of the transfer function being restricted to the third and fourth quadrants of the Nyquist plane over $\omega \in \mathbb{R}_{\geq 0}$. Since this letter considers LTV systems of the state-space form (1), the name 'LTV NI' is chosen for this class of systems to underpin the connection to its LTI counterpart (i.e., although the time-domain definition (2) does not have a Nyquist interpretation, it specialises in the LTI case to the condition $j\omega[\Sigma(j\omega) - \Sigma(j\omega)^*] \geq 0$ $\forall \omega \in \mathbb{R}_{\geq 0}$ which gives the LTI case its 'negative imaginary' name [1]).

We will now define the class of LTV OSNI systems.

Definition 2 (LTV OSNI Systems): Let Σ be a finite-dimensional, square and stable LTV system as described in (1). Then Σ is said to be an LTV OSNI system if there exist the constants $\beta \in \mathbb{R}$ and $\delta > 0$ such that

$$\int_0^T \dot{y}(t)^T u(t)\, \mathrm{d}t \geq \delta \int_0^T \dot{y}(t)^T \dot{y}(t)\, \mathrm{d}t + \beta \tag{3}$$

for any admissible $u \in \mathbb{U}^m$, any initial condition $x_0 \in \mathbb{R}^n$ and all $T \in [0, \infty)$.

Remark 2: Definitions 1 and 2 would remain valid even when the LTV state-space system in (1) had its output equation $y(t) = C(t)x(t)$ with $C(t)$ and $\dot{C}(t)$ assumed to be continuous and bounded for all $t \in \mathbb{R}_{\geq 0}$. We do not however consider that situation due to technical limitations in the results that follow.

B. State-Space Characterization of LTV NI Systems

In this subsection, state-space characterizations are provided for the LTV NI and OSNI systems which involve linear differential matrix inequality (LDMI) conditions.

Lemma 1 (LTV NI Lemma): Let Σ be a finite-dimensional, square, LTV system as described in (1). Then Σ is LTV NI if there exists a continuously differentiable and bounded matrix $P(t) = P(t)^T \geq 0$ for all $t \in \mathbb{R}_{\geq 0}$ such that

$$\begin{bmatrix} \dot{P}(t) + P(t)A(t) + A(t)^T P(t) & P(t)B(t) - A(t)^T C^T \\ B(t)^T P(t) - CA(t) & -CB(t) - B(t)^T C^T \end{bmatrix} \leq 0. \tag{4}$$

Proof: By exploiting the property of block partitioned semidefinite matrix [11], there always exist continuous and bounded matrices $L(t) \in \mathbb{R}^{m \times n}$ and $W(t) \in \mathbb{R}^{m \times m}$ for all $t \in \mathbb{R}_{\geq 0}$ such that

$$\begin{bmatrix} \dot{P}(t) + P(t)A(t) + A(t)^T P(t) & P(t)B(t) - A(t)^T C^T \\ B(t)^T P(t) - CA(t) & -CB(t) - B(t)^T C^T \end{bmatrix}$$
$$= \begin{bmatrix} -L^T(t)L(t) & -L^T(t)W(t) \\ -W^T(t)L(t) & -W^T(t)W(t) \end{bmatrix} \leq 0 \quad \forall t \geq 0. \tag{5}$$

Let $V(t,x) = \frac{1}{2}x^T P(t) x$ with $P(t) = P(t)^T \geq 0$ for all $t \in \mathbb{R}_{\geq 0}$ be a Lyapunov function candidate associated with the system Σ. The time derivative of $V(t,x)$ along the trajectories of Σ, given by (1), subjected to any admissible input $u \in \mathbb{U}^m$, is computed as $\dot{V}(t,x) = \frac{1}{2}x^T(\dot{P} + PA + A^T P)x + x^T PBu$. Integrating the last expression with respect to time t from 0 to $T \in \mathbb{R}_{\geq 0}$ and substituting $\dot{P} + PA + A^T P = -L^T L$ and $PB - A^T C^T = -L^T W$ from (5), we have

$$V(T, x(T)) - V(0, x(0)) = \int_0^T \left[-\frac{1}{2}x^T L^T L x + x^T (A^T C^T - L^T W)u \right] \mathrm{d}t \tag{6}$$

for any admissible $u \in \mathbb{U}^m$ and for all $T \in [0, \infty)$. Now using the fact that $V(T, x(T)) - V(0, x(0)) \geq -V(0, x(0))$, (6) implies (7), where $\beta = -V(0, 0) \in (-\infty, 0]$,

$$\int_0^T x^T A^T C^T u\, \mathrm{d}t \geq \int_0^T \left(\frac{1}{2}x^T L^T L x + x^T L^T W u \right) \mathrm{d}t + \beta. \tag{7}$$

Since CB is a square matrix, it can be expressed as $CB = \frac{1}{2}(CB + B^T C^T) + \frac{1}{2}(CB - B^T C^T)$. Below, we derive the expression for $\dot{y}^T u$ on noting that $\dot{y} = C\dot{x} = CAx + CBu$,

$$\dot{y}^T u = x^T A^T C^T u + \frac{1}{2}u^T \left(CB + B^T C^T\right)u + \frac{1}{2}u^T \left(CB - B^T C^T\right)u$$
$$\Rightarrow \dot{y}^T u = x^T A^T C^T u + \frac{1}{2}u^T \left(CB + B^T C^T\right)u \tag{8}$$

by exploiting the property $u^T(CB - B^T C^T)u = 0$ $\forall u \in \mathbb{R}^m$ since $(CB - B^T C^T)$ is skew-symmetric [11]. Integrating (8) from 0 to $T \in [0, \infty)$ and plugging (7) into it, we find

$$\int_0^T \dot{y}^T u\, \mathrm{d}t \geq \int_0^T \left(\frac{1}{2}x^T L^T L x + x^T L^T W u + \frac{1}{2}u^T W^T W u \right) \mathrm{d}t + \beta$$
$$\Rightarrow \int_0^T \dot{y}^T u\, \mathrm{d}t \geq \beta \qquad \text{[via the completion of squares]}$$

for any admissible $u \in \mathbb{U}^m$ and for all $T \in [0, \infty)$. Hence it is proved that Σ is an LTV NI system via Definition 1. ∎

We will now provide the state-space characterization for LTV OSNI systems.

Lemma 2 (LTV OSNI Lemma): Let Σ be a finite-dimensional, square, LTV system as described in (1). Also let $CB(t) \equiv 0$ and $CA(t) \not\equiv 0$ for all $t \in \mathbb{R}_{\geq 0}$. Then Σ is LTV OSNI if there exists a continuously differentiable and bounded matrix $P(t) = P(t)^T > 0$ for all $t \in \mathbb{R}_{\geq 0}$ such that

$$\dot{P}(t) + P(t)A(t) + A(t)^T P(t) < 0 \text{ and } P(t)B(t) = A(t)^T C^T. \tag{9}$$

Proof: Let there exist a real-valued, continuous and bounded matrix $Q(t) = Q(t) > 0$ for all $t \in \mathbb{R}_{\geq 0}$ such that

$$\dot{P}(t) + P(t)A(t) + A(t)^T P(t) = -Q(t) < 0 \quad \forall t \in \mathbb{R}_{\geq 0}. \tag{10}$$

Inequality (10) implies uniform asymptotic stability of the OSNI system Σ. Let $V(t,x) = \frac{1}{2}x^T P(t) x$ with $P(t) = P(t)^T > 0$ be a Lyapunov function candidate for Σ. Now, utilising (6) from the proof of Lemma 1 and substituting $\dot{V}(t,x) = \frac{1}{2}x^T(\dot{P} + PA + A^T P)x + x^T PBu$, we obtain

$$V(T, x(T)) - V(0, x(0)) = \int_0^T \left(-\frac{1}{2}x^T Q x + x^T A^T C^T u \right) \mathrm{d}t \tag{11}$$

for any admissible $u \in \mathbb{U}^m$ and for all $T \in [0, \infty)$. From (8), we have $\int_0^T \dot{y}^T u \, \mathrm{d}t = \int_0^T x^T A^T C^T u \, \mathrm{d}t$ since $CB(t) = 0 \; \forall t \in \mathbb{R}_{\geq 0}$ via supposition. Utilizing this result, (11) implies

$$\int_0^T \dot{y}^T u \, \mathrm{d}t = \int_0^T x^T A^T C^T u \, \mathrm{d}t \geq \frac{1}{2}\int_0^T x^T Q x \, \mathrm{d}t + \beta. \quad (12)$$

We also have $\dot{y}^T\dot{y} = x^T A^T C^T C A x$ for all $t \in \mathbb{R}_{\geq 0}$ since $CB = 0$ via supposition. Now exploiting the property $\lambda_{\min}[P]||x|| \leq x^T P x \leq \lambda_{\max}[P]||x||$ when $P = P^T \geq 0$ [11], we obtain $\dot{y}^T\dot{y} = x^T A^T C^T C A x \leq \bar{c} x^T x$ and $x^T Q x \geq \underline{q} x^T x$ for all $t \in \mathbb{R}_{\geq 0}$ denoting $\bar{c} = \sup_{\forall t \geq 0} \lambda_{\max}[A^T C^T C A] > 0$ and $\underline{q} = \inf_{\forall t \geq 0} \lambda_{\min}[Q] > 0$. The last two expressions together imply $x^T Q x \geq \frac{\underline{q}}{\bar{c}} \dot{y}^T \dot{y} \; \forall t \in \mathbb{R}_{\geq 0}$. This, in turn, implies from (12) that $\int_0^T \dot{y}^T u \, \mathrm{d}t \geq \delta \int_0^T \dot{y}^T \dot{y} \, \mathrm{d}t + \beta$ for any admissible $u \in \mathbb{U}^m$, for all $T \in [0, \infty)$ and denoting $\delta = \frac{\underline{q}}{2\bar{c}} > 0$. Hence, Σ is an LTV OSNI system according to Definition 2. ∎

Remark 3: The LTV NI lemma specialises to the well-established NI lemma (i.e., in the LTI setting) [1], [12]. Whereas, the LTV OSNI lemma partly captures the LTI OSNI lemma [13], [14] since the latter does not impose the constraint $CB = 0$. Hence, the LTV OSNI result cannot be considered as a generalised version of its LTI counterpart. Moreover, LTV NI and OSNI theory is completely independent of the conventional frequency-domain characterization of the existing NI and OSNI systems.

Remark 4: In contrast to the conventional NI and OSNI theory (i.e., in the LTI setting), the proposed LTV results do not impose the minimality constraint since the LTV NI and OSNI lemma conditions are sufficient-type results. Minimality condition is mainly required to establish the necessity part [15]. The proposed lemmas can be rendered necessary and sufficient if uniform controllability and observability constraints are imposed. However, for LTV systems, it is numerically very difficult to test these properties a priori and hence, the results become less appealing to the readers.

III. Closed-Loop Stability Analysis of LTV NI and OSNI Systems

This section studies an unforced positive feedback closed-loop system shown in Fig. 1 containing two uniformly asymptotically stable LTV NI systems of which, one is LTV OSNI. We show that the closed-loop system has a single globally asymptotically stable equilibrium point which is the origin given by $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Theorem 1: Let Σ_1 and Σ_2 be two finite-dimensional, square and uniformly asymptotically stable LTV systems. Also let $B_2(t)$ has full column rank, $C_2 B_2(t) \equiv 0$ and $C_2 A_2(t) \not\equiv 0$ for all $t \in \mathbb{R}_{\geq 0}$. Suppose Σ_1 is uniformly zero-state detectable and there exist continuously differentiable and bounded matrices $P_1(t) = P_1(t)^T > 0$ and $P_2(t) = P_2(t)^T > 0$ for all $t \in \mathbb{R}_{\geq 0}$ such that Σ_1 satisfies (4) and Σ_2 satisfies (9). Then the origin is a globally uniformly asymptotically stable equilibrium point of the unforced positive feedback interconnection of Σ_1 and Σ_2 shown in Fig. 1 if

$$\begin{bmatrix} P_1(t) & -C_1^T C_2 \\ -C_2^T C_1 & P_2(t) \end{bmatrix} > 0 \quad \forall t \in \mathbb{R}_{\geq 0}. \quad (13)$$

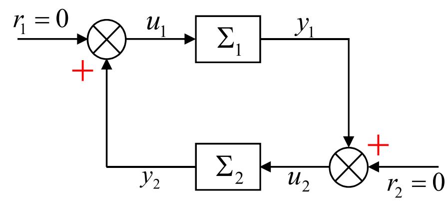

Fig. 1. Positive feedback interconnection of LTV NI systems.

Proof: Let there exist continuous and bounded matrices $L(t) \in \mathbb{R}^{m \times n}$, $W(t) \in \mathbb{R}^{m \times m}$ and $Q(t) = Q(t)^T > 0$ such that Σ_1 satisfies (5) and Σ_2 satisfies (10) for all $t \in \mathbb{R}_{\geq 0}$. We designate $V_1(t, x_1) = \frac{1}{2}x_1^T P_1(t) x_1$ and $V_2(t, x_2) = \frac{1}{2}x_2^T P_2(t) x_2$ be the Lyapunov function candidates associated with Σ_1 and Σ_2 respectively. Let the combined Lyapunov function candidate for the closed-loop system be $V(t, x) = V_1(t, x_1) + V_2(t, x_2) - y_1^T y_2$ where $x = [x_1^T \; x_2^T]^T$. It is apparent that

$$\begin{aligned} V(t, x) &= \frac{1}{2}x_1^T P_1(t) x_1 + \frac{1}{2}x_2^T P_2(t) x_2 - x_1^T C_1^T C_2 x_2 \\ &= \frac{1}{2}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T \begin{bmatrix} P_1(t) & -C_1^T C_2 \\ -C_2^T C_1 & P_2(t) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} > 0 \end{aligned}$$

via (13) and $V(t_0, 0) = 0$ for any $t_0 \in \mathbb{R}_{\geq 0}$. Owing to the continuously differentiable property and boundedness of $P_1(t)$ and $P_2(t)$ $\forall t \in \mathbb{R}_{\geq 0}$, $V(t, x)$ can be characterized as $0 < \alpha_1(||x||) \leq V(t, x) \leq \alpha_2(||x||) < \infty \; \forall x \in \mathbb{R}^{n_1 + n_2}$ where $\alpha_1(\cdot)$ and $\alpha_2(\cdot)$ are class-$\mathscr{K}$ functions with α_1 being radially unbounded in x. Moreover, $V(t, x)$ is a continuously differentiable function since $\dot{V}(t, x)$ remains uniformly continuous in $t \geq 0$ (shown later in the ongoing proof). Now, the time-derivative of $V(t, x)$ is derived as follows:

$$\begin{aligned} \dot{V}(t, x) &= \dot{V}_1(t, x_1) + \dot{V}_2(t, x_2) - \dot{y}_1^T y_2 - y_1^T \dot{y}_2 \\ &= \frac{1}{2}x_1^T(\dot{P}_1 + P_1 A_1 + A_1^T P_1)x_1 + x_1^T P_1 B_1 u_1 + \frac{1}{2}x_2^T(\dot{P}_2 \\ &\quad + P_2 A_2 + A_2^T P_2)x_2 + x_2^T P_2 B_2 u_2 - \dot{y}_1^T y_2 - y_1^T \dot{y}_2 \\ &= -\frac{1}{2}x_2^T Q x_2 + \dot{y}_2^T u_2 - \frac{1}{2}x_1^T(L^T L)x_1 - \frac{1}{2}u_1^T(W^T W)u_1 \\ &\quad - x_1^T L^T W u_1 + \dot{y}_1^T u_1 - \dot{y}_1^T y_2 - y_1^T \dot{y}_2 \;\; \text{[using (5), (10)]} \\ &= -\frac{1}{2}x_2^T Q x_2 + \dot{y}_2^T y_1 - \frac{1}{2}(L x_1 + W u_1)^T (L x_1 + W u_1) \\ &\quad + \dot{y}_1^T y_2 - \dot{y}_1^T y_2 - y_1^T \dot{y}_2 \quad \text{[using } u_1 = y_2 \text{ and } u_2 = y_1\text{]} \\ &= -\frac{1}{2}x_2^T Q x_2 - \frac{1}{2}(L x_1 + W u_1)^T (L x_1 + W u_1) \quad (14) \\ &\leq -\frac{1}{2}x_2^T Q x_2 \leq -q_{\min}\, \chi(||x_2||) \leq 0 \quad (15) \end{aligned}$$

where $q_{\min} = \inf_{\forall t \geq 0} \lambda_{\min}[Q(t)] > 0$ and $\chi(\cdot)$ is a class-$\mathscr{K}$ function. Now, [16, Th. 8.4] and [17, Th. 4.1] guarantee that both $x_1(t)$ and $x_2(t)$ will remain uniformly bounded for all $t \geq 0$. Then, to show uniform asymptotic stability of the states, we will seek to apply Barbalat's lemma [17]. It can be verified that $\ddot{V}(t, x)$ remains bounded $\forall t \geq 0$ since (i) x_1 and x_2 are already proved to be uniformly bounded $\forall t \geq 0$ (and hence, $y_1 = C_1 x_1$ and $y_2 = C_2 x_2$ are also uniformly bounded), (ii) the matrices $A_i(t)$ and $B_i(t)$ for $i \in \{1, 2\}$ are assumed to be bounded $\forall t \geq 0$, and (iii) the input-derivative terms $\dot{u}_1 = \dot{y}_2 = C_2 \dot{x}_2 = C_2 A_2 x_2 + C_2 B_2 y_1$ and $\dot{u}_2 = \dot{y}_1 = C_1 A_1 x_1 + C_1 B_1 y_2$ are also bounded $\forall t \geq 0$.

Therefore, $\dot{V}(t,x)$ is uniformly continuous for all $t \geq 0$ which ultimately implies $\dot{V}(t,x) \to 0$ as $t \to \infty$ by exploiting Barbalat's lemma. Finally, [17, Lemma 4.3] ensures that $\lim_{t\to\infty} x_2(t) = 0$ for any bounded $x_{2,0} \in \mathbb{R}^{n_2}$. This hence implies $\lim_{t\to\infty} y_2(t) = \lim_{t\to\infty} u_1(t) = 0$ as C_2 is constant and $u_1 = y_2$. Furthermore, as $t \to \infty$, the state-space equation of Σ_2, that is, $\dot{x}_2 = A_2(t)x_2(t) + B_2(t)y_1(t)$ implies $\lim_{t\to\infty} y_1(t) = 0$ since $u_2 = y_1$, $B_2(t)$ has full column rank and due to uniform asymptotic convergence of $x_2(t)$. Now, exploiting uniform zero-state detectability (ZSD) [1] and uniform asymptotic stability of Σ_1, $u_1 \equiv 0$ and $y_1 \equiv 0$ imply $\lim_{t\to\infty} x_1(t) = 0$ for any bounded $x_{1,0} \in \mathbb{R}^{n_1}$. Combining the aforementioned arguments, it can be asserted that the positive feedback closed-loop system of Σ_1 and Σ_2 is globally uniformly asymptotically stable. ∎

IV. Linear Parameter-Varying NI Systems

As the LTI NI and OSNI lemmas involve LDMI conditions, it may give rise to computational issues while solving those LDMIs using the SDP solver packages (e.g., CVX, Yalmip, Sedumi). To bypass the LDMIs, in this section, we have specialized the previous results to LPV NI and OSNI systems considering bounded variation of the uncertain system parameters. Now, consider a class of finite-dimensional, square, LPV dynamical systems described by

$$\Sigma_{LPV} : \begin{cases} \dot{x}(t) = A(\rho)x(t) + B(\rho)u(t) \quad x(0) = x_0, \\ y(t) = Cx(t) \end{cases} \tag{16}$$

where $A(\rho)$, $B(\rho)$ depend affinely on the uncertain (possibly time-varying) parameter vector $\rho = [\rho_1, \rho_2, \ldots, \rho_K] \in \mathbb{R}^K$, that is, $A(\rho) = A_0 + \rho_1 A_1 + \cdots + \rho_K A_K$ and $B(\rho) = B_0 + \rho_1 B_1 + \cdots + \rho_K B_K$. Below, we mention two technical assumptions that must be satisfied by the LPV NI systems studied here:

A1. Each parameter ρ_i varies in the known interval $[\rho_{i,\min}, \rho_{i,\max}]$ for all $i \in \{1, 2, \ldots, K\}$. This implies that the parameter vector $\rho \in \mathbb{R}^K$ is valued in a hyper-rectangle described by the set of vertices $\mathscr{V} = \{(v_1, \ldots, v_K): v_i \in \{\rho_{i,\min}, \rho_{i,\max}\}\ \forall i\}$.

A2. The rate of variation $\dot{\rho}_i$ is well defined for all $t \in \mathbb{R}_{\geq 0}$ and $\dot{\rho}_i \in [\gamma_{i,\min}, \gamma_{i,\max}]$ where the range $\gamma_{i,\min} \leq 0 \leq \gamma_{i,\max}$ is known for all $i \in \{1, \ldots, K\}$. This implies that the vector $\dot{\rho} \in \mathbb{R}^K$ varies within a hyper-rectangle having the set of vertices $\mathscr{E} = \{(e_1, \ldots, e_k): e_i \in \{\gamma_{i,\min}, \gamma_{i,\max}\}\ \forall i\}$.

We introduce the following notation $\rho_{\text{mean}} = [\frac{\rho_{1,\min}+\rho_{1,\max}}{2}, \frac{\rho_{2,\min}+\rho_{2,\max}}{2}, \ldots, \frac{\rho_{K,\min}+\rho_{K,\max}}{2}]$ to be used subsequently in Lemmas 3 and 4. Lemma 3 gives a set of sufficient conditions for LPV NI systems and is a specialized result of the LTV NI lemma derived in the previous section.

Lemma 3 (LPV NI Lemma): Consider a finite-dimensional, square, LPV system Σ_{LPV}, as described in (16), that satisfies Assumptions A1 and A2. Suppose $A(\rho_{\text{mean}})$ does not have any pole in the open right-half plane. Then Σ_{LPV} is an LPV NI system if there exist $K+1$ real, symmetric matrices $P_0, P_1, \ldots, P_K$ such that $P(\rho) = P_0 + \rho_1 P_1 + \cdots + \rho_K P_K$ satisfies

$$\begin{bmatrix} P(v)A(v) + A(v)^T P(v) + P(e) - P_0 & P(v)B(v) - A(v)^T C^T \\ \left(P(v)B(v) - A(v)^T C^T\right)^T & -CB(v) - B(v)^T C^T \end{bmatrix} \leq 0 \tag{17a}$$

$$\text{and} \quad \begin{bmatrix} P_i A_i + A_i^T P_i & P_i B_i \\ B_i^T P_i & 0 \end{bmatrix} \geq 0 \tag{17b}$$

$\forall (v, e) \in \mathscr{V} \times \mathscr{E}$ and $\forall i \in \{1, \ldots, K\}$.

Proof: Condition (17a) implies $P(v)A(v) + A(v)^T P(v) + P(e) - P_0 \leq 0\ \forall (v, e) \in \mathscr{V} \times \mathscr{E}$ and $P(\rho) = P_0 + \rho_1 P_1 + \rho_2 P_2 + \cdots + \rho_K P_K \geq 0$ following [19, Th. 3.2] which together ensure the existence of a parameter-dependent Lyapunov function $V(x, \rho) = x^T P(\rho) x$ associated with Σ_{LPV}. We now derive

$$\dot{V}(x,\rho) = \dot{x}^T P(\rho)x + x^T \dot{P}(\rho)x + x^T P(\rho)\dot{x}$$
$$= \begin{bmatrix} x \\ u \end{bmatrix}^T \begin{bmatrix} P(\rho)A(\rho) + A(\rho)^T P(\rho) + P(\dot{\rho}) - P_0 & P(\rho)B(\rho) \\ B(\rho)^T P(\rho) & 0 \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix}.$$

Let $e = \dot{\rho}$, $Z(\rho, e) = \begin{bmatrix} P(\rho)A(\rho) + A(\rho)^T P(\rho) + P(e) - P_0 & P(\rho)B(\rho) \\ B(\rho)^T P(\rho) & 0 \end{bmatrix}$ and $\chi = [x^T \quad u^T]^T$. Now the multi-convexity[2] property of the scalar quadratic function $z(\rho) = \chi^T Z(\rho, e)\chi$, being affine in both ρ and e, is ensured for all admissible ρ and for each fixed $e \in \mathscr{E}$ via (17b) following [19, Th. 3.2]. Consequently, $Z(\rho, e) \leq 0$ holds for all admissible values of ρ and e satisfying Assumptions A1 and A2 due to the affine dependence. This completes the proof. ∎

We will now present the LPV OSNI lemma which is a specialized result of LTV OSNI lemma derived in Section II.

Lemma 4 (LPV OSNI Lemma): Consider a finite-dimensional, square LPV system Σ_{LPV}, as described in (16), that satisfies Assumptions A1 and A2. Suppose $CB(\rho) \equiv 0$ and $CA(\rho) \not\equiv 0$ for all admissible ρ. Suppose further $A(\rho_{\text{mean}})$ is Hurwitz. Then Σ_{LPV} is an LPV OSNI system if there exist $K+1$ symmetric matrices $P_0, P_1, P_2, \ldots, P_K$ such that $P(\rho) = P_0 + \rho_1 P_1 + \cdots + \rho_K P_K$ satisfies

$$P(v)A(v) + A(v)^T P(v) + P(e) - P_0 < 0 \tag{18a}$$

$$P(v)B(v) = A(v)^T C^T \quad \text{and} \tag{18b}$$

$$\begin{bmatrix} P_i A_i + A_i^T P_i & P_i B_i \\ B_i^T P_i & 0 \end{bmatrix} \geq 0 \tag{18c}$$

$\forall (v, e) \in \mathscr{V} \times \mathscr{E}$ and $\forall i \in \{1, \ldots, K\}$.

Proof: The proof can be done in the same spirit of the proof of Lemma 3 and also by following Lemma 2 subjected to the additional assumptions imposed on the LPV OSNI systems. ∎

Finally, we derive a sufficient condition for ensuring global asymptotic stability of the positive feedback interconnection (Fig. 1) comprised of a stable LPV NI system Σ_1 and an LPV OSNI system Σ_2 having state-space realizations $(A_1(\rho_1)$, $B_1(\rho_1)$, $C_1)$ and $(A_2(\rho_2)$, $B_2(\rho_2)$, $C_2)$, respectively, for all $\rho_j = [\rho_{j,1}, \rho_{j,2}, \ldots, \rho_{j,K}] \in \mathbb{R}^K$ with $j \in \{1, 2\}$ where $\rho_{j,i} \in [\rho_{(j,i)_{\min}}, \rho_{(j,i)_{\max}}]$ and $\dot{\rho}_{j,i} \in [\gamma_{(j,i)_{\min}}, \gamma_{(j,i)_{\max}}]$ are known $\forall i \in \{1, 2, \ldots, K\}$. We also define four sets of vertices $\mathscr{V}_j = \{(v_{j,1}, v_{j,2}, \ldots, v_{j,K}) : v_{j,i} \in \{\rho_{(j,i)_{\min}}, \rho_{(j,i)_{\max}}\}\ \forall i\}$

[1] The notion of uniform ZSD is defined for non-autonomous systems [18], analogous to the concept of ZSD applied to autonomous systems [16].

[2] Multi-convexity refers to convexity along each direction $\rho_i\ \forall i \in \{1, 2, \ldots, K\}$ of the parameter space. It is less demanding than convexity with respect to $\rho \in \mathbb{R}^K$ and offers a finite set of LMI constraints [19].

with $j \in \{1, 2\}$ corresponding to ρ_1 and ρ_2 and $\mathscr{C}_j = \{(e_{j,1}, e_{j,2}, \dots, e_{j,K}) : e_{j,i} \in \{\gamma_{(j,i)_{\min}}, \gamma_{(j,i)_{\max}}\}\ \forall i\}$ with $j \in \{1, 2\}$ corresponding to $\dot{\rho}_1$ and $\dot{\rho}_2$.

Theorem 2: Consider Σ_1 and Σ_2 be two finite-dimensional, square, stable LPV systems, as described in (16), that satisfy the Assumptions A1 and A2. Let Σ_1 be observable for all admissible ρ_1 and $A_1(\rho_{1_{\mathrm{mean}}})$, $A_2(\rho_{2_{\mathrm{mean}}})$ be both Hurwitz. Also let $B_2(\rho_2)$ has full column rank, $C_2B_2(\rho_2) \equiv 0$ and $C_2A_2(\rho_2) \not\equiv 0$ for all admissible ρ_2. Assume there exist real, symmetric matrices $P_{1,0}, P_{1,1}, P_{1,2}, \dots, P_{1,K}$ such that Σ_1 satisfies (17a)–(17b) $\forall (v_1, e_1) \in \mathscr{V}_1 \times \mathscr{C}_1$ and $P_{2,0}, P_{2,1}, P_{2,2}, \dots, P_{2,K}$ such that Σ_2 satisfies (18a)–(18c) $\forall (v_2, e_2) \in \mathscr{V}_2 \times \mathscr{C}_2$. Then the origin is a globally asymptotically stable equilibrium point of the unforced positive feedback interconnection of Σ_1 and Σ_2 shown in Fig. 1 if $\begin{bmatrix} P_1(v_1) & -C_1^TC_2 \\ -C_2^TC_1 & P_2(v_2) \end{bmatrix} > 0\ \forall (v_1, v_2) \in \mathscr{V}_1 \times \mathscr{V}_2$.

Proof: This theorem can be readily established by specializing the proof of Theorem 1 to the interconnection of a stable LPV NI and an LPV OSNI systems upon applying Lemmas 3 and 4 instead of Lemmas 1 and 2. ■

V. Case Study

Here, we consider a potential problem of controlling the rectilinear motion of a body with time-varying mass (which prototypes the fuel dynamics of a rocket) being motivated by a similar example taken in [20]. The time-varying mass is expressed by the relation $m(t) = m_0 + m_f\, e^{-\alpha t}$ where $m(t)$ is the total mass of the body, m_f is the initial mass, m_0 is the rest mass and $\alpha > 0$. The equation of motion is given by

$$\Sigma_m : \{m(t)\ddot{q}(t) + (\dot{m}(t) + c + k_2)\dot{q}(t) + k_1q(t) = u(t) \quad (19)$$

where the terms $k_1q(t)$ and $c\dot{q}(t)$ are additionally embedded within the system, to be compatible with the LTV NI framework. The parameter $c > 0$ represents the static drag of the body and $k_2 > 0$ is chosen such that $k_2 > \max_{t\geq 0}(\frac{1}{2}\alpha m_f e^{-\alpha t} - c)$. Now choosing position $q(t) = x_1$ and velocity $\dot{q}(t) = x_2$, the augmented dynamics (19) can be represented in the state-space form $\dot{x}(t) = A_1(t)x(t) + B_1(t)u(t)$ and $y(t) = C_1x(t)$ where $A_1(t) = \begin{bmatrix} 0 & 1 \\ \frac{-k_1}{m(t)} & \frac{-(\dot{m}(t)+c+k_2)}{m(t)} \end{bmatrix}$, $B_1(t) = \begin{bmatrix} 0 \\ \frac{1}{m(t)} \end{bmatrix}$, $C_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}$ and $x = [x_1 \ \ x_2]^T$. First, we will show that the augmented dynamics (19) satisfies the LTV OSNI property. Consider the Hamiltonian function $H(t, x) = \frac{1}{2}k_1x_1(t)^2 + \frac{1}{2}m(t)x_2(t)^2$ associated with Σ_m. It can be verified that $H(t, x) > 0\ \forall t \in \mathbb{R}_{\geq 0}$ and $H(t_0, 0) = 0$ for any $t_0 \geq 0$. The time derivate of $H(t, x)$ is computed as $\dot{H}(t, x) = u(t)x_2 - \frac{1}{2}\dot{m}(t)x_2^2 - cx_2^2 - k_2x_2^2 - k_1x_1x_2 + k_1x_1x_2$. Integrating this with respect to t from 0 to $T \in [0, \infty)$, we have $\int_0^T (u(t)x_2 - \frac{1}{2}\dot{m}(t)x_2^2 - cx_2^2 - k_2x_2^2)\mathrm{d}t = H(T, x(T)) - H(0, x(0)) \geq \beta\ \forall T \in [0, \infty)$ denoting $\beta = -H(0, x(0)) \in (-\infty, 0]$ and since $H(T, x(T)) \geq 0\ \forall T$. The above expression can be rearranged into

$$\int_0^T \dot{y}^T u(t)\, \mathrm{d}t \geq \int_0^T \left[\frac{1}{2}\dot{m}(t) + c + k_2\right]\dot{y}(t)^2\, \mathrm{d}t + \beta \quad (20)$$

which implies LTV OSNI property via Definition 2 with $\delta = \min_{t\geq 0}\{\frac{1}{2}\dot{m}(t) + c + k_2\} > 0$. Next, we will show that there exists a differentiable and bounded matrix $P_1(t) = P_1(t)^T > 0\ \forall t \in \mathbb{R}_{\geq 0}$ such that Σ_m satisfies Lemma 2. We select $P_1(t) = \begin{bmatrix} P_{11}(t) & P_{12}(t) \\ P_{21}(t) & P_{22}(t) \end{bmatrix} = \begin{bmatrix} k_1 + e^{-t}\frac{1}{\frac{t}{\eta}+a_0} & 0 \\ 0 & m(t) \end{bmatrix}$ where $k_1 \geq 1$, $a_0 > 0$ and $\eta = 2\delta$. It is evident that $P_1(t) = P_1(t)^T > 0$ and $P_1(t)B_1(t) = A_1(t)^TC_1^T\ \forall t \in \mathbb{R}_{\geq 0}$ since $P_1(t)B_1(t) = \begin{bmatrix} P_{11}(t) & P_{12}(t) \\ P_{12}(t) & P_{22}(t) \end{bmatrix}\begin{bmatrix} 0 \\ \frac{1}{m(t)} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $A_1(t)^TC_1^T = \begin{bmatrix} 0 & 1 \\ \frac{-k_1}{m(t)} & \frac{-(\dot{m}(t)+c+k_2)}{m(t)} \end{bmatrix}^T\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. We then simplify the expression

$$\begin{aligned}\dot{P}_1(t) + P_1(t)A_1(t) + A_1(t)^TP_1(t) &= \begin{bmatrix} \dot{P}_{11}(t) & 0 \\ 0 & \dot{m}(t) \end{bmatrix} \\ &+ \begin{bmatrix} P_{11}(t) & 0 \\ 0 & m(t) \end{bmatrix}\begin{bmatrix} 0 & 1 \\ \frac{-k_1}{m(t)} & \frac{-(\dot{m}(t)+c+k_2)}{m(t)} \end{bmatrix} \\ &+ \begin{bmatrix} 0 & 1 \\ \frac{-k_1}{m(t)} & \frac{-(\dot{m}(t)+c+k_2)}{m(t)} \end{bmatrix}^T\begin{bmatrix} P_{11}(t) & 0 \\ 0 & m(t) \end{bmatrix} \\ &= \begin{bmatrix} \dot{P}_{11}(t) & P_{11}(t) - k_1 \\ P_{11}(t) - k_1 & -\dot{m}(t) - 2(c + k_2) \end{bmatrix}. \end{aligned} \quad (21)$$

Now on taking Schur complement with respect to the term $-[\dot{m}(t) + 2(c + k_2)]$ of (21), which remains negative $\forall t \in \mathbb{R}_{\geq 0}$ via choice of k_2, we find that $\dot{P}_{11}(t) + \frac{(P_{11}(t)-k_1)^2}{\dot{m}(t)+2(c+k_2)} = -\frac{e^{-t}}{\frac{t}{\eta}+a_0} - \frac{1}{\eta(\frac{t}{\eta}+a_0)^2}[e^{-t} - e^{-2t}] < 0\ \forall t \in \mathbb{R}_{\geq 0}$ since $k_1 \geq 1$, $\eta = 2\delta > 0$, $a_0 > 0$ and $e^{-t} - e^{-2t} \geq 0\ \forall t \in \mathbb{R}_{\geq 0}$. Therefore, via Schur Complement lemma, (21) is guaranteed to be negative definite $\forall t \in \mathbb{R}_{\geq 0}$ and hence, the augmented dynamics Σ_m is an LTV OSNI system via Lemma 2.

Finally, in order to ensure robust stability of Σ_m, we choose a simple LTI OSNI controller $K(s) = \dfrac{1}{s+1}$ with a minimal state-space realisation $(A_2, B_2, C_2, D_2) = (-1, 1, 1, 0)$. $K(s)$ satisfies Lemma 2 with $P_2 = 1$. We will now check whether Theorem 1 holds in this case or not. Inequality (13) holds $\forall t \in \mathbb{R}_{\geq 0}$ since, via Schur complement lemma, $P_2 = 1$ and $P_1(t) - C_1^TC_2P_2^{-1}C_2^TC_1 = \begin{bmatrix} k_1 + \frac{e^{-t}}{\frac{t}{\eta}+a_0} & 0 \\ 0 & m(t) \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} > 0$ $\forall t \in \mathbb{R}_{\geq 0}$ on noting that $m(t) > 0$ and $k_1 + \frac{e^{-t}}{\frac{t}{\eta}+a_0} > 1$ as $k_1 \geq 1$ via design.

MATLAB Simulation Results: We choose $m_0 = 1.5$kg, $m_f = 1$kg, $\alpha = 0.1$, $c = 10^{-2}$Ns/m, $k_2 = 0.1$Ns/m and $k_1 = 5$N/m. Fig. 2(a) and Fig. 2(b) show a comparative study of the step response (position and velocity) of the closed-loop dynamics (19) in presence of the LTI OSNI controller $K(s) = \frac{1}{s+1}$ [indicated by the Blue curves] and with only unity positive feedback [indicated by the Red curves]. The figures suggest that due to the influence of the controller, the dynamic response has improved to a significant extent compared to that obtained by using only unity feedback (used as an arbitrary baseline for comparison). Fig. 3(a) and Fig. 3(b) depict respectively the phase portraits (x_2 vs. x_1) of the closed-loop dynamics (19) in presence of the controller $K(s)$ and with only unity feedback. In Fig. 3(a), the rate of convergence of the phase trajectory is much faster than shown in Fig. 3(b). Apart from the phase portrait analysis, we have also analysed the rate of decay of a standard cost

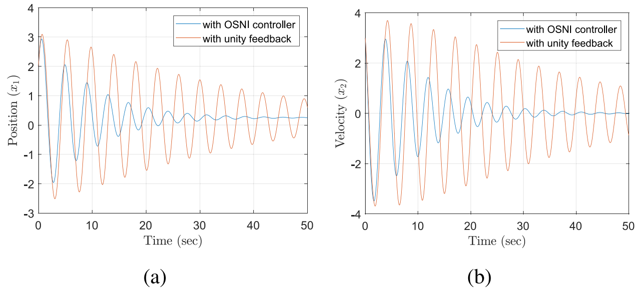

Fig. 2. Closed-loop step responses of the body with time-varying mass: (a) Position ($x_1 = q$) and (b) Velocity ($x_2 = \dot{q}$).

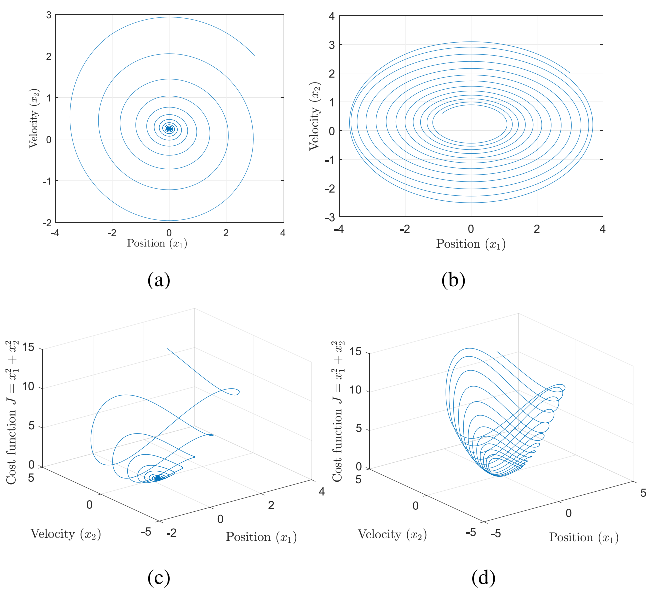

Fig. 3. Phase portrait (x_2 vs. x_1) of the closed-loop dynamics in presence of (a) the LTI OSNI controller and (b) with only unity feedback. Level curves of the cost function $J = x_1^2 + x_2^2$ evaluated in presence of (c) the LTI OSNI controller and (d) with only unity feedback.

function $J = x_1^2 + x_2^2$ evaluated along the closed-loop dynamics (19). From Fig. 3(c) and Fig. 3(d), it is evident that the rate of decay of cost function J in presence of the LTI OSNI controller is much faster than that with only unity positive feedback.

VI. Conclusion

This letter extends the NI theory to linear time-varying (LTV) systems. This letter also formally introduces the time-domain definition of NI systems using a particular dissipative supply rate that involves the input to the system (u) and the time derivative of the system's output ($\dot{y}$). A strict subset within this LTV NI class, termed as LTV OSNI systems, is also introduced which resembles the LTI OSNI results [13], [14]. To test the LTV NI and OSNI properties of a given system, state-space characterizations are given which involve LDMIs. Later, the LTV NI and OSNI results have been specialized to LPV cases which replace the LDMIs by LMIs and thereby eliminates the computational issues caused by the SDP solver packages while solving the LDMIs.

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