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Feedback Stability Analysis via Dissipativity with Dynamic Supply Rates ⋆

Sei Zhen Khong [a], Chao Chen [b,1], Alexander Lanzon [c]

[a] *Department of Electrical Engineering, National Sun Yat-sen University, Kaohsiung 80424, Taiwan*

[b] *Department of Electrical Engineering (ESAT), KU Leuven, Kasteelpark Arenberg 10, Leuven 3001, Belgium*

[c] *Department of Electrical and Electronic Engineering, School of Engineering, University of Manchester, Manchester M13 9PL, UK*

Abstract

We propose a general notion of dissipativity with dynamic supply rates for nonlinear systems. This extends classical dissipativity with static supply rates and dynamic supply rates of miscellaneous quadratic forms. The main results of this paper concern Lyapunov and asymptotic stability analysis for nonlinear feedback dissipative systems that are characterised by dissipation inequalities with respect to compatible dynamic supply rates but involving possibly different and independent auxiliary systems. Importantly, dissipativity conditions guaranteeing stability of the state of the feedback systems, without concerns on the stability of the state of the auxiliary systems, are provided. The key results also specialise to a simple coupling test for the interconnection of two nonlinear systems described by dynamic $(\Psi, \Pi, \Upsilon, \Omega)$-dissipativity, and are shown to recover several existing results in the literature, including small-gain, passivity indices, static (Q, S, R)-dissipativity, dissipativity with terminal costs, etc. Comparison with the input-output approach to feedback stability analysis based on integral quadratic constraints is also made.

Key words: Dissipativity, dynamic supply rates, nonlinear feedback systems, asymptotic stability.

1 Introduction

The notion of dissipativity of dynamical systems was first introduced by Jan C. Willems in [56]. The seminal work has profoundly influenced research in the systems and control community, so much so that the *IEEE Control Systems Magazine* recently published a special two-part issue [45] commemorating the 50^{th} anniversary of the papers [56]. Dissipativity theory abstracts the notion of energy and its dissipation in dynamical systems, and may be viewed as a generalisation of Lyapunov theory for autonomous systems to open systems with input and outputs [59]. Importantly, the construction of storage functions for linear systems with quadratic supply rates led to the emergence of linear matrix inequalities (LMIs) [5] in the field of control. The literature on dissipativity is vast. It incorporates [22–24, 36] on stability theory and basic properties, [51, 57] on cyclo-dissipativity, [58] on behavioural systems, [47] on synchronisation of nonlinear oscillators, [19] on mixed small-gain/passive systems, and [46, 53] on incremental dissipativity, to name a few. Several books have also been written on the topic [3, 6, 50]. The importance of dissipativity to the field of systems and control is thus self-evident from the large existing literature on it. Not only is dissipativity a notion that is physically motivated by energy dissipation, it is also interrelated with a broad range of systems tools including Riccati equations [56,

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Email addresses: `szkhong@mail.nsysu.edu.tw` (Sei Zhen Khong), `chao.chen@kuleuven.be` (Chao Chen), `Alexander.Lanzon@manchester.ac.uk` (Alexander Lanzon).

[1] Corresponding author: C. Chen

65], the Kalman–Yakubovich–Popov lemma [24,38], the Iwasaki–Hara lemma [26] and LMIs. When applied to robust closed-loop stability analysis, dissipativity theory unifies the small-gain and passivity results [54].

Traditionally, dissipativity has been defined in terms of static and time-invariant supply rates and dynamics are confined to the corresponding storage functions of the states of the dynamical input-state-output systems in question. When it comes to robust closed-loop (asymptotic/exponential) stability analysis based on dissipativity, conservatism may be reduced with the aid of stable and stably invertible dynamical multipliers [17, Sec. 3.5.1]. This method suffers from arguably serious drawbacks in that the need for the multipliers to be stable and stably invertible substantially restricts their usefulness. On the contrary, the input-output approach to robust feedback (input-output finite-gain) stability analysis [12, 62] has been shown to naturally accommodate a wider class of multipliers in a direct fashion, for example, via the notion of integral quadratic constraints (IQCs) [35]. Under well-posedness assumptions, graph separation is necessary and sufficient for input-output closed-loop stability [13, 21, 48, 49]. The theory of IQCs [7,8,28,35,39] provides a powerful and unifying framework for establishing graph separation, and thus input-output closed-loop stability, e.g. [31, 32, 64]. The type of multipliers that may be accommodated in this theory is extensive. It includes Zames-Falb [63], small-gain, passivity, (Q, S, R)-dissipativity, and circle criterion, to mention just a few. In the linear time-invariant (LTI) setting, it is also known that IQCs are nonconservative for robust stability analysis [25, 29, 30, 40].

Motivated in part by the prowess and utility of multipliers, the notion of dissipativity with dynamic supply rates has been considered for robust stability analysis in various contexts [1, 2, 4, 9, 15, 33, 42, 44, 52, 60, 61]. In [60], dynamic supply rates of quadratic differential forms are considered and physically motivated by numerous examples. Dynamic supply rates of quadratic forms based on either affine nonlinear or LTI auxiliary systems are investigated in [2,9,42,44]. Notably, dynamic supply rates for differential dissipativity [14, 15, 53], differential passivity [52], counterclockwise dynamics [1], negative imaginariness [4, 33] and system phase [10] have also been examined in the literature.

In this paper, we propose a dissipativity notion involving dynamic supply rates of general form for nonlinear dynamical systems. The proposed notion generalises static supply rates and the dynamic supply rates of quadratic (and differential) forms considered in [2, 9, 42, 44, 60, 61], and may be used to capture the class of input-output negative imaginary systems in [4, 33]. It relies on the dynamics of a possibly nonlinear auxiliary system that may be independent of the dynamics in the supply rate, marking a remarkable departure from the literature. The dynamics in the supply rate may be seen as counterparts to the multipliers used in the definitions of IQCs, whereas the dynamics of the auxiliary system facilitate the verification of the dissipativity of the system with respect to the supply rate in question.

Lyapunov stability and asymptotic stability of a feedback interconnection consisting of two nonlinear dissipative systems sharing the same dynamic supply rate and utilising possibly different auxiliary systems are established in this paper. The use of dynamic supply rates has the advantage of reducing conservatism in feedback stability analysis similar to using multipliers in input-output stability analysis. An interesting specialisation of the key results of this paper is a simple coupling test to check the feedback stability of two nonlinear systems that are described by dynamic $(\Psi, \Pi, \Upsilon, \Omega)$-dissipation inequalities. We also specialise our main results to several existing results in the literature [9, 42, 50] on static and dynamic supply rates of miscellaneous quadratic forms. Last, but not least, it is noteworthy that in many aspects, our main results are distinct from the IQC result with incrementally bounded multipliers for input-output closed-loop stability analysis [28], and their differences and relation are also discussed in detail.

The remainder of this paper is organised as follows. In Section 2, we propose a novel general notion of dissipativity with dynamic supply rates and provide illustrating examples of systems that it can capture. The main results on feedback Lyapunov and asymptotic stability via the newly proposed general dissipativity notion are given in Section 3. Section 4 shows that several existing results in the literature can be stated as corollaries of the main results in this paper. Section 5 provides the differences and relation between the main results and IQC based feedback input-output stability. Section 6 provides a numerical example to demonstrate the utility of the main results and Section 7 summarises this paper.

Notation: Let $\mathbb{N}$, $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{R}^n$ and $\mathbb{R}^{p\times m}$ denote the sets of natural numbers excluding 0, real numbers, nonnegative real numbers, n-dimensional real vectors and $p \times m$ real matrices, respectively. Given a matrix M, its transpose is denoted by $M^\top$. An identity matrix of compatible size is denoted by I. Let $\|x\| = \sqrt{x^\top x}$ for $x \in \mathbb{R}^n$. A function $\alpha : \mathbb{R}^n \to \mathbb{R}$ is said to be positive definite if $\alpha(0) = 0$ and $\alpha(r) > 0$ for all $0 \neq r \in \mathbb{R}^n$. A function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ is said to belong to class $\mathcal{K}$ if it is continuous, strictly increasing, and $\alpha(0) = 0$. It is said to belong to class $\mathcal{K}_\infty$ if it belongs to class $\mathcal{K}$ and $\lim_{r\to\infty} \alpha(r) = \infty$. The extended space of $\mathbb{R}$-valued Lebesgue absolutely integrable functions is defined as

$$\mathbf{L}_{1e} = \left\{ v : \mathbb{R}_+ \to \mathbb{R} \;\middle|\; \int_0^T \|v(t)\|\, dt < \infty \;\forall T \in [0, \infty) \right\}.$$

An operator Ψ maps an input u in some signal space to an output y in another space via $y = \Psi(u)$. An operator can capture any static, dynamic, linear or nonlinear system. Define the truncation operator $(P_T u)(t) = u(t)$ for $t \le T$ and $(P_T u)(t) = 0$ for $t > T$. An operator Ψ is said to be *causal* if $P_T \Psi P_T = P_T \Psi$ for all $T \ge 0$. We denote an LTI system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned}$$

by its realisation (A, B, C, D), where A, B, C and D are real matrices with compatible dimensions.

2 Dissipativity with Dynamic Supply Rates

All dynamics considered in this paper are time-invariant. Consider a nonlinear input-state-output system

$$\Sigma : \begin{array}{l} \dot{x}(t) = f(x(t), u(t)),\ x(0) = x_0,\ x(t) \in \mathcal{X},\ u(t) \in \mathcal{U} \\ y(t) = h(x(t), u(t)),\ y(t) \in \mathcal{Y}, \end{array} \tag{1}$$

with $\mathcal{X} = \mathbb{R}^n$, $\mathcal{U} = \mathbb{R}^m$, $\mathcal{Y} = \mathbb{R}^p$, locally Lipschitz $f : \mathcal{X} \times \mathcal{U} \to \mathcal{X}$ and continuous $h : \mathcal{X} \times \mathcal{U} \to \mathcal{Y}$. An input u to Σ in (1) is called *admissible* if there exists a unique solution $x(t)$ on $t \in [0, \infty)$ for every initial condition $x(0)$. The set of all admissible inputs to Σ is denoted by $\mathscr{U}$ and the set of outputs of Σ over $\mathscr{U}$ is denoted as $\mathscr{Y}$. Based on the admissible sets $\mathscr{U}$ and $\mathscr{Y}$, we propose the following definition of a dynamic supply rate:

Definition 1 (Dynamic supply rate) *A time function ξ is called a* dynamic supply rate for Σ *if it is the output of a causal time-invariant dynamic operator*

$$\Xi : \mathscr{U} \times \mathscr{Y} \times \bar{\mathcal{X}} \to \mathbf{L}_{1e}. \tag{2}$$

In other words, $\xi(t) = \Xi(u, y, \bar{x})(t)$, where $u \in \mathscr{U}$, $y \in \mathscr{Y}$ and $\bar{x} \in \bar{\mathcal{X}} = \mathbb{R}^{n_{\bar{x}}}$.

Note that the operator Ξ in (2) can capture any causal system (whether it is static or dynamic, linear or nonlinear). A characterisation of Ξ via, for example, state-space equations, is not necessary. The argument $\bar{x} \in \bar{\mathcal{X}}$ in (2) is only used for specifying initial conditions and is significant for the following reason. In the case where Ξ has a state-space representation, its initial condition may be taken to be an arbitrary function of $\bar{x} \in \bar{\mathcal{X}}$. In the case where Ξ has no state-space representation or the initial condition of its state-space representation is fixed (e.g. at 0), the dependence on $\bar{x}$ is inconsequential.

Throughout this paper, the dynamic supply rate ξ and its associated operator Ξ may be used interchangeably without ambiguity since they represent the same object. A supply rate $\xi(t) = \Xi(u, y, \bar{x})(t)$ is *static* when there exists $\tilde{\Xi} : \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}$ such that $\xi(t) = \Xi(u, y, \bar{x})(t) = \tilde{\Xi}(u(t), y(t))$ for all $t \ge 0$, $u \in \mathscr{U}$, $y \in \mathscr{Y}$ and $\bar{x} \in \bar{\mathcal{X}}$, i.e. the supply rate is independent of $\bar{x} \in \bar{\mathcal{X}}$ and the dependency on $u \in \mathscr{U}$ and $y \in \mathscr{Y}$ is that of a static function.

An example of a dynamic supply rate ξ of the quadratic form is

$$\xi(t) = \Xi(u, y, \bar{x})(t) = \left(\Psi \begin{bmatrix} u \\ y \end{bmatrix} \right)(t)^\top \left(\Pi \begin{bmatrix} u \\ y \end{bmatrix} \right)(t), \tag{3}$$

where Ψ and Π are two causal, nonlinear, dynamic operators whose initial conditions may depend on $\bar{x}$. This is a generalisation of the supply rates considered in [2, 9, 42, 44], where Ψ and Π are taken to be either affine nonlinear or linear time-invariant operators and $\bar{\mathcal{X}} = \emptyset$. Another example of a dynamic supply rate is the quadratic differential form considered in [60, 61]:

$$\xi(t) = \Xi(u, y, \bar{x})(t) = \sum_{k,\ell} \left(\frac{d^k \left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right]}{dt^k}(t) \right)^\top P_{k\ell} \left(\frac{d^\ell \left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right]}{dt^\ell}(t) \right),$$

where $P_{k\ell} \in \mathbb{R}^{(m+p)\times(m+p)}$.

For Σ in (1), we associate with it an *auxiliary system* [1]

$$\Phi : \begin{array}{l} \dot{z}(t) = g(z(t), x(t), u(t)),\ z(0) = z_0, z(t) \in \mathcal{Z} \\ \phi(t) = h_\Phi(z(t), x(t), u(t)),\ \phi(t) \in \mathcal{O}, \end{array} \tag{4}$$

with $\mathcal{Z} = \mathbb{R}^{n_z}$ and $\mathcal{O} = \mathbb{R}^{p_\phi}$, locally Lipschitz $g : \mathcal{Z} \times \mathcal{X} \times \mathcal{U} \to \mathcal{Z}$ and continuous $h_\Phi : \mathcal{Z} \times \mathcal{X} \times \mathcal{U} \to \mathcal{O}$. It is assumed throughout that there exists a unique solution $z(t)$ on $t \in [0, \infty)$ to (4) for all $u \in \mathscr{U}$. Equipped with the dynamic supply rates and auxiliary systems, we are ready to generalise the classical notion of dissipativity.

Definition 2 (Dynamic dissipativity) *Let $\Xi : \mathscr{U} \times \mathscr{Y} \times \bar{\mathcal{X}} \to \mathbf{L}_{1e}$ be causal. Σ in* (1) *is said to be* Ξ-dissipative on $(\mathcal{X}, \mathscr{U})$ *if there exist an auxiliary system* (4) *and a* storage function $S : \mathcal{X} \times \mathcal{Z} \to \mathbb{R}$ *such that the following* dissipation inequality

$$S(x(T), z(T)) \le S(x(0), z(0)) + \int_0^T \xi(t)\, dt \tag{5}$$

holds for all $T > 0$, $u \in \mathscr{U}$, $x(0) \in \mathcal{X}$, and $\bar{x} \in \bar{\mathcal{X}}$, where $\xi(t) = \Xi(u, y, \bar{x})(t)$ and x, z, y satisfy (1) *and* (4). *Furthermore, Σ is said to be* Ξ'-dissipative *if* (5) *holds for all $T > 0$, $u \in \mathscr{U}$, and $x(0) \in \mathcal{X}$, where $\xi(t) = \Xi(u, y, x(0))(t)$ and x, z, y satisfy* (1) *and* (4).

Note from the definition above that Ξ'-dissipativity is an *easier* property to satisfy than Ξ-dissipativity, since

[1] One may adopt the form $\dot{z}(t) = \hat{g}(z(t), y(t), u(t))$ and $\phi(t) = \hat{h}_\Phi(z(t), y(t), u(t))$, which is a special case of (4) by noting $y(t) = h(x(t), u(t))$.

the former is required to hold only for $\bar{x} = x(0)$ in the supply rate while the latter needs to hold for all $\bar{x} \in \bar{\mathcal{X}}$, independently of $x(0) \in \mathcal{X}$. The purpose of defining both Ξ-dissipativity and Ξ'-dissipativity will be made clear when feedback stability is examined in Section 3. When the chosen supply rate $\Xi(u, y, \bar{x})$ is independent of $\bar{x}$, the two notions Ξ-dissipativity and Ξ'-dissipativity are identical.

When Σ is static, $\mathcal{X} = \emptyset$ and thus only Ξ-dissipativity is sensible in Definition 2 with $S(z)$ replacing $S(x, z)$ in (5), i.e. a static system Σ is Ξ-dissipative if there exist an auxiliary system (4) and $S : \mathcal{Z} \to \mathbb{R}$ such that

$$S(z(T)) \leq S(z(0)) + \int_0^T \xi(t)\, dt \qquad (6)$$

for all $T > 0$, $u \in \mathscr{U}$ and $\bar{x} \in \bar{\mathcal{X}}$, where $\xi(t) = \Xi(u, y, \bar{x})(t)$.

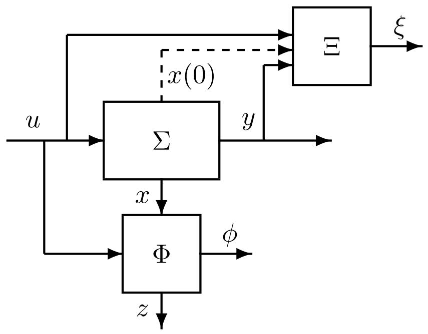

Fig. 1. A physical system Σ with state variable x, input u and output y. It is associated with a dynamic supply rate $\xi = \Xi(u, y, \bar{x})$ and an auxiliary system Φ with state variable z and output ϕ. The dotted line signifies that the initial condition of Ξ may be taken to be a function of $\bar{x} = x(0)$.

The auxiliary system Φ in Definition 2 may be related to the dynamics in the operator Ξ. It may also be empty with $\mathcal{Z} = \emptyset$, in which case we have $S : \mathcal{X} \to \mathbb{R}$ and an obvious simplification in Definition 2. In general, the auxiliary system Φ and dynamic operator Ξ may be independent of each other. See Fig. 1 for an illustration. One notably motivating example involves the class of input-output negative imaginary (IONI) systems [33], which is elaborated in Example 4. The purpose of introducing an auxiliary system (4) is to facilitate the verification of the dissipation inequality (5) with a dynamic supply rate. This idea enjoys certain similarities with that of using prolonged systems [11] in differential dissipativity variational analysis in [14, Def. 3] and [53], but is also fundamentally different since the present paper is concerned with dissipativity as opposed to its differential or incremental form [46, Sec. 2].

If S is continuously differentiable, we say that it is a C^1 storage function. To illustrate the verification of dissipativity with dynamic supply rates, we note the following lemma when S is a C^1 storage function.

Lemma 3 *Let Σ be given by (1) and the auxiliary system Φ be given by (4). Then Σ is Ξ-dissipative with a C^1 storage function S if*

$$\left(\left[\frac{\partial}{\partial x} S(x,z) \right]^\top f(x,u) \right)(t) + \left(\left[\frac{\partial}{\partial z} S(x,z) \right]^\top g(z,x,u) \right)(t) \leq \xi(t) \qquad (7)$$

for all $t \geq 0$, $u \in \mathscr{U}$, $x(0) \in \mathcal{X}$ and $\bar{x} \in \bar{\mathcal{X}}$, where $\xi(t) = \Xi(u, y, \bar{x})(t)$ and x, z, y satisfying (1) and (4). Similarly, Σ is Ξ'-dissipative if (7) holds with $\xi(t) = \Xi(u, y, x(0))(t)$.

PROOF. The proof follows from Definition 2 by taking integrals on both sides of (7). $\square$

Assuming that Ξ has a state-space representation which shares the same state equation with that of the auxiliary system Φ in (4), then (7) may be verified via an algebraic inequality. Specifically, let Φ be LTI with realisation $(A_\Phi, B_\Phi, C_\Phi, D_\Phi)$ and the supply rate be $\xi(t) = \Xi(u, y, \bar{x})(t) = (\Phi \left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right])(t)^\top P (\Phi \left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right])(t)$, where $P \in \mathbb{R}^{(m+p)\times(m+p)}$ and the initial condition of Φ may depend on $\bar{x} \in \bar{\mathcal{X}}$, then (7) holds if

$$\left[\frac{\partial}{\partial x} S(x,z) \right]^\top f(x,u) + \left[\frac{\partial}{\partial z} S(x,z) \right]^\top \left(A_\Phi z + B_\Phi \begin{bmatrix} u \\ h(x,u) \end{bmatrix} \right) \leq \left(C_\Phi z + D_\Phi \begin{bmatrix} u \\ h(x,u) \end{bmatrix} \right)^\top P \left(C_\Phi z + D_\Phi \begin{bmatrix} u \\ h(x,u) \end{bmatrix} \right)$$

for all $x \in \mathcal{X}$, $z \in \mathcal{Z}$ and $u \in \mathcal{U}$. In the case where Φ is stable, this coincides with that considered in [2, Sec. 8.1]. More generally, let the state-equation of the auxiliary system Φ be $\dot{z}(t) = \hat{g}(z(t), y(t), u(t))$, where $y(t) = h(x(t), u(t))$. Let the supply rate $\xi(t)$ be of the general quadratic form (3), where the outputs of causal operators $\Psi = \left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right] \mapsto \phi_1$ and $\Pi = \left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right] \mapsto \phi_2$ are described by $\phi_1(t) = \hat{h}_{\Phi_1}(z(t), y(t), u(t))$ and $\phi_2(t) = \hat{h}_{\Phi_2}(z(t), y(t), u(t))$, respectively, and they share the same state-equation involving z as in Φ, i.e. the output of Φ is $\left[\begin{smallmatrix} \phi_1 \\ \phi_2 \end{smallmatrix}\right]$. Then, (7) holds if

$$\left[\frac{\partial}{\partial x} S(x,z) \right]^\top f(x,u) + \left[\frac{\partial}{\partial z} S(x,z) \right]^\top \hat{g}(z, h(x,u), u) \leq \hat{h}_{\Phi_1}(z, h(x,u), u)^\top \hat{h}_{\Phi_2}(z, h(x,u), u)$$

for all $x \in \mathcal{X}$, $z \in \mathcal{Z}$ and $u \in \mathcal{U}$.

2.1 Examples of Dynamic Dissipativity

Several motivating examples are provided in this subsection. The first is adopted from [33], which provides a class of negative imaginary systems characterised by an LTI auxiliary system and a dynamic supply rate. The example is paraphrased in terms of Definition 2.

Example 4 (Input-output negative imaginariness) *Let Σ be a stable LTI system with a minimal realisation (A, B, C, D) in which $D = D^\top$. Then, $\Sigma(s) = C(sI - A)^{-1}B + D$ is said to be IONI with a level of output strictness $\delta \geq 0$ and a level of input strictness $\epsilon \geq 0$ having an arrival rate $\alpha \in \mathbb{N}$ and departure rate $\beta \in \mathbb{N}$ (i.e. $\mathrm{IONI}_{(\delta,\epsilon,\alpha,\beta)}$) [33, Def. 1] if*

$$j\omega[\Sigma(j\omega) - \Sigma(j\omega)^*] - \delta\omega^2\bar{\Sigma}(j\omega)^*\bar{\Sigma}(j\omega) - \epsilon\left(\frac{\omega^{2\beta}}{1+\omega^{2(\alpha+\beta-1)}}\right) I \geq 0 \quad \forall\omega \in \mathbb{R}\cup\{\infty\},$$

where $\bar{\Sigma}(j\omega) = \Sigma(j\omega) - D$. Moreover, Σ is said to be output strictly negative imaginary (OSNI) if it is $\mathrm{IONI}_{(\delta,\epsilon,\alpha,\beta)}$ with $\delta > 0$, $\epsilon \geq 0$, $\alpha, \beta \in \mathbb{N}$ and $\Sigma(s) - \Sigma(-s)^\top$ has full normal rank [33, Def. 7]. Let Φ be a stable LTI auxiliary system with state z for which $z(0) = 0$, minimal realisation $(A_\Phi, B_\Phi, C_\Phi, D_\Phi)$, and transfer function $\Phi(s) = C_\Phi(sI - A_\Phi)^{-1}B_\Phi + D_\Phi$ satisfying

$$\Phi(-s)\Phi(s) = \frac{(-s)^\beta s^\beta}{1 + (-s)^{(\alpha+\beta-1)}s^{(\alpha+\beta-1)}};$$

see [33, Lem. 1]. For instance, in the special case of $\alpha = \beta = 1$, the auxiliary system $\Phi(s) = \frac{s}{s+1}$ has a realisation $(-1, 1, -1, 1)$. By [33, Th. 4] and Lemma 3, it may be verified that an $\mathrm{IONI}_{(\delta,\epsilon,\alpha,\beta)}$ Σ is Ξ-dissipative with the dynamic supply rate

$$\Xi(u, y, \bar{x})(t) = 2\dot{\bar{y}}(t)^\top u(t) - \delta\dot{\bar{y}}(t)^\top\dot{\bar{y}}(t) - \epsilon(\Phi u)(t)^\top(\Phi u)(t) \tag{8}$$

and storage function of the form $S(x, z) = \begin{bmatrix} x \\ z \end{bmatrix}^\top P \begin{bmatrix} x \\ z \end{bmatrix}$, where $P = P^\top$. Note that the operator Ξ in (8) involves not only the auxiliary system Φ but also the differentiation operation on $\bar{y}$, i.e. $\dot{\bar{y}}$. Furthermore, by [33, Cor. 8] and Lemma 3, it may be verified that an OSNI Σ is Ξ-dissipative with the supply rate

$$\Xi(u, y, \bar{x})(t) = 2\dot{\bar{y}}(t)^\top u(t) - \delta\dot{\bar{y}}(t)^\top\dot{\bar{y}}(t)$$

and storage function $S(x) = x^\top P x$ with $P = P^\top$. IONI and OSNI systems are examples in which the dynamics of the auxiliary system Φ are not necessarily those of the supply rate Ξ.

Next, we provide a couple of examples of nonlinear systems that are dissipative with respect to dynamic supply rates. They may serve the purpose of illustrating the search for auxiliary systems in order to satisfy certain desired dissipation inequalities. The following example provides a Ξ'-dissipative system Σ whose supply rate Ξ depends on the initial condition $x(0)$ of Σ.

Example 5 *Let Σ in (1) be described by*

$$\Sigma: \begin{aligned} \dot{x}_1(t) &= -ax_1(t) - \psi(x_1(t)) + 2x_2(t) \\ &\quad - \textstyle\sum_{k=-N}^{M} b_k(x_1(t))^{2k+1} + u(t) \\ \dot{x}_2(t) &= -x_2(t) + u(t) \\ y(t) &= x_1(t) - x_2(t) \end{aligned} \tag{9}$$

with $x(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} \in \mathbb{R}^2$, where $a \geq 1$, $b_k \geq 0$ for all $k \in \{-N, -N+1, \ldots, M\}$, N and M are nonnegative integers, and $\psi : \mathbb{R} \to \mathbb{R}$ is a locally integrable nonlinearity satisfying $\psi(0) = 0$ and $\psi(r)r \geq 0$ for all $r \in \mathbb{R}$. Let the dynamic supply rate $\Xi = \begin{bmatrix} u \\ y \\ x(0) \end{bmatrix} \mapsto \xi$ be given by

$$\Xi: \begin{aligned} \dot{z}(t) &= -z(t) + u(t), \quad z(0) = [\,0\ 1\,]\,x(0) \\ \xi(t) &= u(t)(3z(t) + y(t)). \end{aligned} \tag{10}$$

Consider the candidate storage function $S(x_1, x_2, z) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 + \frac{1}{2}z^2$. In view of (10), let the auxiliary system Φ have dynamics $\dot{z}(t) = -z(t) + u(t)$ with $z(0) = x_2(0)$, i.e. the same as the state equation in (10). Observe that $x_2(t) = z(t)$ $\forall t \geq 0$. Along the solutions to Σ and Φ,

$$\begin{aligned} &\frac{d}{dt}S(x_1, x_2, z) \\ &= -ax_1^2 + 2x_1x_2 - x_1\psi(x_1) - \textstyle\sum_{k=-N}^{M} b_k(x_1)^{2k+2} + ux_1 \\ &\quad - x_2^2 + ux_2 + z(-z + u) \\ &\leq -(a-1)x_1^2 - (x_1^2 - 2x_1x_2 + x_2^2) - z^2 + u(x_1 + x_2 + z) \\ &\leq u(x_1 + x_2 + z) - y^2 \leq u(x_1 + 2x_2) = \xi. \end{aligned} \tag{11}$$

Integrating both sides gives that Σ is Ξ'-dissipative by Lemma 3.

Example 5 will be revisited in Section 6 where a simulation example is provided. The next two nonlinear examples concern Ξ-dissipativity.

Example 6 *Let Σ be defined by $y(t) = \sigma(u(t))$, where $\sigma : \mathbb{R} \to \mathbb{R}$ is locally integrable and satisfies $\sigma(0) = 0$ and*

$$\sigma(r)(br - \sigma(r)) \geq 0 \ \forall r \in \mathbb{R}, \tag{12}$$

$$(\sigma(r) - \sigma(q))(r - q) \geq 0 \ \forall r, q \in \mathbb{R}, \tag{13}$$

with $b > 0$; i.e., Σ is sector-bounded in the sector $[0, b]$ and monotonically nondecreasing. Adding (12) to (13) and rearranging the inequality yield

$$(1+b)\sigma(r)r - (\sigma(r))^2 - \sigma(r)q \geq -\sigma(q)q + \sigma(q)r \tag{14}$$

for $r, q \in \mathbb{R}$. Let the supply rate $\Xi = \left[\begin{smallmatrix} u \\ y \\ \bar{x} \end{smallmatrix}\right] \mapsto \xi$ be given by

$$\Xi : \begin{array}{l} \dot{z}(t) = -z(t) + u(t), \ z(0) = z_0 \in \mathbb{R} \\ \xi(t) = -y(t)\left[z(t) - (1+b)u(t) + y(t)\right]. \end{array}$$

In view of the staticity of Σ, consider the candidate storage function $S(z) = \int_0^z \sigma(r)\,dr$. Note that S belongs to C^1 and $\frac{d}{dt}S(z) = \sigma(z)\dot{z}$. In light of (14), let the auxiliary system Φ have dynamics $\dot{z}(t) = -z(t) + u(t)$ with $z(0) = z_0$. Then, along the solutions to Φ and by using (14), we have $\frac{d}{dt}S(z) \leq (1+b)yu - y^2 - yz = \xi$. Integrating both sides yields that Σ is Ξ-dissipative by Lemma 3.

Example 7 *Let Σ in (1) be given by*

$$\Sigma : \begin{array}{l} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = -(x_1(t))^3 + (\psi(x_2(t)))^2 + u(t) \\ y(t) = x_2(t) \end{array}$$

with $x(0) = \left[\begin{smallmatrix} x_{1,0} \\ x_{2,0} \end{smallmatrix}\right]$, where $\psi : \mathbb{R} \to \mathbb{R}$ is locally integrable and satisfies $\psi(0) = 0$ and $\psi(r)r \geq 0$ for all $r \in \mathbb{R}$. Let the dynamic supply rate $\Xi = \left[\begin{smallmatrix} u \\ y \\ \bar{x} \end{smallmatrix}\right] \mapsto \xi$ be given by

$$\Xi : \begin{array}{l} \dot{z}(t) = -z(t) + \psi(z(t))(u(t))^2 + y(t), \ z(0) = z_0 \in \mathbb{R} \\ \xi(t) = y(t)\left[z(t) + u(t) + (\psi(y(t)))^2\right]. \end{array}$$

Consider the candidate storage function $S(x_1, x_2, z) = \frac{1}{4}x_1^4 + \frac{1}{2}x_2^2 + \frac{1}{2}z^2$. Then, we obtain $\frac{d}{dt}S(x_1, x_2, z) = y(\psi(y))^2 + yu + z\dot{z}$. Note that $\psi(z(t))z(t) \geq 0$ for all $z(t) \in \mathbb{R}$ and thus $z(t)\left[-z(t) - \psi(z(t))(u(t))^2\right] \leq 0$ for all $z(t) \in \mathbb{R}$. Using the auxiliary system Φ with dynamics $\dot{z}(t) = -z(t) + \psi(z(t))(u(t))^2 + y(t)$ with $z(0) = z_0$, it then follows that $\frac{d}{dt}S(x_1, x_2, z) \leq y(\psi(y))^2 + yu + zy = \xi$. By Lemma 3, Σ is Ξ-dissipative. Alternatively, one may choose an empty Φ with $\mathcal{Z} = \emptyset$. In such a case, a viable supply rate is $\tilde{\xi}(t) = \tilde{\Xi}(u(t), y(t)) = u(t)y(t) + y(t)(\phi(y(t)))^2$, which is static.

In both Examples 6 and 7, even though the systems may be described by dissipativity with respect to some static supply rates, the advantage of using dynamic supply rates lies in offering great flexibility in system characterisation as well as reducing conservatism in feedback stability analysis, similarly to the benefit of using dynamic multipliers, or IQCs, in an input-output setting.

3 Main Results on Feedback Stability

In this section, we present the main results of this paper — feedback stability analysis via the proposed dissipativity with dynamic supply rates. The results involve Lyapunov and asymptotic stability presented in the order stated. First, some technical definitions are stated and the configuration under study is introduced.

3.1 Definitions

Consider the system Σ described in (1).

Definition 8 *The system Σ in (1) is said to be* zero-state detectable *if $u(t) = 0$ and $y(t) = 0$ for all $t \geq 0$ implies $\lim_{t\to\infty} x(t) = 0$.*

Definition 9 *Let $x(t)$ be a solution of Σ in (1) with $u = 0$. A point $p \in \mathcal{X}$ is said to be a* positive limit point *of $x(t)$ if there exists a sequence $\{t_n\}_{n=1}^{\infty}$, with $t_n \to \infty$ as $n \to \infty$, such that $x(t_n) \to p$ as $n \to \infty$. The set of all positive limit points of $x(t)$ is called the* positive limit set *of $x(t)$.*

Definition 10 *A set $M \subset \mathcal{X}$ is said to be a* positively invariant set *with respect to Σ in (1) if*

$$u = 0 \quad \text{and} \quad x(0) \in M \implies x(t) \in M \quad \forall t \geq 0.$$

Definition 11 *Σ in (1) with $u = 0$ is said to be:*

(i) *Lyapunov stable with respect to x if, for every $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ such that $\|x(0)\| < \delta$ implies that $\|x(t)\| < \epsilon$ for all $t \geq 0$;*
(ii) *asymptotically stable with respect to x if it is Lyapunov stable with respect to x and there exists $\delta > 0$ such that $\|x(0)\| < \delta$ implies that $\lim_{t\to\infty} x(t) = 0$.*
(iii) *globally asymptotically stable with respect to x if it is Lyapunov stable with respect to x and $\lim_{t\to\infty} x(t) = 0$ for all $x(0) \in \mathcal{X}$.*

3.2 Problem Formulation and Preliminaries

Consider two nonlinear input-state-output systems

$$\Sigma_i : \begin{array}{l} \dot{x}_i(t) = f_i(x_i(t), u_i(t)), \ x_i(t) \in \mathcal{X}_i, \ u_i(t) \in \mathcal{U}_i, \\ y_i(t) = h_i(x_i(t), u_i(t)), \ y_i(t) \in \mathcal{Y}_i \end{array} \tag{15}$$

for $i \in \{1, 2\}$, with $x_i(0) = x_{i,0}$, $\mathcal{X}_i = \mathbb{R}^{n_i}$, $\mathcal{U}_1 = \mathcal{Y}_2 = \mathbb{R}^m$, $\mathcal{U}_2 = \mathcal{Y}_1 = \mathbb{R}^p$, locally Lipschitz $f_i : \mathcal{X}_i \times \mathcal{U}_i \to \mathcal{X}_i$ and continuous $h_i : \mathcal{X}_i \times \mathcal{U}_i \to \mathcal{Y}_i$ interconnected in a feedback configuration shown in Fig. 2 and described by

$$u_1 = w_1 + y_2; \quad u_2 = w_2 + y_1. \tag{16}$$

Denote the admissible inputs set by $\mathscr{U}_i$ and the outputs set over $\mathscr{U}_i$ by $\mathscr{Y}_i$ for $i \in \{1, 2\}$.

Henceforth, the feedback interconnection of Fig. 2 is denoted by $\Sigma_1 \| \Sigma_2$ and written as

$$\Sigma_1 \| \Sigma_2 : \begin{array}{l} \dot{x}(t) = \check{f}(x(t), w(t)), \ x(t) \in \mathcal{X}, w(t) \in \mathcal{W}, \\ y(t) = \check{h}(x(t), w(t)), \ y(t) \in \mathcal{Y}, \end{array} \tag{17}$$

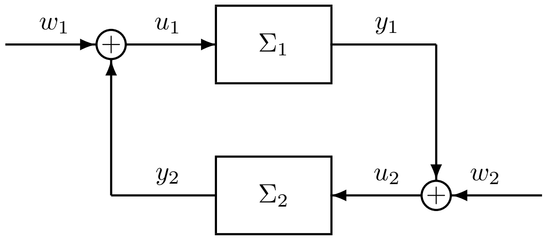

Fig. 2. Feedback configuration of Σ_1 and Σ_2.

where $x(0) = (x_{1,0}, x_{2,0})$, $\mathcal{X} = \mathcal{X}_1 \times \mathcal{X}_2$, $\mathcal{W} = \mathcal{U}_1 \times \mathcal{U}_2$, $\mathcal{Y} = \mathcal{Y}_1 \times \mathcal{Y}_2$, $x(t) = (x_1(t), x_2(t))$, $w(t) = (w_1(t), w_2(t))$ and $y(t) = (y_1(t), y_2(t))$.

We associate an auxiliary system Φ_i of the form (4) with state variable $z_i \in \mathcal{Z}_i = \mathbb{R}^{n_{z_i}}$ for $i \in \{1, 2\}$ with each system Σ_i. That is,

$$\Phi_i : \begin{array}{l} \dot{z}_i(t) = g_i(z_i(t), x_i(t), u_i(t)),\ z_i(t) \in \mathcal{Z}_i, z_i(0) = z_{i,0} \\ \phi_i(t) = h_{\Phi_i}(z_i(t), x_i(t), u_i(t)),\ \phi_i(t) \in \mathcal{O}_i. \end{array}$$

In what follows, let $\mathcal{Z} = \mathcal{Z}_1 \times \mathcal{Z}_2$, $\mathcal{O} = \mathcal{O}_1 \times \mathcal{O}_2$, $z(t) = (z_1(t), z_2(t))$ and $\phi(t) = (\phi_1(t), \phi_2(t))$. The two auxiliary systems to the feedback system $\Sigma_1 \| \Sigma_2$ may thus be described succinctly as

$$\begin{aligned} \dot{z}(t) &= \check{g}(z(t), x(t), w(t)),\ z(t) \in \mathcal{Z}, z(0) = (z_{1,0}, z_{2,0}) \\ \phi(t) &= \check{h}_\Phi(z(t), x(t), w(t)),\ \phi(t) \in \mathcal{O}. \end{aligned} \tag{18}$$

Define the operator $\Gamma : \mathscr{U} \times \mathscr{Y} \times \bar{\mathcal{X}} \to \mathscr{Y} \times \mathscr{U} \times \bar{\mathcal{X}}$ by $\Gamma(u, y, \bar{x}) = (y, u, \bar{x})$, i.e. Γ swaps the order of its first two arguments. Denote by $\circ$ the composition operation.

3.3 Lyapunov Stability

Main results on feedback Lyapunov stability are derived in this section. First, an assumption on the storage functions is stated.

Assumption 12 *The storage functions S_1 and S_2 are C^1 and there exist $\delta > 0$ and class $\mathcal{K}$ functions α and β such that*

$$\alpha(\|x\|) \leq \sum_{i=1}^{2} S_i(x_i, z_i) \leq \beta(\|x\|) \tag{19}$$

for all $x \in \mathcal{X}$ with $\|x\| < \delta$ and $z \in \mathcal{Z}$.

Assumption 12 requires boundedness on the sum of two storage functions in terms of parts (but not all) of their arguments. This resembles boundedness on a time-varying Lyapunov candidate function [27, Th. 4.8] and a Lyapunov candidate function for partial stability [20, Th. 4.1]. An alternative assumption where the lower and upper bounds are functions of $\|z\|$ will be considered later.

Theorem 13 *Suppose there exists a causal operator $\Xi : \mathscr{U}_1 \times \mathscr{Y}_1 \times \mathcal{X}_1 \to \mathbf{L}_{1e}$ such that Σ_1 is Ξ'-dissipative and Σ_2 is $(-\Xi \circ \Gamma)$-dissipative. Furthermore, suppose the corresponding storage functions S_1 and S_2 satisfy Assumption 12. Then, $x^* = (x_1^*, x_2^*) = (0, 0)$ is a Lyapunov stable equilibrium of the closed-loop system $\Sigma_1 \| \Sigma_2$ with $w_1 = 0$ and $w_2 = 0$. If additionally Σ_2 is static, then $\mathcal{X}_2 = \emptyset$ and $x_1^* = 0$ is a Lyapunov stable equilibrium.*

PROOF. By hypothesis and using the time-invariant properties of Σ_1 and Σ_2, we have

$$\begin{aligned} S_1(x_1(t_2), z_1(t_2)) &\leq S_1(x_1(t_1), z_1(t_1)) \\ &\quad + \int_{t_1}^{t_2} \Xi(u_1, y_1, x_1(t_1))(t)\, dt \\ S_2(x_2(t_2), z_2(t_2)) &\leq S_2(x_2(t_1), z_2(t_1)) \\ &\quad - \int_{t_1}^{t_2} \Xi(y_2, u_2, x_1(t_1))(t)\, dt \end{aligned} \tag{20}$$

for all $t_2 \geq t_1$, all initial conditions $x_1(t_1) \in \mathcal{X}_1, x_2(t_1) \in \mathcal{X}_2$, and all input functions $u_1 \in \mathscr{U}_1$, $u_2 \in \mathscr{U}_2$. Substituting the feedback equations (16) with $w_1 = 0, w_2 = 0$ into the inequalities in (20), summing them, dividing both sides by $t_2 - t_1$, and taking $t_2 \to t_1$ then yields

$$\frac{d}{dt}(S_1(x_1, z_1) + S_2(x_2, z_2)) \leq 0 \tag{21}$$

along the solutions to (17) and (18). Define $V(x, z) = S_1(x_1, z_1) + S_2(x_2, z_2)$. We have from (21) that $\frac{d}{dt}V(x, z) = \dot{V}(x, z) \leq 0$ along the solutions to (17) and (18). This implies that V is nonincreasing along the solutions to (17) and (18). From (19), we have $V(x^*, z) = 0$ and $V(x, z) > 0$ for all $x \in \mathcal{X}$ with $\|x\| < \delta$, $x \neq x^*$, and $z \in \mathcal{Z}$. This means (x^*, z) is a strict minimum of V for all $z \in \mathcal{Z}$, whereby $\check{f}(x^*, 0) = 0$, i.e. $x^* = 0$ is an equilibrium in (17) with $w_1 = 0$ and $w_2 = 0$. That $x^* = 0$ is Lyapunov stable then follows from [20, Th. 4.1(ii)]. For the case when Σ_2 is additionally static, $\mathcal{X}_2 = \emptyset$ and we have $S_2(z_2(t_2)) \leq S_2(z_2(t_1)) - \int_{t_1}^{t_2} \Xi(y_2, u_2, \bar{x})(t)\, dt$ for all $t_2 \geq t_1$, $\bar{x} \in \bar{\mathcal{X}}$, $u_2 \in \mathscr{U}_2$ and $y_2(t) = \Sigma_2(u_2(t))$. With $V(x_1, z) = S_1(x_1, z_1) + S_2(z_2)$, the arguments above may be repeated to show $x_1^* = 0$ is Lyapunov stable. □

The purpose of the operator Γ in Theorem 13 is to represent the "*inverse supply rate*" for Σ_2, namely, $(\Xi \circ \Gamma)(u_2, y_2, \bar{x}) = \Xi(y_2, u_2, \bar{x})$. Theorem 13 indicates the main idea of this paper: *If Σ_1 and Σ_2 can be simultaneously described by "complementary" dynamic supply rates Ξ and $-\Xi \circ \Gamma$, then the closed-loop system $\Sigma_1 \| \Sigma_2$ will have Lyapunov stability with respect to x.* This generalises the classical idea of feedback stability via static dissipativity, and the connection is elaborated in Section 4. It is noteworthy that Σ_1 and Σ_2 may be associated with different auxiliary systems Φ_1 and Φ_2 with state variables z_1 and z_2, respectively, as long as the dissipativity conditions on Σ_1 and Σ_2 hold. More importantly, the dynamics involving z_1 and z_2 are not necessarily stable since their stability is irrelevant as far as closed-loop stability is concerned.

Next, we present a feedback Lyapunov stability result under a different assumption on the storage functions that may be more useful in certain circumstances than Assumption 12.

Assumption 14 *The storage functions S_1 and S_2 are C^1 and there exist $\delta > 0$ and class $\mathcal{K}$ functions α and β such that*

$$\alpha(\|(x,z)\|) \leq \sum_{i=1}^{2} S_i(x_i,z_i) \leq \beta(\|(x,z)\|)$$

for all $x \in \mathcal{X}$ and $z \in \mathcal{Z}$ with $\|(x,z)\| < \delta$.

In contrast to Assumption 12, the lower and upper bounds in Assumption 14 depend on both x and z. A byproduct of this assumption is that the stability with respect to both $x = 0$ and $z = 0$ in (17) and (18) may be established, even though we are only concerned with the former. This resembles the theory of dynamic Lyapunov functions proposed in [41, Def. 1].

Theorem 15 *Suppose there exists a causal operator $\Xi : \mathscr{U}_1 \times \mathscr{Y}_1 \times \mathcal{X}_1 \to \mathbf{L}_{1e}$ such that Σ_1 is Ξ'-dissipative and Σ_2 is $(-\Xi\circ\Gamma)$-dissipative. Furthermore, suppose the corresponding storage functions S_1 and S_2 satisfy Assumption 14 and $(x^*,z^*) = (x_1^*,x_2^*,z_1^*,z_2^*) = (0,0,0,0)$ is an equilibrium of (17) and (18) with $w_1 = 0$ and $w_2 = 0$. Then, the equilibrium $x^* = (0,0)$ of $\Sigma_1\|\Sigma_2$ in (17) is Lyapunov stable. If additionally Σ_2 is static, then $\mathcal{X}_2 = \emptyset$ and the equilibrium $x_1^* = 0$ is Lyapunov stable.*

PROOF. Define $V(x,z) = S_1(x_1,z_1)+S_2(x_2,z_2)$. Following the same arguments in the proof of Theorem 13, we can obtain that $\frac{d}{dt}V(x,z) = \dot{V}(x,z) \leq 0$ along the solutions to (17) and (18). By the standard Lyapunov stability theorem [50, Th. 3.2.4], the equilibrium (x^*,z^*) of (17) and (18) is Lyapunov stable. By [27, Lem. 4.5], this is equivalent to the existence of $c > 0$ and class $\mathcal{K}$ function κ such that

$$\|(x(t),z(t))\| \leq \kappa(\|(x(0),z(0))\|)$$

for all $t \geq 0$ and all $(x(0),z(0))$ satisfying $\|(x(0),z(0))\| < c$. This implies

$$\|x(t)\| \leq \|(x(t),z(t))\| \leq \kappa(\|(x(0),0)\|) = \kappa(\|x(0)\|)$$

for all $t \geq 0$ and $\|x(0)\| < c$, from which Lyapunov stability of $x^* = (0,0)$ of (17) follows again by [27, Lem. 4.5]. The static case can be similarly proved by using $V(x_1,z) = S_1(x_1,z_1) + S_2(z_2)$. □

Remark 16 *In Theorem 15, (x^*,z^*) is presumed to be an equilibrium. By contrast, Theorem 13 establishes that x^* is an equilibrium using properties of the storage functions in Assumption 12.*

Feedback stability in the sense of Lyapunov often leaves much to be desired. Next, we examine the stronger notion of asymptotic feedback stability via dissipativity.

3.4 Asymptotic Stability

In this subsection, we establish feedback asymptotic stability via dissipativity with dynamic supply rates. The following technical lemma is needed in the proof of Theorem 20. It mimics [50, Prop. 3.2.16] and establishes asymptotic stability for an open-loop system through dissipativity.

Lemma 17 *Let Σ in* (1) *be zero-state detectable and Ξ-dissipative on $(\mathcal{X},\mathscr{U})$ with an auxiliary system Φ given in* (4)*, and a* static *supply rate $\tilde{\Xi}(u(t),y(t))$ that is continuous in $y(t) \in \mathcal{Y}$ and satisfies $\tilde{\Xi}(0,y(t)) \leq 0$ for all $y(t) \in \mathcal{Y}$. Let $(0,0) \in (\mathcal{X},\mathscr{U})$ and $\tilde{\Xi}(0,y(t)) = 0$ imply $y(t) = 0$. Suppose also that the corresponding storage function $S(x,z)$ is C^1 and there exist $\delta > 0$ and class $\mathcal{K}$ functions α,β such that*

$$\alpha(\|x\|) \leq S(x,z) \leq \beta(\|x\|) \tag{22}$$

for all $x \in \mathcal{X}$ with $\|x\| < \delta$ and $z \in \mathcal{Z}$. Then, $x^* = 0$ is an asymptotically stable equilibrium of Σ in (1) *with $u = 0$. If, additionally, α,β are class $\mathcal{K}_\infty$ functions and* (22) *holds for all $x \in \mathcal{X}$ and $z \in \mathcal{Z}$, then $x^* = 0$ is globally asymptotically stable.*

PROOF. By hypothesis and using the time-invariant property of Σ, with $u = 0$, we have the following dissipation inequality

$$S(x(t_2),z(t_2)) \leq S(x(t_1),z(t_1)) + \int_{t_1}^{t_2} \tilde{\Xi}(0,y(t))\,dt.$$

Dividing both sides by t_2-t_1 and taking the limit as $t_2 \to t_1$, we get $\frac{d}{dt}S(x(t),z(t)) \leq \tilde{\Xi}(0,y(t)) \leq 0$, and thus $S(x,z)$ is nonincreasing along the solutions of (1) with $u = 0$. Since $S(x,z)$ has a strict local minimum in x at $x^* = 0$ and $S(x^*,z) = 0$ for all $z \in \mathcal{Z}$ according to (22), it follows that $x^* = 0$ is an equilibrium, i.e. $f(x^*,0) = 0$. Note that $\beta^{-1}\circ\alpha$ is a class $\mathcal{K}$ function [27, Lem. 4.2]. Given any $\epsilon > 0$, let $\delta = \delta(\epsilon)$ be chosen such that $\delta = \beta^{-1}(\alpha(\epsilon)) > 0$. Then by hypothesis and since $S(x,z)$ is nonincreasing along the solutions of (1) with $u = 0$, for $\|x(0)\| < \delta$, we have $\alpha(\|x(t)\|) \leq S(x(t),z(t)) \leq$

$S(x(0), z(0)) \leq \beta(\|x(0)\|) < \beta(\delta)$ for all $t > 0$. This implies $\|x(t)\| < \alpha^{-1}(\beta(\delta)) = \epsilon$ for all $t > 0$. Then $x^* = 0$ is a Lyapunov stable equilibrium.

To further show asymptotic stability of $x^* = 0$, let $W(x) = -\tilde{\Xi}(0, h(x, 0))$. Since W is continuous, $W(x) \geq 0$ and $W(0) = 0$, by [20, Th. 4.2], the trajectory $x(t)$ with $\|x(0)\| < \delta$ approaches the set $C = \{x \in \mathcal{X} \mid \tilde{\Xi}(0, h(x, 0)) = 0\}$ as $t \to \infty$ and thus the positive limit set of $x(t)$ is a subset of the set C. Furthermore, by [27, Lem. 4.1], the positive limit set of $x(t)$ is a positively invariant set. Suppose $x(t)$ as a solution to (1) stays identically for all $t \geq 0$ in the set C. Since $\tilde{\Xi}(0, h(x(t), 0)) = 0$ implies $y(t) = h(x(t), 0) = 0$ for all $t \geq 0$ and Σ is zero-state detectable, we have $\lim_{t\to\infty} x(t) = 0$. In other words, no solution other than $x^* = 0$ stays identically in the set C for all $t \geq 0$. Asymptotic stability of $x^* = 0$ then follows.

If, additionally, α, β are class $\mathcal{K}_\infty$ functions and (22) holds for all $x \in \mathcal{X}$ and $z \in \mathcal{Z}$, global asymptotic stability of $x^* = 0$ can be concluded by using the same arguments as above. □

A strict version of dynamic dissipativity beyond Definition 2 is given below. It is needed to establish asymptotic closed-loop stability.

Definition 18 (I/O-strict dynamic dissipativity) *Let $\Xi : \mathscr{U} \times \mathscr{Y} \times \bar{\mathcal{X}} \to \mathbf{L}_{1e}$ be causal, and γ_1 and γ_2 be locally integrable continuous positive definite functions. Σ in* (1) *is called* very strictly Ξ-dissipative *if there exist an auxiliary system* (4) *and $S : \mathcal{X} \times \mathcal{Z} \to \mathbb{R}$ such that*

$$S(x(T), z(T)) + \int_0^T [\gamma_1(u(t)) + \gamma_2(y(t))]\, dt \leq S(x(0), z(0)) + \int_0^T \xi(t)\, dt \tag{23}$$

holds for all $T > 0$, $u \in \mathscr{U}$, $x(0) \in \mathcal{X}$ and $\bar{x} \in \bar{\mathcal{X}}$, where $\xi(t) = \Xi(u, y, \bar{x})(t)$ and x, z, y satisfy (1) *and* (4)*. In addition, it is called* input (resp. output) strictly Ξ-dissipative *if* (23) *holds without the term $\gamma_2(\cdot)$ (resp. $\gamma_1(\cdot)$). Strict Ξ'-dissipativity may be defined similarly as in Definition 2 by requiring $\xi(t) = \Xi(u, y, x(0))(t)$.*

Definition 18 lists three different types of dissipativity strictness. It reminisces the definition of strict passivity in the literature; see, for example, [27, Def. 6.3].

Remark 19 *We have seen earlier in Example 5 that Σ in* (9) *is Ξ'-dissipative. We may now further characterise its dissipativity strictness via Definition 18. Specifically, according to the first inequality in* (11)*, Σ is output strictly Ξ'-dissipative.*

Equipped with Definition 18 and Lemma 17, we are ready to present the main result on asymptotic stability of the closed-loop system $\Sigma_1 \| \Sigma_2$.

Theorem 20 *Suppose Σ_1 and Σ_2 in* (15) *are zero-state detectable and there exists a causal operator $\Xi : \mathscr{U}_1 \times \mathscr{Y}_1 \times \mathcal{X}_1 \to \mathbf{L}_{1e}$ such that any one of the following dissipativity conditions holds:*

(i) *Σ_1 is Ξ'-dissipative and Σ_2 is very strictly $(-\Xi \circ \Gamma)$-dissipative;*
(ii) *Σ_1 is very strictly Ξ'-dissipative and Σ_2 is $(-\Xi \circ \Gamma)$-dissipative;*
(iii) *Σ_1 is input strictly Ξ'-dissipative and Σ_2 is input strictly $(-\Xi \circ \Gamma)$-dissipative;*
(iv) *Σ_1 is output strictly Ξ'-dissipative and Σ_2 is output strictly $(-\Xi \circ \Gamma)$-dissipative.*

Furthermore, suppose the corresponding storage functions S_1 and S_2 satisfy Assumption 12 (resp. Assumption 14 and $(x^*, z^*) = (x_1^*, x_2^*, z_1^*, z_2^*) = (0, 0, 0, 0)$ is an equilibrium of $\Sigma_1 \| \Sigma_2$ and auxiliary system (18) *with $w_1 = 0$ and $w_2 = 0$). Then, $x^* = (x_1^*, x_2^*) = (0, 0)$ is an asymptotically stable equilibrium of the closed-loop system $\Sigma_1 \| \Sigma_2$ with $w_1 = 0$ and $w_2 = 0$. Moreover, if Assumption 12 (resp. Assumption 14) holds for all $x \in \mathcal{X}$ and $z \in \mathcal{Z}$ for class $\mathcal{K}_\infty$ functions α and β, then $x^* = 0$ is a globally asymptotically stable equilibrium.*

PROOF. Let S_1 and S_2 satisfy Assumption 12. We show that $x^* = 0$ is an asymptotically stable equilibrium when (i) holds. Stability involving (ii), (iii) or (iv) may be established similarly. By hypothesis, we have

$$\begin{aligned} S_1(x_1(T), z_1(T)) &\leq S_1(x_1(0), z_1(0)) \\ &\quad + \int_0^T \Xi(u_1, y_1, x_1(0))(t)\, dt \\ S_2(x_2(T), z_2(T)) &\leq S_2(x_2(0), z_2(0)) \\ - \int_0^T \Xi(y_2, u_2, x_1(0))(t)\, dt &- \int_0^T [\gamma_1(u_2(t)) + \gamma_2(y_2(t))]\, dt \end{aligned}$$

for all $T > 0$, initial conditions $x_1(0) \in \mathcal{X}_1$, $x_2(0) \in \mathcal{X}_2$, and input functions $u_1 \in \mathscr{U}_1$, $u_2 \in \mathscr{U}_2$. By substituting the feedback equations (16) with $w_1 = 0$ and $w_2 = 0$, i.e. $u_1 = y_2$, $u_2 = y_1$, into the conditions above and summing the inequalities, we get

$$\begin{aligned} S_1(x_1(T), z_1(T)) + S_2(x_2(T), z_2(T)) &\leq S_1(x_1(0), z_1(0)) \\ + S_2(x_2(0), z_2(0)) - \int_0^T & [\gamma_1(y_1(t)) + \gamma_2(y_2(t))]\, dt. \end{aligned} \tag{24}$$

Let $S(x, z) = S_1(x_1, z_1) + S_2(x_2, z_2)$ and $\tilde{\xi}(t) = \tilde{\Xi}(w(t), y(t)) = -\gamma_1(y_1(t)) - \gamma_2(y_2(t))$, where $x = (x_1, x_2)$, $z = (z_1, z_2)$, $w = (w_1, w_2)$, and $y = (y_1, y_2)$. Note that $\tilde{\Xi}(0, y(t)) \leq 0$, $\tilde{\Xi}(0, y(t)) = 0$ implies that

$y(t) = 0$, and $\tilde{\Xi}(0, y(t))$ is static and continuous in $y(t) \in \mathcal{Y}$ since γ_1 and γ_2 are continuous positive definite functions. Moreover, by (24), the closed-loop system $\Sigma_1 \| \Sigma_2$ with input $w = 0$ and output y is dissipative with a static supply rate $\tilde{\xi}(t) = \tilde{\Xi}(w(t), y(t))$ via the auxiliary system (18), where the storage function S is in C_1. Therefore, $x^* = 0$ is an asymptotically stable equilibrium via Lemma 17. In addition, when Assumption 12 holds for all $x_i \in \mathcal{X}_i$ and $z_i \in \mathcal{Z}_i$ and α_i, β_i are class $\mathcal{K}_\infty$ functions, global asymptotic stability of $x^* = 0$ follows again from Lemma 17.

Now suppose S_1 and S_2 satisfy Assumption 14 and $(x^*, z^*) = (0, 0)$ is an equilibrium. Using (24), one obtains that (x^*, z^*) is Lyapunov stable by the standard Lyapunov theorem [50, Th. 3.2.4]. Let $S(x, z) = S_1(x_1, z_1) + S_2(x_2, z_2)$. Noting that (24) and the zero-state detectability of Σ_1 and Σ_2 imply that the only solution x that can stay identically in $\{x \in \mathcal{X} : \dot{S}(x, z) = 0\}$ is $x(t) = 0$ for all $t \geq 0$, asymptotic stability of $x^* = 0$ then follows from the LaSalle's invariance principle [27, Th. 4.4]. □

Notice that in conditions (i)-(iv) of Theorem 20, both Σ_1 and Σ_2 share complementary supply rates Ξ and $-\Xi \circ \Gamma$, which are used to characterise dynamic dissipativity. In comparison with the Lyapunov stability result (Theorem 13), Theorem 20 requires extra positive definite terms $\gamma_i(\cdot)$ as in (23) for describing the strictness of dissipativity so that the stronger notion of asymptotic stability of $\Sigma_1 \| \Sigma_2$ can be established. It is worth noting that conditions (i)-(iv) of Theorem 20 involve permutations of the positive definite terms $\gamma_i(\cdot)$ on the inputs and outputs of the open-loop systems. Some important special cases of Theorem 20 relating to the literature are detailed in Section 4.

Remark 21 *In the case where Σ_2 is static or stability of $x_2^* = 0$ is of no concern, the dissipativity conditions* (i)-(iv) *in Theorem 20 for Σ_2 can be simplified by omitting x_2 as in* (6) *and restricting $\mathcal{X}$ to be $\mathcal{X}_1$ in Assumption 12 or 14 and Theorem 20. In this case, stability of $x_1^* = 0$ may be established with $S(x_1, z) = S_1(x_1, z_1) + S_2(z_2)$ by looking at the closed-loop map from w_1 to y_1.*

Interestingly, asymptotic stability of the feedback system may be established using a type of strict dissipativity where the strictness is derived from the state. In this case, the assumption on the zero-state detectability of Σ_1 and Σ_2 is not needed.

Definition 22 (State-strict dynamic dissipativity) *Let $\Xi : \mathcal{U} \times \mathcal{Y} \times \bar{\mathcal{X}} \to \mathbf{L}_{1e}$ be causal, and γ be a class $\mathcal{K}$ function. Σ in* (1) *is called* state strictly Ξ-dissipative *if there exist an auxiliary system* (4) *and $S : \mathcal{X} \times \mathcal{Z} \to \mathbb{R}$ such that*

$$S(x(T), z(T)) + \int_0^T \gamma(\|x(t)\|)\, dt \leq S(x(0), z(0)) + \int_0^T \xi(t)\, dt \quad (25)$$

holds for all $T > 0$, $u \in \mathcal{U}$, $x(0) \in \mathcal{X}$, and $\bar{x} \in \bar{\mathcal{X}}$, where $\xi(t) = \Xi(u, y, \bar{x})(t)$ and x, z, y satisfy (1) *and* (4)*. Furthermore, Σ is said to be* state strictly Ξ'-dissipative *if* (25) *holds for all $T > 0$, $u \in \mathcal{U}$, and $x(0) \in \mathcal{X}$, with $\xi(t) = \Xi(u, y, x(0))(t)$.*

Theorem 23 *The conclusions of Theorem 20 still hold if the dissipativity conditions* (i)-(iv) *are replaced by:*

(v) *Σ_1 is state strictly Ξ'-dissipative and Σ_2 is state strictly $(-\Xi \circ \Gamma)$-dissipative*

and the supposition of zero-state detectability of Σ_1 and Σ_2 is removed.

PROOF. Using time-invariance of Σ_1 and Σ_2 and summing the two dissipation inequalities yield that

$$\begin{aligned} &S_1(x_1(t_2), z_1(t_2)) + S_2(x_2(t_2), z_2(t_2)) \\ &\quad \leq S_1(x_1(t_1), z_1(t_1)) + S_2(x_2(t_1), z_2(t_1)) \\ &\quad - \int_{t_1}^{t_2} (\gamma_1(\|x_1(t)\|) + \gamma_2(\|x_2(t)\|))\, dt \end{aligned}$$

along the solutions to (17) and (18). Dividing both sides by $t_2 - t_1$ and taking the limit as $t_2 \to t_1$ yields

$$\frac{d}{dt} S(x, z) \leq -\gamma(\|x(t)\|) \leq 0 \quad (26)$$

along the solutions to (17) and (18), where $S(x, z) = S_1(x_1, z_1) + S_2(x_2, z_2)$ and $\gamma(\|x(t)\|) = \gamma_1(\|x_1(t)\|) + \gamma_2(\|x_2(t)\|)$.

For the case where Assumption 12 holds: By the same arguments as in the proof of Theorem 13, $x^* = 0$ is an equilibrium of (17) with $w_1 = 0$ and $w_2 = 0$. Moreover, observe that γ is a class $\mathcal{K}$ function because so are γ_1 and γ_2. It follows that $x^* = 0$ is an asymptotically stable equilibrium via [20, Th. 4.1(iv)] on noting (19) and (26).

Next, for the case when Assumption 14 holds and $(x^*, z^*) = (0, 0)$ is an equilibrium: Lyapunov stability of $(x^*, z^*) = (0, 0)$ follows from (26). Since the only solution x of $\Sigma_1 \| \Sigma_2$ that can stay identically in $\{x \in \mathcal{X} : \dot{S}(x, z) = 0\}$ is $x(t) = 0$ for all $t \geq 0$, asymptotic stability of $x^* = 0$ then holds by the LaSalle's invariance principle [27, Th. 4.4].

Finally, for the case when α and β are class $\mathcal{K}_\infty$ functions, global asymptotic stability of $x^* = 0$ can be concluded using similar arguments as above. □

3.5 *Exponential Stability*

While asymptotic stability guarantees convergence to the origin as time progresses, it gives no a priori rate of convergence. We may investigate the even stronger notion of exponential closed-loop stability via dissipativity. To this end, a stronger notion of dissipativity called exponential dissipativity is warranted. In light of Definition 2, the exponential Ξ-dissipativity of a system with decay rate $\lambda > 0$ can be naturally defined by changing the dissipation inequality in (5) to

$$e^{\lambda T} S(x(T), z(T)) \leq S(x(0), z(0)) + \int_0^T e^{\lambda t} \xi(t)\, dt, \quad (27)$$

where $\xi(t) = \Xi(u, y, \bar{x})(t)$. Exponential dissipativity with respect to static and dynamic supply rates with a quadratic form under $\bar{\mathcal{X}} = \emptyset$ has been established in [9]. We note that by requiring (27) for both Σ_1 and Σ_2, and under mild assumptions on the storage functions S_1 and S_2, one can easily establish exponential stability of $\Sigma_1 \| \Sigma_2$ via generic dynamic supply rates that need not have a quadratic form in a similar fashion to Theorem 20. We omit the details of such a result for brevity.

4 Specialisation of the Main Results

In this section, we specialise the main results — Theorems 13 and 20 — from the preceding section to obtain several corollaries pertinent to static and dynamic dissipativity results.

4.1 $(\Psi, \Pi, \Upsilon, \Omega)$-*dissipativity*

The celebrated $(Q, \hat{S}, R)$-dissipativity [22] has made a profound impact on the theory of dissipativity over the past half-century. It involves using a static matrix triplet $(Q, \hat{S}, R)$ in a static quadratic supply rate. The following specialisation of Theorem 20 generalises the static matrix triplet $(Q, \hat{S}, R)$ to a dynamic operator quadruplet $(\Psi, \Pi, \Upsilon, \Omega)$.

Let the operators Ψ_i, Π_i, Υ_i, Ω_i be causal and time-invariant for $i \in \{1, 2\}$ and let

$$\Theta_i = \begin{bmatrix} \Psi_i & \Pi_i \\ \Upsilon_i & \Omega_i \end{bmatrix}. \quad (28)$$

The causal operators Θ_i for $i \in \{1, 2\}$ do not need to be bounded on positive time support. Hence, they do not need to be stable operators. Furthermore, the operators Θ_i for $i \in \{1, 2\}$ can be any causal system (whether static or dynamic, linear or nonlinear) and do not need to have a state-space representation (e.g. may have pure derivatives). Define the dynamic supply rates for Σ_1 and Σ_2 by

$$\Xi_i(u_i, y_i, \bar{x}) = \left[\begin{smallmatrix} u_i \\ y_i \end{smallmatrix}\right]^\top \left(\Theta_i \left[\begin{smallmatrix} u_i \\ y_i \end{smallmatrix}\right]\right), \quad (29)$$

where Θ_i do not depend on $\bar{x}$. Such a special form of Ξ-dissipativity may be referred to as "$(\Psi, \Pi, \Upsilon, \Omega)$-dissipativity" and the following result can be derived.

Theorem 24 *For $i \in \{1, 2\}$, let Σ_i in (15) be zero-state detectable and Ξ_i-dissipative with Ξ_i given by (29) and causal time-invariant Θ_i given by (28) where Θ_i does not depend on $\bar{x}$. Let the corresponding storage functions S_i satisfy Assumption 12 (resp. Assumption 14 and suppose $(x^*, z^*) = (0, 0)$ is an equilibrium of (17) and (18) with $w_1 = 0$, $w_2 = 0$) and define $H = \left[\begin{smallmatrix} 0 & I_p \\ I_m & 0 \end{smallmatrix}\right]$. Suppose there exists $\tau > 0$ such that the causal operator*

$$\Theta = -(\Theta_1 + \tau H^\top \Theta_2 H) \quad (30)$$

is input strictly passive in the input-output sense [2] *. Then $x^* = (x_1^*, x_2^*) = 0$ is an asymptotically stable equilibrium of the closed-loop system $\Sigma_1 \| \Sigma_2$ with $w_1 = 0$, $w_2 = 0$. Moreover, if Assumption 12 (resp. Assumption 14) holds for all $x \in \mathcal{X}$ and $z \in \mathcal{Z}$ for class $\mathcal{K}_\infty$ functions α and β, $x^* = 0$ is a globally asymptotically stable equilibrium.*

PROOF. Since $\Theta_2 = \frac{1}{\tau} H(-\Theta - \Theta_1) H^\top$, we have that Σ_1 and Σ_2 satisfy the following dissipation inequalities:

$$\begin{aligned}
&S_1(x_1(T), z_1(T)) - S_1(x_1(0), z_1(0)) \\
&\qquad \leq \int_0^T \begin{bmatrix} u_1(t) \\ y_1(t) \end{bmatrix}^\top \left(\Theta_1 \left[\begin{smallmatrix} u_1 \\ y_1 \end{smallmatrix}\right]\right)(t)\, dt \\
&\tau S_2(x_2(T), z_2(T)) - \tau S_2(x_2(0), z_2(0)) \\
&\leq -\int_0^T \begin{bmatrix} y_2(t) \\ u_2(t) \end{bmatrix}^\top \left(\Theta \left[\begin{smallmatrix} y_2 \\ u_2 \end{smallmatrix}\right]\right)(t)\, dt - \int_0^T \begin{bmatrix} y_2(t) \\ u_2(t) \end{bmatrix}^\top \left(\Theta_1 \left[\begin{smallmatrix} y_2 \\ u_2 \end{smallmatrix}\right]\right)(t)\, dt \\
&\leq -\delta \left\| \left[\begin{smallmatrix} y_2 \\ u_2 \end{smallmatrix}\right] \right\|_T^2 - \int_0^T \begin{bmatrix} y_2(t) \\ u_2(t) \end{bmatrix}^\top \left(\Theta_1 \left[\begin{smallmatrix} y_2 \\ u_2 \end{smallmatrix}\right]\right)(t)\, dt
\end{aligned}$$

for some $\delta > 0$, where the last inequality follows from input strict passivity of the operator Θ. Now let $\hat{S}_2 = \tau S_2$. Since $\Xi_1(u_1, y_1, \bar{x}) = \left[\begin{smallmatrix} u_1 \\ y_1 \end{smallmatrix}\right]^\top \left(\Theta_1 \left[\begin{smallmatrix} u_1 \\ y_1 \end{smallmatrix}\right]\right)$ and $(-\Xi_1 \circ \Gamma)(u_2, y_2, \bar{x}) = -\left[\begin{smallmatrix} y_2 \\ u_2 \end{smallmatrix}\right]^\top \left(\Theta_1 \left[\begin{smallmatrix} y_2 \\ u_2 \end{smallmatrix}\right]\right)$ and both do not depend on $\bar{x}$, it can be seen that condition (i) in Theorem 20 holds and the claim follows from the same theorem. □

If the causal dynamic operators Ψ_i, Π_i, Υ_i, Ω_i in (28) are LTI with frequency-domain representations $\Psi_i(\omega)$, $\Pi_i(\omega)$, $\Upsilon_i(\omega)$, $\Omega_i(\omega)$ respectively, then Ξ_i-dissipativity in Theorem 24 captures the $(Q(\omega), S(\omega), R(\omega))$-dissipativity notions in [18, 33, 37] and also relates to the theory of integral quadratic constraints (IQCs) [35].

[2] A causal operator $\Theta : \mathbf{L}_{2e}^{m+p} \to \mathbf{L}_{2e}^{m+p}$ is said to be *input strictly passive in the input-output sense* if there exists $\delta > 0$ such that $\langle f, \Theta f \rangle_T \geq \delta \|f\|_T^2$ for all $f \in \mathbf{L}_{2e}^{m+p}$ and $T > 0$.

Such a connection is discussed in more detail in Section 5.

While Σ_1 and Σ_2 may be dissipative with respect to different supply rates in Theorem 24, the supply rates are related via the *coupling term* (30) that requires Θ defined therein to be input strictly passive. Furthermore, if the causal operators Ψ_i, Π_i, Υ_i, Ω_i in (28) are LTI with corresponding state-space realisations, the input strict passivity requirement on Θ can be easily tested numerically via LMIs; see, for example, [6, Sec. 3.1].

Several existing static dissipativity results in the literature may be established using Theorem 24 with empty auxiliary systems, wherein $\mathscr{Z}_i = \emptyset$. First, the small-gain theorem in [50, Th. 8.2.1] assumes that Σ_i is Ξ_i-dissipative with static $\Theta_i = \left[\begin{smallmatrix} r_i^2 I & 0 \\ 0 & -I \end{smallmatrix}\right]$ in (29), $r_i > 0$ and $r_1 r_2 < 1$. Choosing $\tau \in (r_1^2, 1/r_2^2)$ in Theorem 24 ensures that coupling term (30) is input strictly passive. The small-gain theorem in [50, Th. 8.2.1] is thus a specialisation of Theorem 24.

Second, the passivity theorem in [50, Prop. 4.3.1(iv)], which is applicable to a negative feedback configuration, assumes that Σ_1 and $-\Sigma_2$ are output strictly passive, i.e. Σ_1 is Ξ_1-dissipative and $-\Sigma_2$ is Ξ_2-dissipative with static $\Theta_i = \left[\begin{smallmatrix} 0 & 1/2I \\ 1/2I & -\epsilon_i I \end{smallmatrix}\right]$ in (29) and $\epsilon_i > 0$. Choosing $\tau = 1$ in Theorem 24 ensures that coupling term (30) is input strictly passive after absorbing the negative sign of $-\Sigma_2$ into y_2. Therefore, the passivity theorem in [50, Prop. 4.3.1(iv)] is also a specialisation of Theorem 24.

Third, one can also derive the following immediate result on passivity indices. Let Σ_1 and $-\Sigma_2$ be input-feedforward and output-feedback passive; i.e. Σ_1 is Ξ_1-dissipative and $-\Sigma_2$ is Ξ_2-dissipative with static $\Theta_i = \left[\begin{smallmatrix} -\delta_i I & 1/2I \\ 1/2I & -\epsilon_i I \end{smallmatrix}\right]$ in (29). If $\delta_1 + \epsilon_2 > 0$ and $\delta_2 + \epsilon_1 > 0$, then choosing $\tau = 1$ ensures that (30) is input strictly passive after absorbing the negative sign of $-\Sigma_2$ into y_2 and asymptotic stability of $\Sigma_1 \| (-\Sigma_2)$ can be established via Theorem 24. Such a result reminisces the finite-gain input-output closed-loop stability result based on passivity indices in [54, Thm. 6.6.58].

4.2 Dissipation with Terminal Costs

Specific types of dynamic supply rates have appeared in the study of finite-gain input-output stability of feedback systems via dissipation inequalities [43, 44] as a means to recover the standard theory of IQCs [35]. In [43, 44], feedback interconnections of a nonlinear system and an LTI system are considered and canonical factorisations of the multipliers are crucial. Similar dynamic supply rates can also be located in [2, 42], where asymptotic stability of feedback systems is examined.

It is demonstrated in [42, Th. 30] that the dynamic dissipativity with terminal costs result [42, Th. 13] can be used to recover the renowned IQC-based input-output stability result [35, Th. 1] for a feedback interconnection of a stable nonlinearity and a stable LTI system. Next, we demonstrate that [42, Th. 13] on dissipativity with terminal costs, restated in Corollary 25 below for convenience, is a specialisation of Theorem 20.

Corollary 25 *Let Σ_2 be an LTI system with minimal realisation (A, B, C, D). Let an auxiliary system Φ be LTI with realisation $\left(A_\Phi, \begin{bmatrix} B_{\Phi_1} & B_{\Phi_2} \end{bmatrix}, C_\Phi, \begin{bmatrix} D_{\Phi_1} & D_{\Phi_2} \end{bmatrix}\right)$ and state variable z with $z(0) = 0$. Given $P = P^\top$, suppose there exist $X = X^\top$, $Z = Z^\top$, and $\epsilon > 0$ such that Σ_1 satisfies*

$$z(T)^\top Z z(T) \le \int_0^T \left(\Phi \left[\begin{smallmatrix} u_1 \\ y_1 \end{smallmatrix}\right]\right)(t)^\top P \left(\Phi \left[\begin{smallmatrix} u_1 \\ y_1 \end{smallmatrix}\right]\right)(t)\, dt$$

for all $T > 0$, $u_1 \in \mathscr{U}_1$, and y_1 being a solution to (15), and Σ_2 satisfies

$$\begin{aligned} \left[\begin{smallmatrix} z(T) \\ x_2(T) \end{smallmatrix}\right]^\top X \left[\begin{smallmatrix} z(T) \\ x_2(T) \end{smallmatrix}\right] \le{} & \left[\begin{smallmatrix} 0 \\ x_2(0) \end{smallmatrix}\right]^\top X \left[\begin{smallmatrix} 0 \\ x_2(0) \end{smallmatrix}\right] \\ & - \int_0^T \left(\Phi \left[\begin{smallmatrix} y_2 \\ u_2 \end{smallmatrix}\right]\right)(t)^\top P \left(\Phi \left[\begin{smallmatrix} y_2 \\ u_2 \end{smallmatrix}\right]\right)(t)\, dt \\ & - \epsilon \int_0^T \left(\|u_2(t)\|^2 + \|y_2(t)\|^2\right) dt \end{aligned}$$

for all $T > 0$, $u_2 \in \mathscr{U}_2$, $x_2(0) \in \mathscr{X}_2$, and x_2 and y_2 being a solution to (15). If $X + \operatorname{diag}(Z, 0) > 0$, then $x_2^* = 0$ is an asymptotically stable equilibrium of the closed-loop system $\Sigma_1 \| \Sigma_2$ with $w_1 = 0$ and $w_2 = 0$.

PROOF. Note that Σ_2 is zero-state detectable because (A, B, C, D) is a minimal realisation. Let the supply rate

$$\Xi(u, y, \bar{x})(t) = \left(\Phi \left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right]\right)(t)^\top P \left(\Phi \left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right]\right)(t),$$

where Ξ does not depend on $\bar{x}$ and the storage functions

$$S_1(z) = z^\top Z z \text{ and } S_2(x_2, z) = \left[\begin{smallmatrix} z \\ x_2 \end{smallmatrix}\right]^\top X \left[\begin{smallmatrix} z \\ x_2 \end{smallmatrix}\right].$$

Here, Σ_1 and Σ_2 "share" the same auxiliary system Φ and state z, and the operator Ξ is constructed from Φ. Since stability of $x_1^* = 0$ is of no concern, together with $X + \operatorname{diag}(Z, 0) > 0$, one can easily verify that Assumption 14, condition (i) in Theorem 20 and Remark 21 hold, from which the result holds. □

4.3 Dissipation for Affine-Nonlinear Systems

We show next that the dynamic dissipativity setting in [9], where control-affine nonlinear systems are considered, may be recovered from Theorems 13 and 20. To this end, consider $\Sigma_1 \| \Sigma_2$, where Σ_i is of the affine form:

$$\Sigma_i : \begin{array}{l} \dot{x}_i(t) = F_i(x_i(t)) + G_i(x_i(t))u_i(t), \\ y_i(t) = H_i(x_i(t)) + J_i(x_i(t))u_i(t) \end{array}$$

and $F_i(\cdot)$, $G_i(\cdot)$, $H_i(\cdot)$ and $J_i(\cdot)$ map $\mathcal{X}_i$ to real matrices and vectors with compatible dimensions. Next, associate Σ_1 and Σ_2 with an auxiliary system $\Phi = \left[\begin{smallmatrix} u_\Phi \\ y_\Phi \end{smallmatrix}\right] \mapsto \phi$ represented by the following affine form:

$$\begin{aligned} \dot{z}(t) &= F_\Phi(z(t)) + G_\Phi(z(t))u_\Phi(t) + I_\Phi(z(t))y_\Phi(t) \\ \phi(t) &= H_\Phi(z(t)) + J_\Phi(z(t))u_\Phi(t) + K_\Phi(z(t))y_\Phi(t) \end{aligned} \tag{31}$$

with $z(0) = 0$, and $F_\Phi(\cdot), G_\Phi(\cdot), I_\Phi(\cdot), H_\Phi(\cdot), J_\Phi(\cdot)$ and $K_\Phi(\cdot)$ map $\mathcal{Z}$ to real matrices and vectors with compatible dimensions. In [9], the same Φ is adopted for both Σ_1 and Σ_2, with inputs $\left[\begin{smallmatrix} u_\Phi \\ y_\Phi \end{smallmatrix}\right] = \left[\begin{smallmatrix} u_1 \\ y_1 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} u_\Phi \\ y_\Phi \end{smallmatrix}\right] = \left[\begin{smallmatrix} y_2 \\ u_2 \end{smallmatrix}\right]$, respectively. Define the following two operators:

$$\begin{aligned} \Xi_1(u_1, y_1, \bar{x})(t) &= \left(\Phi \left[\begin{smallmatrix} u_1 \\ y_1 \end{smallmatrix}\right]\right)(t)^\top P_1 \left(\Phi \left[\begin{smallmatrix} u_1 \\ y_1 \end{smallmatrix}\right]\right)(t), \\ \Xi_2(u_2, y_2, \bar{x})(t) &= \left(\Phi \left[\begin{smallmatrix} y_2 \\ u_2 \end{smallmatrix}\right]\right)(t)^\top P_2 \left(\Phi \left[\begin{smallmatrix} y_2 \\ u_2 \end{smallmatrix}\right]\right)(t), \end{aligned} \tag{32}$$

where $P_1 = P_1^\top$ and $P_2 = P_2^\top$ are static matrices and both Ξ_1 and Ξ_2 are independent of $\bar{x}$. The following corollary, which first appeared in [9, Th. 3.2], is a specialisation of Theorem 13.

Corollary 26 *Suppose there exist a common auxiliary system Φ in (31), $P_1 = P_1^\top$ and $P_2 = P_2^\top$ such that Σ_1 is Ξ_1-dissipative and Σ_2 is Ξ_2-dissipative with quadratic dynamic supply rates of the form (32). Suppose further that the corresponding storage functions S_1 and S_2 satisfy Assumption 12. If there exists $\tau > 0$ such that $P_1 + \tau P_2 \le 0$, then $x^* = (x_1^*, x_2^*) = 0$ is a Lyapunov stable equilibrium of $\Sigma_1 \| \Sigma_2$ with $w_1 = 0$ and $w_2 = 0$.*

PROOF. Let $P = P_1 + \tau P_2$ and note that $P_2 = (P - P_1)/\tau$. By hypothesis, Σ_1 and Σ_2 satisfy the following dissipation inequalities

$$\begin{aligned} S_1(x_1(T), z_1(T)) &\le S_1(x_1(0), 0) + \int_0^T \Xi_1(u_1, y_1, \bar{x})(t)\, dt, \\ \tau S_2(x_2(T), z_2(T)) &\le \tau S_2(x_2(0), 0) \\ &\quad - \int_0^T \left(\Phi \left[\begin{smallmatrix} y_2 \\ u_2 \end{smallmatrix}\right]\right)(t)^\top (P_1 - P) \left(\Phi \left[\begin{smallmatrix} y_2 \\ u_2 \end{smallmatrix}\right]\right)(t)\, dt \\ &\le \tau S_2(x_2(0), 0) - \int_0^T \Xi_1 \circ \Gamma(u_2, y_2, \bar{x})(t)\, dt, \end{aligned}$$

where the last inequality follows from the fact $P \le 0$. By defining the supply rate $\Xi(u, y, \bar{x})(t) = \Xi_1(u, y, \bar{x})(t)$, we conclude that $x^* = 0$ is a Lyapunov stable equilibrium with $w_1 = 0$, $w_2 = 0$ via an invocation of Theorem 13.□

The asymptotic stability version of Corollary 26 can be similarly established by specialising Theorem 20. It is omitted here for conciseness.

The notion of dynamic dissipativity introduced in this paper is more general than that in [9] as explained next. First, the supply rate (32) considered in [9] is constructed directly from the auxiliary system Φ, while we treat the supply rate Ξ and auxiliary system Φ as two independent objects, cf. Fig. 1. Second, the same auxiliary system Φ in (31) is adopted for both Σ_1 and Σ_2 for feedback stability analysis in [9], while auxiliary systems Φ_i in (18) can be different for Σ_1 and Σ_2 in Theorems 13 and 20. Third, dissipativity of the quadratic form in (32) is considered in [9] and defined via control-affine auxiliary systems in (31), whereas in this paper, dissipativity of the general form in (5) via general auxiliary systems in (4) are considered for general nonlinear systems in (1).

5 Relations with Integral Quadratic Constraint Theory

This section elaborates the relation between the dissipativity results for robust feedback Lyapunov-type stability in Section 3 and IQC results for robust feedback finite-gain input-output stability [28, 35].

We first define some notation required for finite-gain input-output feedback stability. Denote by $\mathbf{L}_2^n$ the set of $\mathbb{R}^n$-valued Lebesgue square-integrable functions: $\mathbf{L}_2^n = \{v : \mathbb{R} \to \mathbb{R}^n \mid \|v\|_{\mathbf{L}_2}^2 = \int_{-\infty}^{\infty} \|v(t)\|^2\, dt < \infty\}$. Let $\mathbf{L}_{2+}^n = \{v \in \mathbf{L}_2^n : v(t) = 0 \text{ for all } t < 0\}$. Recall the truncation operator P_T and define the extended space $\mathbf{L}_{2e}^n = \{v : \mathbb{R} \to \mathbb{R}^n \mid P_T v \in \mathbf{L}_{2+}^n\ \forall T \in [0, \infty)\}$. Given $v, w \in \mathbf{L}_{2e}^n$, let $\langle v, w \rangle_T = \int_0^T v(t)^\top w(t)\, dt$ and $\|v\|_T^2 = \langle v, v \rangle_T$. An operator $\Sigma : \mathbf{L}_{2e}^n \to \mathbf{L}_{2e}^n$ is said to be *incrementally $\mathbf{L}_{2e}$-bounded* if $\sup_{\substack{T>0; x,y \in \mathbf{L}_{2e}^n; \\ P_T x \ne P_T y}} \frac{\|P_T(\Sigma x - \Sigma y)\|_{\mathbf{L}_2}}{\|P_T(x-y)\|_{\mathbf{L}_2}} < \infty$. Note that an incrementally $\mathbf{L}_{2e}$-bounded Σ is necessarily causal [50, Prop. 2.1.6]. A causal Σ is called *bounded* if its bound is finite [55, Sec. 2.4], i.e. $\|\Sigma\| = \sup_{\substack{T>0; u \in \mathbf{L}_{2e}^n; \\ \|u\|_T \ne 0}} \frac{\|\Sigma u\|_T}{\|u\|_T} = \sup_{0 \ne u \in \mathbf{L}_{2+}^n} \frac{\|\Sigma u\|_{\mathbf{L}_2}}{\|u\|_{\mathbf{L}_2}} < \infty$.

The following definitions of well-posedness and finite-gain input-output stability for feedback system $\Sigma_1 \| \Sigma_2$ are standard.

Definition 27 *$\Sigma_1 \| \Sigma_2$ is said to be* well-posed *if the map $(u_1, u_2) \mapsto (w_1, w_2)$ defined by (16) has a causal inverse on $\mathbf{L}_{2e}^{m+p}$. $\Sigma_1 \| \Sigma_2$ is said to be finite-gain* $\mathbf{L}_{2+}$-stable *if it is well-posed and the map $\left[\begin{smallmatrix} w_1 \\ w_2 \end{smallmatrix}\right] \in \mathbf{L}_{2e}^{m+p} \mapsto \left[\begin{smallmatrix} u_1 \\ u_2 \end{smallmatrix}\right] \in \mathbf{L}_{2e}^{m+p}$ is bounded, i.e. there exists $C > 0$ such that $\int_0^T \|u_1(t)\|^2 + \|u_2(t)\|^2\, dt \le C \int_0^T \|w_1(t)\|^2 + \|w_2(t)\|^2\, dt$ for all $w_1 \in \mathbf{L}_{2e}^m$, $w_2 \in \mathbf{L}_{2e}^p$ and $T > 0$.*

The above notions of feedback well-posedness and finite-gain input-output stability are well studied; see [12, 16, 55]. Define the extended graph of Σ_1 as $\mathcal{G}_\epsilon(\Sigma_1) = \left\{\left[\begin{smallmatrix} u_1 \\ y_1 \end{smallmatrix}\right] \in \mathbf{L}_{2e}^{m+p} : y_1 = \Sigma_1 u_1\right\}$. Likewise, define the extended inverse graph of Σ_2 as $\mathcal{G}'_\epsilon(\Sigma_2) = \left\{\left[\begin{smallmatrix} y_2 \\ u_2 \end{smallmatrix}\right] \in \mathbf{L}_{2e}^{m+p} : y_2 = \Sigma_2 u_2\right\}$.

We restate below an IQC result from [28] for comparison purposes.

Theorem 28 ([28, Th. III.1]) *Given causal systems* $\Sigma_1 : \mathbf{L}_{2e}^m \to \mathbf{L}_{2e}^p$ *and* $\Sigma_2 : \mathbf{L}_{2e}^p \to \mathbf{L}_{2e}^m$ *satisfying* $\Sigma_i 0 = 0$, $i \in \{1, 2\}$*, suppose* $\Sigma_1 \| \Sigma_2$ *is well-posed and there exist incrementally bounded multipliers* $\Psi : \mathbf{L}_{2e}^{m+p} \to \mathbf{L}_{2e}^q$ *and* $\Pi : \mathbf{L}_{2e}^{m+p} \to \mathbf{L}_{2e}^q$ *such that*

$$\begin{aligned} &\langle \Psi v_1, \Pi v_1 \rangle_T \geq 0 \; \forall v_1 \in \mathcal{G}_\epsilon(\Sigma_1), T > 0 \\ \textit{and} \quad &\langle \Psi v_2, \Pi v_2 \rangle_T \leq -\epsilon \|v_2\|_T^2 \; \forall v_2 \in \mathcal{G}'_\epsilon(\Sigma_2), T > 0. \end{aligned} \tag{33}$$

Then $\Sigma_1 \| \Sigma_2$ *is finite-gain* $\mathbf{L}_{2+}$*-stable.*

Observe that by taking a quadratic supply rate of the form in (3), Theorems 20 and 28 are similar on important grounds and differ in a few significant aspects. In terms of similarities, Theorem 28 relies on quadratic graph separation enforced by (33). Such a separation is captured by conditions (i)-(iv) in Theorem 20 with the aid of storage functions that possess properties listed therein. On the other hand, some important differences include:

- (i) The IQC based Theorem 28 makes use of a *quadratic* form involving incrementally bounded multipliers, while the supply rate in the dissipativity based Theorem 20 may accommodate more general forms;
- (ii) Theorem 28 is an input-output feedback stability result, whereas Theorem 20 is a Lyapunov-type stability result on the state of the closed-loop system;
- (iii) Theorem 20 requires the existence of storage functions that satisfy several properties. Therefore, in practice, the conditions in Theorem 28 may be easier to verify. In particular, the use of dynamic multipliers is both natural and well known in the theory of IQCs as a means to reduce conservatism [28, 35]. By contrast, introducing dynamics into the supply rate in a dissipation inequality often complicates the search for a suitable storage function. Fortunately, the distinct auxiliary systems aid with the satisfaction of the dissipation inequalities.

  Theorem 28 is a hard (a.k.a. unconditional) IQC theorem, where the integrals are taken from 0 to T for all $T > 0$ for signals in extended spaces [34]. This has been shown in [28] to be recoverable by a more powerful soft (a.k.a. conditional) IQC theorem [28, Th. IV.2], where integrals are taken from 0 to ∞ for square-integrable signals, when equipped with homotopies that are continuous in a gap distance measure [16]. There, the use of noncausal multipliers is readily accommodated.

Despite the aforementioned dissimilarities, Corollary 25, or its sister result [42, Th. 13], as a specialised form of Theorem 20, has been shown in [42, Th. 30] to recover a limited version of the soft IQC theorem for a feedback interconnection of a stable nonlinearity and a stable LTI system [35, Th. 1]. The proof relies on the fact that dissipativity of the stable LTI component with a quadratic storage function is equivalent to one that involves the additional exogenous signals w_1 and w_2 in Fig. 2; see [42, Lem. 17] and [44, Lem. 1]. Such a result is not known to hold for nonlinear systems, and hence in the nonlinear setting, dissipativity and IQC approaches to feedback stability analysis remain distinct.

It is worth noting that certain types of dissipation inequalities may be used to show input-output (finite-gain) stability as mentioned above; see, e.g., [44] and [43]. Since such results typically involve some LTI dynamics and input-output stability is not the main focus of this paper, we do not discuss them here in detail. It is known that under certain conditions, global exponential stability implies input-output finite-gain stability [54, Sec. 6.3] [20, Sec. 7.6]. On the other hand, for Lur'e feedback systems involving a static nonlinearity, it has been shown that under certain Lipschitz continuity (resp. boundedness) condition, input-output finite gain feedback stability implies global attractivity [54, Sec. 6.3] (resp. exponential stability [35, Prop. 1]) of the origin.

6 A Numerical Example

This section provides a numerical example to demonstrate the stability result established in Theorem 20.

Consider a feedback system $\Sigma_1 \| \Sigma_2$ in Fig. 2, where $\Sigma_1 = u_1 \mapsto y_1$ is provided in (9) with $x(0) = \left[\begin{smallmatrix} x_1(0) \\ x_2(0) \end{smallmatrix}\right] \in \mathcal{X} = \mathbb{R}^2$ and $\Sigma_2 = u_2 \mapsto y_2$ is described by

$$\Sigma_2 : \begin{aligned} \dot{x}_3(t) &= -5x_3(t) - \psi_2(x_3(t)) + u_2(t) \\ y_2(t) &= x_3(t) - 0.2u_2(t) \end{aligned}$$

with $x_3(0) \in \mathbb{R}$, where $\psi_2 : \mathbb{R} \to \mathbb{R}$ is locally integrable and satisfies $\psi_2(0) = 0$ and $\psi_2(r)r \geq 0$ for all $r \in \mathbb{R}$. We have shown in Example 5 and Remark 19 that Σ_1 is output strictly Ξ'-dissipative with respect to the supply rate $\Xi(u_1, y_1, x(0))$ provided in (10). Next, we attempt to validate a complementary dissipation inequality for Σ_2. Consider the candidate storage function $S_2(x_3, z_2) = \frac{1}{2}x_3^2 + \frac{1}{2}z_2^2$. The complementary supply rate $(-\Xi \circ \Gamma) = \left[\begin{smallmatrix} u_2 \\ y_2 \\ \bar{x} \end{smallmatrix}\right] \to \xi_2$ is given by

$$-\Xi \circ \Gamma : \begin{aligned} \dot{z}_2(t) &= -z_2(t) + y_2(t), \quad z_2(0) = [\,0 \;\; 1\,]\,\bar{x} \\ \xi_2(t) &= -y_2(t)(3z_2(t) + u_2(t)) \end{aligned}$$

with $\bar{x} \in \mathbb{R}^2$. Observe that for all $x_3(0) \in \mathbb{R}$ and $\bar{x} \in \mathbb{R}^2$,

$$\begin{aligned}
&\frac{d}{dt}S_2(x_3, z_2) + y_2^2 - \xi_2 = x_3\dot{x}_3 + z_2\dot{z}_2 + y_2^2 - \xi_2 \\
=& -5x_3^2 - x_3\psi_2(x_3) + u_2x_3 - z_2^2 + z_2y_2 + y_2^2 + y_2(3z_2 + u_2) \\
\leq& -5x_3^2 + u_2(y_2 + 0.2u_2) - z_2^2 + y_2^2 + 4z_2y_2 + u_2y_2 \\
=& -z_2^2 + 4z_2y_2 + y_2^2 + 0.2u_2^2 + 2u_2y_2 - 5(y_2 + 0.2u_2)^2 \\
=& -z_2^2 + 4z_2y_2 - 4y_2^2 = -(z_2 - 2y_2)^2 \leq 0
\end{aligned}$$

along the solutions to Σ_2 and $-\Xi \circ \Gamma$. This indicates that Σ_2 is output strictly $(-\Xi \circ \Gamma)$-dissipative by Lemma 3. Applying Theorem 20(iv), it can then be concluded that the equilibrium $(x_1^*, x_2^*, x_3^*) = (0, 0, 0)$ of $\Sigma_1 \| \Sigma_2$ is globally asymptotically stable.

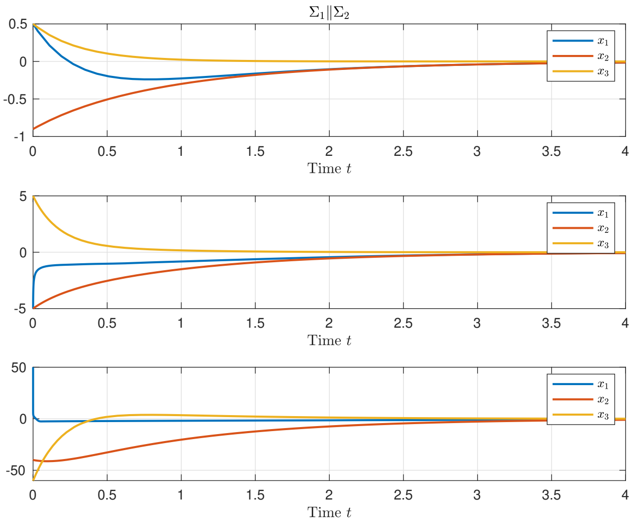

Fig. 3. Simulation results of the state trajectories of $\Sigma_1 \| \Sigma_2$.

As a simulation example, for Σ_1, let $a = 1$, $N = 0$, $M = 2$, $b_k = 1$ for all k and suppose ψ is the saturation function described by $\psi(r) = \min(\max(r, -5), 5)$. For Σ_2, suppose $\psi_2(r) = \min(\max(r, -8), 8)$. The state trajectories of $\Sigma_1 \| \Sigma_2$ are illustrated by Fig. 3 under three different sets of initial conditions.

7 Conclusion

In this paper, a general notion of dissipativity with dynamic supply rates was introduced for nonlinear systems, extending the notion of classical dissipativity. Lyapunov and asymptotic stability analyses were performed for feedback interconnections of two dissipative systems satisfying dissipativity with respect to dynamic supply rates. In these results, both dynamical systems are characterised by compatible dissipation inequalities with respect to "coupled" dynamic supply rates. Satisfaction of the dissipation inequalities is aided by the dynamics of possibly distinct auxiliary systems. The results were shown to recover several knowns results in the literature.

A noteworthy specialisation of the results is a simple coupling test to verify whether the feedback interconnection of two nonlinear systems, each satisfying independent $(\Psi, \Pi, \Upsilon, \Omega)$-dissipation inequalities, is asymptotically stable. This coupling test is simple to compute if the supply rate operators are chosen to be LTI. Moreover, a meaningful comparison with the integral quadratic constraint based input-output approach to feedback stability was made.

Future research directions include exploring physically illustrating examples for specific dynamic supply rates, such as those manifesting negative imaginary dynamics, and developing dissipativity with dynamic supply rates for more general systems, for example those taking hybrid forms and large-scale interconnected networks.

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