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# Dynamic dissipative characterisation of time-domain input-output negative imaginary systems ⋆

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**Abstract**

This paper introduces the class of Time-Domain Input-Output Negative Imaginary (TD-IONI) systems. The new TD-IONI definition unifies the class of the existing Negative Imaginary (NI) systems, including those that have imaginary-axis poles. A new dynamic dissipative framework is proposed to define and characterise the TD-IONI systems. This framework does not impose any *a priori* conditions on the system, such as asymptotic stability, minimality, full normal rank constraint, etc., which are commonly used in the NI literature. Dynamic dissipativity of TD-IONI systems also leads to an LMI-based state-space characterisation, which can be conveniently used to classify the strict/non-strict TD-IONI properties of a given system. This paper also reveals the connections amongst the NI theory, dynamic dissipativity and classical dissipativity. Subsequently, a frequency-domain dissipative supply rate is also proposed to describe the whole class of TD-IONI systems, which is defined with respect to a shifted imaginary axis to capture particularly the systems having poles on the imaginary axis. This trick overcomes the limitation of earlier frequency-domain dissipative frameworks to capture systems with imaginary-axis poles. Finally, the derived results are specialised for the Time-Domain Output (Strictly) Negative Imaginary subclass since such systems exhibit useful closed-loop stability properties when connected in a positive feedback loop. Several illustrative numerical examples are provided to make the results intuitive and useful.

*Key words:* Input-output negative imaginary systems; output strictly negative imaginary systems; dynamic dissipativity; shifted $(Q_a(\omega), S_a(\omega), R_a(\omega))$-dissipativity; quadratic supply rate; storage function; Lyapunov stability.

## 1 Introduction

Negative Imaginary (NI) systems theory has drawn attention from control theorists and practising engineers due to its potential in solving a variety of real-world engineering problems such as vibration control of lightly-damped mechanical systems [26, 41], cantilever beams [2], large space structures [30], robotic manipulators [30]; nano-positioning applications [31, 33]; train platooning [28]; control of various multi-agent systems [38, 37, 36, 1], etc. NI theory has become appealing due to its simple robust stability condition that depends on the loop gain only at the zero frequency. In a SISO setting, an NI transfer function's imaginary part remains non-positive for all $\omega \geq 0$. Among the strict subclasses within the NI class, Strictly NI (SNI [26]), Strongly Strict NI (SSNI [27]) and Output Strictly NI (OSNI [6, 4, 5, 24, 3]) appear quite often in the literature. SNI systems are defined by the property $\Im\{M(j\omega)\} < 0 \ \forall \omega \in (0, \infty)$ in the SISO setting. SSNI systems form a particular subset within the SNI class that satisfies two additional frequency-domain conditions in the neighbourhood of $\omega = 0$ and $\omega = \infty$. In contrast, a transfer function $M(s)$ is said to have OSNI property if the transformed system $F(s) = s[M(s) - M(\infty)]$ is Output Strictly Passive.

NI literature has witnessed persistent progress both in theory [26, 25, 40, 29, 12, 13, 14] and applications [31, 10, 11, 33] over the past fourteen years since its inception. However, the connections between NI systems theory and classical dissipativity have not yet been fully explored. For passive systems, a complete characterisation

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already exists in the literature, which was built on the Willems's dissipative framework [39] and Hill-Moylan's $(Q,S,R)$-dissipative framework [18, 19]. In the NI literature, [35] took the first step to define a frequency-domain dissipative supply rate for characterising the systems with 'mixed' NI and finite-gain properties, taking inspiration from a similar framework used in [17] for defining 'mixed' passive and finite-gain systems. The ideas of [35] were later expanded in [9, 10, 11] to describe the systems with 'mixed' NI, passive and finite-gain properties. However, [35, 9, 10, 11] did not address the following crucial issues: (i) how to define an appropriate and unifying supply rate that will capture the full class of NI systems; (ii) how to theoretically establish the dissipative property of an NI system; (iii) how to define a compatible time-domain dissipative supply rate; and (iv) how to establish the equivalence between the conventional frequency-domain definitions of NI systems and the proposed dissipative characterisation. Recently, [4] has proposed a new frequency-domain dissipative framework for characterising the class of stable Output Negative Imaginary [1] (ONI) systems, including OSNI systems (introduced in [6]) as a strict subset. Of late, [5, 24] has first shed some light on defining a time-domain dissipative supply rate to capture ONI systems allowing poles on the imaginary axis. Note that time-domain dissipativity is more powerful than frequency-domain dissipativity since the latter can handle only stable, LTI systems. In contrast, the former applies to even marginally-stable and nonlinear cases.

Drawn by the above facts and limitations, this paper defines the notion of TD-IONI systems utilising a time-domain dissipative supply rate $w(u,\bar{u},\dot{\bar{y}})$ that involves the input to the system $(u)$, a filtered version of the input $(\bar{u})$ and the time-derivative of an auxiliary output of the system $(\dot{\bar{y}})$. The auxiliary output $\bar{y} = y - Du$ has been used instead of the physical output $y$ to capture particularly bi-proper cases. The proposed approach differs from the classical dissipative framework [39] in the sense that it contains a time-derivative term $\dot{\bar{y}}$ and an additional input $\bar{u}$, whereas in the classical approach, a supply rate consists of only $u$ and $y$. The IONI class encompasses the existing ONI/OSNI systems [6, 4, 5] and the entire class of NI systems (i.e. allowing poles on the imaginary axis) [30, 25]. A dynamic dissipative framework, termed as $(\Sigma, \bar{Q})$-dissipativity [2] where $\Sigma$ is a specific LTI operator and $\bar{Q} = \bar{Q}^\top$ is a real matrix of appropriate dimension, is developed for characterising the TD-IONI systems (discussed in Section 3). The dynamic dissipative framework also gives rise to a necessary and sufficient LMI condition to check the strict/non-strict TD-IONI properties of a given system. Unlike [35, 9, 10] and [11], this paper theoretically establishes that the TD-IONI systems are dissipative with respect to a particular supply rate $w(u,\bar{u},\dot{\bar{y}})$ by proving the existence of a positive semidefinite storage function (refer to Section 4). These findings reveal the missing links amongst the TD-IONI (and NI as well) systems theory, dynamic dissipativity and the classical dissipativity (in the sense of Willems [39]). Interestingly, according to the new definition, the TD-OSNI and TD-ISNI systems may contain a simple pole at the origin [e.g. $\frac{1}{s}$, $\frac{s+4}{s(s+2)}$], in contrast to the earlier notions wherein OSNI and ISNI properties can be defined only for stable systems [35, 9, 10, 11, 4].

This paper also introduces a frequency-domain dissipative framework (refer to Section 5) to characterise the TD-IONI systems allowing poles on the $j\omega$-axis. Note here that so far in the NI literature [35, 9, 11, 4], the notion of frequency-domain dissipativity [i.e. $(Q(\omega), S(\omega), R(\omega))$-dissipativity] has been used to describe only stable systems since this supply rate relies on the Fourier transformation and hence, cannot handle any transfer function or signal not bounded on the $j\omega$-axis. To overcome this limitation, we will introduce the notion of a "shifted $(Q_a(\omega), S_a(\omega), R_a(\omega))$-dissipativity" by exploiting the idea of the Fourier transform with respect to a shifted $j\omega$-axis [i.e. the Fourier integral is now evaluated on the $(a+j\omega)$-axis for a specific $a>0$]. The "shifted $(Q_a(\omega), S_a(\omega), R_a(\omega))$-dissipative" supply rate complies with the proposed time-domain supply rate $w(u,\bar{u},\dot{\bar{y}})$. Moreover, when restricted to stable systems, the proposed frequency-domain dissipative characterisation resembles the existing frequency-domain definitions of NI [26, 25] and OSNI systems [6, 4]. Finally, an asymptotic stability result for an unforced positive feedback interconnection containing a TD-ONI system without poles at the origin and a stable TD-OSNI (or an SNI) system is derived (in Section 6) utilising the dissipative approach based on the Lyapunov stability concept, which is entirely different from the approaches adopted so far in the NI literature [26, 41, 25].

*Notation:* The notations are standard throughout. $\mathbb{R}_{\geq 0}$ and $\mathbb{R}_{>0}$ denote respectively the sets of all non-negative and all positive real numbers. $\mathbb{C}$ denotes the set of all complex numbers. The set of all natural numbers (excluding 0) is denoted as $\mathbb{N} = \{1, 2, 3, \cdots\}$. $\mathbb{C}^-$ and $\bar{\mathbb{C}}^-$ denote respectively the open left-half and the closed left-half of the complex plane. $A^\top$, $A^*$ and $\bar{A}$ denote the transpose, the complex conjugate transpose and the complex conjugate of a matrix $A$. $A^{-*}$ and $A^{-\top}$ represent shorthand for $\left(A^{-1}\right)^*$ and $\left(A^{-1}\right)^\top$ respectively. $\lambda_{\max}(A)$ denotes the maximum eigenvalue of a matrix $A$ that has only real eigenvalues. $\mathscr{RH}_\infty^{m\times n}$ denotes the set of all real, rational, proper and asymptotically stable transfer function matrices of dimension $(m \times n)$. For a real, rational transfer function matrix $M(s)$, $M(j\omega)^* = M(-j\omega)^\top$. $(A,B,C,D)$ gives a state-space

[1] ONI systems are defined for finite-dimensional, square and causal systems. OSNI systems form a strict subset within the ONI class and may contain a pole on the origin.

[2] The notion of $(\Sigma, \bar{Q})$-dissipativity specialises to $(Q,S,R)$-dissipativity in the sense of Hill-Moylan [18] when $\Sigma$ is a linear and static operator.

realisation of a real, rational, proper transfer function matrix $M(s) = D + C(sI - A)^{-1}B$. The space of all real-valued, absolutely square integrable, time-domain functions is defined by $\mathbb{L}_2^m = \{f : \mathbb{R} \to \mathbb{R}^m : f(t) = 0 \text{ when } t < 0, \int_0^\infty f(t)^\top f(t)\,\mathrm{d}t < \infty\}$, while the space of all real-valued, locally square integrable, time-domain functions is defined as $\mathbb{L}_{2e}^m = \{f : \mathbb{R} \to \mathbb{R}^m : f(t) = 0 \text{ when } t < 0, \int_0^T f(t)^\top f(t)\,\mathrm{d}t < \infty\ \forall T \in [0,\infty)\}$. An energy supply rate function $w(u,y)$ is an abstraction of the energy inflow into a physical system which is expressed by the mapping $w : \mathbb{U} \times \mathbb{Y} \to \mathbb{R}$ where the input space $\mathbb{U} \in \mathbb{L}_{2e}^m$ and the output space $\mathbb{Y} \in \mathbb{L}_{2e}^p$. $w(u,y)$ satisfies the property $\int_0^T w(u,y)\,\mathrm{d}t < \infty$ for all admissible $(u,y) \in \mathbb{U} \times \mathbb{Y}$ and $\forall T \in [0,\infty)$. In particular, $\int_0^\infty w(u,y)\,\mathrm{d}t < \infty$ when $(u,y) \in \mathbb{L}_2^m \times \mathbb{L}_2^p$. A storage function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is said to be a $C^1$ storage function if it is continuously differentiable in its argument over the entire domain. $\mathscr{F}[\cdot]$ and $\mathscr{L}[\cdot]$ represent the Fourier and the Laplace transform operators respectively. $\mathscr{L}_2^m(j\mathbb{R})$ denotes the frequency-domain Lebesgue space [16, 35] under the inner product $\langle f, g\rangle = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(j\omega)^* g(j\omega)\,\mathrm{d}\omega < \infty$ when $f, g \in \mathscr{L}_2^m(j\mathbb{R})$. For a signal $f \in \mathscr{L}_2^m(j\mathbb{R})$, the norm is given by $\|f\| = \sqrt{\langle f, f\rangle} = \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty} f(j\omega)^* f(j\omega)\,\mathrm{d}\omega} < \infty$. Following the notion of $\mathscr{L}_2^m(j\mathbb{R})$ space, another frequency-domain space $\mathscr{L}_2^m(a + j\mathbb{R})$ with a given $a > 0$ is defined under the inner product $\langle f, g\rangle_a = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(a + j\omega)^* g(a + j\omega)\,\mathrm{d}\omega < \infty$ for the signals that are not bounded on the $j\omega$-axis but are bounded on the $(a + j\omega)$-axis for a specific $a > 0$. Note that an energy supply rate can also be defined in frequency-domain [i.e. $(Q(\omega), S(\omega), R(\omega))$-dissipativity [17, 35, 4]] for stable LTI systems and it remains equivalent to the corresponding time-domain supply rate via Parseval's theorem [7]. The symbol $\star$ stands for the time-domain convolution operator. $A \otimes B$ represents the Kronecker product of two real matrices $A$ and $B$ of any dimensions.

## 2 Essential preliminaries

In this section, we present essential technical preliminaries which underpin the proofs of the main results.

### *2.1 Definitions of NI systems*

We will now recall the definitions of NI and SNI systems.

**Definition 1** *(**NI System**) [30, 25] Let $M(s)$ be the real, rational and proper transfer function matrix of a finite-dimensional, square and causal system with no poles in $\{s \in \mathbb{C} : \Re[s] > 0\}$. Then, $M(s)$ is said to be NI if*

- *$j[M(j\omega) - M(j\omega)^*] \geq 0$ for all $\omega \in (0,\infty)$ except the values of $\omega$ where $s = j\omega$ is a pole of $M(s)$;*
- *If $s = j\omega_0$ with $\omega_0 \in (0,\infty)$ is a pole of $M(s)$, then it is at most a simple pole and the residue matrix $\lim_{s \to j\omega_0}(s - j\omega_0)jM(s)$ is Hermitian and positive semidefinite;*
- *If $s = 0$ is a pole of $M(s)$, then $\lim_{s\to 0} s^k M(s) = 0$ for all $k \geq 3$ and $\lim_{s\to 0} s^2 M(s)$ is Hermitian and positive semidefinite.*

**Definition 2** *(**SNI System**) [26, 25] Let $M(s)$ be the real, rational and proper transfer function matrix of a finite-dimensional, square and causal system. Then, $M(s)$ is said to be SNI if $M(s)$ has no poles in $\{s \in \mathbb{C} : \Re[s] \geq 0\}$ and $j[M(j\omega) - M(j\omega)^*] > 0$ for all $\omega \in (0,\infty)$.*

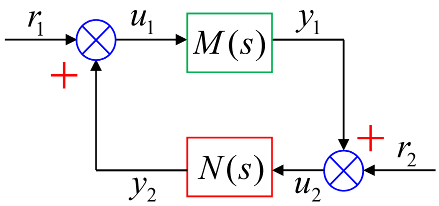


Fig. 1. Interconnection of NI systems with positive feedback.

We will now present the internal stability theorem for a positive feedback interconnection of an NI and an SNI system that removes all the previous restrictions on the gains of the systems at infinite frequency (which imposed strict properness of the loop).

**Theorem 1** *[25] Let $M(s)$ be an NI system without poles at the origin and $N(s)$ be an SNI system. Then, the positive feedback interconnection of $M(s)$ and $N(s)$, shown in Fig. 1, is internally stable if and only if*

$$\begin{cases} \det[I - M(\infty)N(\infty)] \neq 0, \\ \lambda_{\max}\left[(I - M(\infty)N(\infty))^{-1}(M(\infty)N(0) - I)\right] < 0, \\ \lambda_{\max}\left[(I - N(0)M(\infty))^{-1}(N(0)M(0) - I)\right] < 0. \end{cases} \tag{1}$$

### *2.2 Dissipative systems notations and definition*

The class of finite-dimensional, causal, LTI systems studied in this paper are described by the state-space equations

$$M : \begin{cases} \dot{x} = Ax + Bu, & x(0) = x_0; \\ y = Cx + Du. \end{cases} \tag{2}$$

The admissible inputs are considered to be in the space $\mathbb{L}_{2e}^m$ such that the unique solution of the state trajectory $x(t)$ exists forward in time $t \geq 0$ and $x \in \mathbb{L}_{2e}^n$. Therefore, the output $y(t)$ also exists and $y \in \mathbb{L}_{2e}^p$. We introduce the state transition function $\Phi$, associated with $M$, being a mapping from $\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \times \mathbb{R}^n \times \mathbb{R}^m$ onto $\mathbb{R}^n$. Here, $\Phi(t_1, t_0, x(t_0), u(t))$ denotes the state $x(t_1) \in \mathbb{R}^n$ at time

$t_1$ when the system $M$ starts from an initial state $x(t_0) \in \mathbb{R}^n$ at time $t_0$ and an admissible input $u(t)$ is applied on $M$ for the time interval $t \in [t_0, t_1]$.

We will now recall the notion of classical dissipativity of finite-dimensional, causal, dynamical systems, as introduced in [39].

**Definition 3** ***(Dissipative systems)*** *[39] A dynamical system $M$, given in (2), is said to be dissipative with respect to an energy supply rate $w(u,\, y)$ if there exists a function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, called the storage function, such that*

$$V\left(x(0)\right) + \int_0^T w\left(u, y\right) \mathrm{d}t \geq V\left(x\left(T\right)\right) \tag{3}$$

*for any $T \in [0, \infty)$, any initial condition $x(0) \in \mathbb{R}^n$ and any admissible input $u \in \mathbb{L}_{2e}^m$ where $x(T) = \Phi(T, 0, x(0), u(t))$ and $w(u, y)$ has been evaluated along any trajectory of (2).*

Inequality (3) is known as the 'dissipation inequality' in the sense of Willems. If $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is a differentiable storage function, then the dissipation inequality (3) can be expressed in the differential form as

$$w(u, y) \geq \dot{V}(x). \tag{4}$$

Note that for finite-dimensional LTI systems with a minimal state-space realisation, the storage function $V(x)$ can be characterized by a quadratic form $x^\top P x$, without loss of generality, where $P = P^\top > 0$ [39], [21]. Moreover, in the LTI setting, the storage function $V(x)$ can always be assumed to be a differentiable function of $x$ [18, 23].

If the supply rate function in (4) takes the form $w(u, y) = y^\top Q y + 2y^\top S u + u^\top R u$ where $Q = Q^\top \in \mathbb{R}^{p\times p}$, $S \in \mathbb{R}^{p\times m}$ and $M = M^\top \in \mathbb{R}^{m\times m}$, then (4) specialises to

$$\left(y^\top Q y + 2y^\top S u + u^\top R u\right) \geq \dot{V}(x). \tag{5}$$

So far we have discussed only time-domain dissipativity. However, dissipative characterization can also be expressed in the frequency-domain. The following definition articulates the notion of frequency-domain $(Q(\omega), S(\omega), R(\omega))$-dissipativity which may be regarded as a frequency-domain counterpart of the Hill-Moylan's $(Q, S, R)$-dissipativity [18].

**Definition 4** ***($(Q(\omega), S(\omega), R(\omega))$-dissipativity)*** *[17, 35] Let $M(s) \in \mathscr{RH}_\infty^{p\times m}$ be the transfer function matrix of a finite-dimensional and causal system $M$ with the input-output relationship $Y(s) = M(s)U(s)$. Then, $M$ is said to be $(Q(\omega), S(\omega), R(\omega))$-dissipative with respect to the frequency-dependent triplet $(Q(\omega), S(\omega), R(\omega))$ where $Q(\omega) = Q(\omega)^\top \in \mathbb{R}^{p\times p}$, $S(\omega) \in \mathbb{C}^{p\times m}$ and $R(\omega) = R(\omega)^\top \in \mathbb{R}^{m\times m}$ $\forall \omega \in \mathbb{R}$ if*

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\Big[Y(j\omega)^* Q(\omega) Y(j\omega) + Y(j\omega)^* S(\omega) U(j\omega) + U(j\omega)^* S(\omega)^* Y(j\omega) + U(j\omega)^* R(\omega) U(j\omega)\Big]\mathrm{d}\omega \geq 0 \tag{6}$$

*for all admissible $U \in \mathscr{L}_2^m(j\mathbb{R})$.*

Note that for marginally-stable systems, the frequency-domain integral in (6) does not remain bounded on the $j\omega$-axis and hence, $(Q(\omega), S(\omega), R(\omega))$-dissipative supply rate cannot be defined. To overcome this limitation, in this paper, we introduce the notion of a "shifted $(Q_a(\omega), S_a(\omega), R_a(\omega))$-dissipativity" that relies on the Fourier transformation evaluated with respect to a shifted $j\omega$-axis, denoted as the $(a + j\omega)$-axis, where the parameter $a$ is chosen to be a positive constant such that $a \geq 0$ and $a > \Re[\lambda_i(A)]$ $\forall i$. In such cases, inequality (6) takes the form (7), as introduced via the following definition.

**Definition 5** ***(Shifted $(Q_a(\omega), S_a(\omega), R_a(\omega))$-dissipativity)*** *Let $M(s) \in \mathscr{R}^{p\times m}$ be the transfer function matrix of a finite-dimensional and causal system $M$ with the input-output relationship $Y(s) = M(s)U(s)$. Then, $M$ is said to be a "shifted $(Q_a(\omega), S_a(\omega), R_a(\omega))$-dissipative" system with respect to the frequency-dependent triplet $Q_a(\omega) = Q_a(\omega)^\top \in \mathbb{R}^{p\times p}$, $S_a(\omega) \in \mathbb{C}^{p\times m}$ and $R_a(\omega) = R_a(\omega)^\top \in \mathbb{R}^{m\times m}$ $\forall \omega \in \mathbb{R}$ if*

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\Big[Y(a+j\omega)^* Q_a(\omega) Y(a+j\omega) + Y(a+j\omega)^* S_a(\omega) U(a+j\omega) + U(a+j\omega)^* S_a(\omega)^* Y(a+j\omega) + U(a+j\omega)^* R_a(\omega) U(a+j\omega)\Big]\mathrm{d}\omega \geq 0 \tag{7}$$

*for all admissible $U(a + j\omega) \in \mathscr{L}_2^m(j\mathbb{R})$, where $U(a + j\omega) = \mathscr{F}[e^{-at}u(t)]$ and $Y(a + j\omega) = \mathscr{F}[e^{-at}y(t)]$ on noting that $u_a = e^{-at}u(t) \in \mathbb{L}_2^m$ and $y_a = e^{-at}y(t) \in \mathbb{L}_2^m$ for all $t \geq 0$, subject to an appropriate choice of $a > 0$ and restricting the time-domain input signals $u \in \mathbb{L}_2^m$.*

### *2.3 Dynamic dissipativity theory*

We will now set the notation and definition for dynamic dissipativity. A system $M$ is said to be *dynamically dissipative* if the system $M$, cascaded with another given dynamic system $\Sigma$, is dissipative [8]. The concept is depicted through Fig. 2.

**Definition 6** *A finite-dimensional, causal and square system $M$ is said to be dynamically dissipative, termed as $(\Sigma, \bar{Q})$-dissipative where $\bar{Q} = \bar{Q}^\top \in \mathbb{R}^{\hat{l}\times\hat{l}}$ and the LTI*

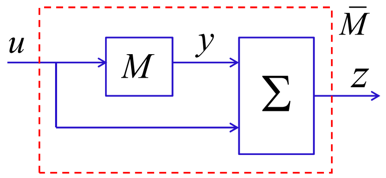


Fig. 2. Cascaded interconnection of $M$ and $\Sigma$.

*operator $\Sigma$ has a real, rational transfer function representation $\Sigma(s)$, if the cascade combination of $M$ and $\Sigma$, designated as $\bar{M}$ in Fig. 2, is dissipative with respect to the supply rate $z^\top \bar{Q} z$, where $z = \Sigma \begin{pmatrix} y \\ u \end{pmatrix}$ is evaluated along the trajectories of the combined system $\bar{M}$.*

Let $x = x(t) \in \mathbb{R}^n$ and $\hat{x} = \hat{x}(t) \in \mathbb{R}^{\hat{n}}$ denote respectively the states of $M$ and $\Sigma$. We denote $\bar{x} = \begin{bmatrix} x^\top & \hat{x}^\top \end{bmatrix}^\top$. Definition 6 implies that for a $(\Sigma, \bar{Q})$-dissipative system with respect to a particular $\bar{Q} = \bar{Q}^\top \in \mathbb{R}^{\hat{l} \times \hat{l}}$ and $\Sigma(s)$, there always exists a storage function $V : \mathbb{R}^{n+\hat{n}} \to \mathbb{R}_{\geq 0}$ such that

$$V(\bar{x}(0)) + \int_0^T z^\top \bar{Q} z \, \mathrm{d}t \geq V(\bar{x}(T)) \tag{8}$$

for all $T \in [0, \infty)$ and all $u \in \mathbb{L}_{2e}^m$. In general, for LTI systems, the storage function $V(\bar{x})$ can be assumed to be a $C^1$ function of $\bar{x} \in \mathbb{R}^{n+\hat{n}}$ and hence, (8) becomes equivalent to $z^\top \bar{Q} z \geq \dot{V}(\bar{x})$. Note that in particular circumstances, $\Sigma$ can also be a linear dynamical operator, such as $\frac{\mathrm{d}}{\mathrm{d}t}(\cdot)$ or $\int (\cdot) \, \mathrm{d}t$.

## 3 TD-IONI systems and their connections to dynamic dissipativity

At its inception, input and/or output negative imaginary system properties were defined only for stable LTI systems relying on the frequency-domain $(Q(\omega), S(\omega), R(\omega))$-dissipative approach [35, 9, 4]. However, these definitions were not uniform across the literature and moreover, the supply rates used in [35, 9] could not capture bi-proper OSNI systems. Drawn by these issues, very recently, [24] has proposed a frequency-domain condition for defining the class of *stable* IONI systems that involves a particular band-pass filter function, mentioned in (10), for capturing the frequency-domain behaviour [3] of an IONI (or NI) system in the neighbourhood of $\omega = 0$ and $\omega = \infty$. The present paper is motivated by the ideas developed in [24] and introduces the notion of Time-Domain Input-Output Negative Imaginary (TD-IONI) systems, which is defined completely in the time-domain utilising the concept of dynamic dissipativity. The definition and characterisation of the proposed TD-IONI systems do not impose any *a priori* conditions (such as stability, minimality, full normal rank constraint, etc. – commonly used in the NI literature) on the system to be defined and thereby, pose no difficulty in acquiring systems containing poles on the imaginary axis, even at the origin. Interestingly, it is found that the TD-IONI class captures the full set of the existing NI systems [26, 41, 25]. Note that in this section, the admissible inputs $u$ are considered to be in the space $\mathbb{L}_{2e}^m$ along with sufficient smoothness properties such that a unique solution of the state trajectory $x(t)$ exists forward in time $t \geq 0$ and also, $x \in \mathbb{L}_{2e}^n$. Hence, $\dot{y}(t) = C\dot{x}(t) = CAx(t) + CBu(t)$ does also exist forward in time $\forall t \geq 0$ and $\dot{y} \in \mathbb{L}_{2e}^m$.

**Definition 7** ***(TD-IONI$_{(\delta,\varepsilon,\alpha,\beta)}$** *systems) Let $M$ be a finite-dimensional, causal and square system governed by the minimal state-space equations $\dot{x} = Ax + Bu$ and $y = Cx + Du$ with zero initial condition. Let the associated transfer function matrix be $M(s) \in \mathscr{R}^{m \times m}$. Define $\bar{y} = y - Du$ and $\bar{u} = \mathscr{L}^{-1}[f_s(s)I_m] \star u$ where $f_s(s) \in \mathscr{RH}_\infty$ is defined in* (11)*. Let $\delta \geq 0$, $\varepsilon \geq 0$, $\alpha \in \mathbb{N}$ and $\beta \in \mathbb{N}$. Then, $M$ is said to be a Time-Domain Input-Output Negative Imaginary system with a level of output strictness $\delta \geq 0$, level of input strictness $\varepsilon \geq 0$ and having arrival rate specified by $\alpha \in \mathbb{N}$ and departure rate specified by $\beta \in \mathbb{N}$, denoted by* TD-IONI$_{(\delta,\varepsilon,\alpha,\beta)}$*, if*

$$\int_0^T \left( 2\dot{\bar{y}}^\top u - \delta \, \dot{\bar{y}}^\top \dot{\bar{y}} - \varepsilon \bar{u}^\top \bar{u} \right) \mathrm{d}t \geq 0 \tag{9}$$

*for all admissible $u \in \mathbb{L}_{2e}^m$ and all $T \in [0, \infty)$.*

Inequality (9) is referred to as the "TD-IONI$_{(\delta,\varepsilon,\alpha,\beta)}$ inequality" [4] in this paper. TD-IONI$_{(\delta,\varepsilon,\alpha,\beta)}$ definition unifies all the existing versions of the input and/or output negative imaginary systems [35, 10, 6, 4, 5, 24] and opens the door to accept the systems with poles on the $j\omega$-axis, even at the origin. TD-IONI$_{(\delta,\varepsilon,\alpha,\beta)}$ systems have been defined in the spirit of a new time-domain dissipative supply rate $w(u, \bar{u}, \dot{\bar{y}})$ [refer to Section 4 for details] instead of relying on conventional frequency-domain definitions.

*Classification of* TD-IONI$_{(\delta,\varepsilon,\alpha,\beta)}$ *systems:* We will now describe the strict and non-strict subclasses within the TD-IONI$_{(\delta,\varepsilon,\alpha,\beta)}$ class on the basis of the values of the parameters $\delta$, $\varepsilon$, $\alpha$ and $\beta$:

---

[3] Note that, in the SISO setting, the frequency-domain behaviour around $\omega = 0$ and $\omega = \infty$ indicates respectively the departure rate and the arrival rate of the Nyquist plot of a transfer function.

[4] The time-domain inequality condition (9) is also applicable to linear time-varying and nonlinear input-affine IONI systems. However, this paper deals with only LTI TD-IONI systems (denoted by TD-IONI$_{(\delta,\varepsilon,\alpha,\beta)}$).

- NI if it belongs to $\{M(s) : M(s)$ is TD-IONI$_{(\delta,\varepsilon,\alpha,\beta)}$, $\delta \geq 0,\ \varepsilon \geq 0,\ \alpha \in \mathbb{N},\ \beta \in \mathbb{N}\}$;
- TD-ISNI if it belongs to $\{M(s) : M(s)$ is TD-IONI$_{(\delta,\varepsilon,\alpha,\beta)}$, $\delta \geq 0,\ \varepsilon > 0,\ \alpha \in \mathbb{N},\ \beta \in \mathbb{N}\}$;
- TD-VSNI if it belongs to $\{M(s) : M(s)$ is TD-IONI$_{(\delta,\varepsilon,\alpha,\beta)}$, $\delta > 0,\ \varepsilon > 0,\ \alpha \in \mathbb{N},\ \beta \in \mathbb{N}\}$;
- TD-OSNI if it belongs to $\{M(s) : M(s)$ is TD-IONI$_{(\delta,\varepsilon,\alpha,\beta)}$, $\delta > 0,\ \varepsilon \geq 0,\ \alpha \in \mathbb{N},\ \beta \in \mathbb{N}\}$.

Fig. 3 illustrates the classification through a comprehensive Venn diagram.

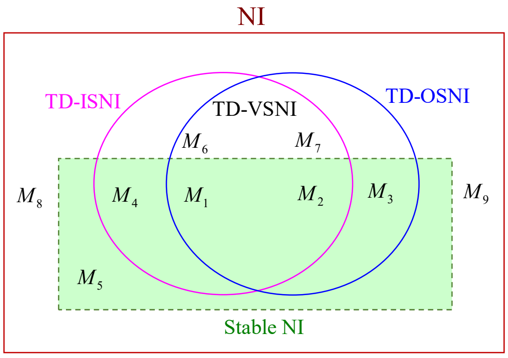


Fig. 3. Relationship among the strict and non-strict subclasses within the TD-IONI systems. $M_1, M_2, \cdots, M_9$ denote the examples (given in Subsection 3.4) corresponding to the major subsets of TD-IONI systems.

**Remark 1** *According to Definition 7, the sets of* TD-OSNI *and* TD-ISNI *systems capture some systems that are not asymptotically stable (e.g.* $\frac{1}{s}$, $\frac{s+4}{s(s+2)}$*). This attribute makes a significant contrast with the earlier literature on the OSNI and ISNI systems because these properties were originally defined only for stable LTI systems [35, 9, 11, 4]. It may be noted that Definition 7 can conveniently capture the marginally-stable NI systems since the time-domain approach does not need to impose an asymptotic stability constraint on the system to be defined, contrary to the conventional frequency-domain definitions. The Venn diagram in Fig. 3 clearly shows the set-theoretic relationship among the strict and non-strict, and the stable and marginally-stable subsets within the* TD-IONI *class.*

### 3.1 *Analysis of the filter term used in Definition 7*

The time-domain inequality condition introduced in Definition 7 for TD-IONI$_{(\delta,\varepsilon,\alpha,\beta)}$ systems involves an auxiliary input $(\bar{u})$ that is a band-pass filtered version of the actual input $u$. The transfer function of the filter $f(s)$ comes from [24] and is given by

$$f(s) = \frac{(-s)^{\beta} s^{\beta}}{1 + (-s)^{(\alpha+\beta-1)}\, s^{(\alpha+\beta-1)}} \qquad (10)$$

where $\alpha \in \mathbb{N}$ and $\beta \in \mathbb{N}$. Note that $\bar{u} = \mathscr{L}^{-1}[f_s(s) I_m] \star u$ where $f_s(s)$ is stable and minimum-phase spectral factor of $f(s)$. The next lemma shows the spectral factorisation of the filter function $f(s)$.

**Lemma 1** *[24] Let $f(s)$ be defined in* (10)*. Then, $f(s)$ can be spectral factorised as $f(s) = f_s^{\sim}(s) f_s(s)$ where $f_s(s) \in \mathscr{RH}_\infty$ is given by*

$$f_s(s) = \begin{cases} \dfrac{s}{s+1} \quad \text{when } \alpha = \beta = 1, \\[2ex] \dfrac{s^{\beta}}{\prod_{i=0}^{\left(\frac{\alpha+\beta-1}{2}-1\right)} \left(s^2 + 2\sin\left[\frac{(2i+1)\pi}{2(\alpha+\beta-1)}\right] s + 1\right)} \\ \text{when } \alpha+\beta \text{ is odd}, \\[2ex] \dfrac{s^{\beta}}{(s+1) \prod_{i=0}^{\left(\frac{\alpha+\beta}{2}-2\right)} \left(s^2 + 2\sin\left[\frac{(2i+1)\pi}{2(\alpha+\beta-1)}\right] s + 1\right)} \\ \text{when } \alpha+\beta \text{ is even and } \alpha+\beta > 2. \end{cases} \qquad (11)$$

### 3.2 *Characterisation of the* TD-IONI *systems in a dynamic dissipative framework*

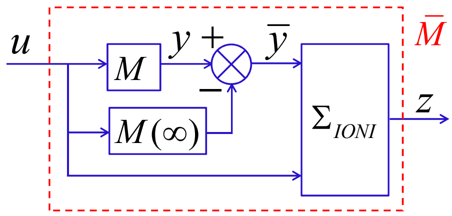


Fig. 4. A dynamic dissipative framework for characterising the class of TD-IONI systems.

The idea of utilising the dynamic dissipativity theory to characterise the class of TD-IONI systems has been found to be quite useful because the dissipative supply rate for defining such systems requires the input to the system $(u)$, a bandpass filtered version of the input $(\bar{u})$ and the time-derivative of an auxiliary output of the system $(\dot{\bar{y}})$. Owing to the presence of a time-derivative term and the additional dynamics associated with the input filter $f(s)I_m$ in the supply rate, the classical $(Q,S,R)$-dissipative framework [18, 20], where $Q$, $S$ and $R$ are all real and constant matrices, cannot directly capture TD-IONI$_{(\delta,\varepsilon,\alpha,\beta)}$ systems. Interestingly, the time-derivative operator and the filter dynamics $f(s)I_m$, defined in (10), can be embedded inside a separate module $\Sigma_{IONI}$ producing a new output $z = \begin{bmatrix} \dot{\bar{y}} \\ \bar{u} \\ u \end{bmatrix} = \Sigma_{IONI}\begin{pmatrix} \bar{y} \\ u \end{pmatrix}$, as shown in Fig. 4. With this structural modification, the overall cascaded system $\bar{M}$ can still be characterised using the Willems's dissipative framework with respect to the input $u$ and the combined output $z$. This is the primary motivation behind adapting the dynamic dissipative approach for characterising TD-IONI$_{(\delta,\varepsilon,\alpha,\beta)}$ systems. This strategy resembles the dynamic dissipative framework proposed in [8].

In the proposed scheme (Fig. 4), the linear operator $\Sigma_{IONI}$ has been designed to have a particular configuration represented by the real, rational transfer function $\Sigma_{IONI}(s) = \begin{bmatrix} sI_m & 0 \\ 0 & f_s(s)I_m \\ 0 & I_m \end{bmatrix}$, such that the output of the cascaded system $\bar{M}$ becomes $z = \begin{bmatrix} \dot{\bar{y}}^\top & \bar{u}^\top & u^\top \end{bmatrix}^\top$. The combined system $\bar{M}$ shown in Fig. 4 has the following state-space realization:

$$\bar{M} : \begin{cases} \dot{\bar{x}} = \mathscr{A}\bar{x} + \mathscr{B}u, \quad \bar{x}(0) = \bar{x}_0; \\ z = \mathscr{C}\bar{x} + \mathscr{D}u, \end{cases} \tag{12}$$

where $\mathscr{A} = \begin{bmatrix} A & 0 \\ 0 & I_m \otimes A_f \end{bmatrix}$, $\mathscr{B} = \begin{bmatrix} B \\ I_m \otimes B_f \end{bmatrix}$, $\mathscr{C} = \begin{bmatrix} CA & 0 \\ 0 & I_m \otimes C_f \\ 0 & 0 \end{bmatrix}$, $\mathscr{D} = \begin{bmatrix} CB \\ I_m \otimes D_f \\ I_m \end{bmatrix}$ and $\bar{x} = \begin{bmatrix} x \\ x_s \end{bmatrix}$. $x \in \mathbb{R}^n$ and $x_s \in \mathbb{R}^{\hat{n}}$ denote respectively the states of the system $M$ and the spectral factor $f_s(s) \in \mathscr{RH}_\infty$ of the bandpass filter $f(s)$, as defined in (11). Note that $\bar{y} = y - Du$ and $\bar{u} = \mathscr{L}^{-1}[f_s(s)I_m] \star u$. Let $f_s(s)$ have a minimal state-space representation $(A_f, B_f, C_f, D_f)$. The admissible set of inputs $u(t)$ belongs to the space $\mathbb{L}_{2e}^m$ with sufficient smoothness properties such that the combined state trajectory $\bar{x}(t)$ exists forward in time $t \geq 0$ and $\bar{x} \in \mathbb{L}_{2e}^{n+\hat{n}}$. We also define the following shorthand notations $\mathscr{C}_1 = \begin{bmatrix} CA & 0 \end{bmatrix}$, $\mathscr{C}_2 = \begin{bmatrix} 0 & I_m \otimes C_f \end{bmatrix}$, $\mathscr{D}_1 = CB$ and $\mathscr{D}_2 = I_m \otimes D_f$, which will be used in Theorem 2 and Lemma 2.

**Remark 2** *It can be noted that the combined state-space realization $(\mathscr{A}, \mathscr{B}, \mathscr{C}, \mathscr{D})$ retains minimality when $(A, B, C, D)$ and $(A_f, B_f, C_f, D_f)$ are minimal and $M(s)$ does not have any poles at the origin. If $M(s)$ has poles at the origin then it will give rise to a pole-zero cancellation at $s = 0$ between $M(s)$ and $\Sigma_{IONI}(s) = \begin{bmatrix} sI_m & 0 \\ 0 & f_s(s)I_m \\ 0 & I_m \end{bmatrix}$. As a consequence, $(\mathscr{A}, \mathscr{B}, \mathscr{C}, \mathscr{D})$ loses its state observability via the PBH test [42], despite the pair $(A, C)$ being completely observable. In such cases, $\begin{bmatrix} \lambda_i I - \mathscr{A} \\ \mathscr{C} \end{bmatrix} v_i = 0$ where $v_i \in \mathbb{R}^{n+\hat{n}}$, $v_i \neq 0$ is the eigenvector of $\mathscr{A}$ corresponding to $\lambda_i = 0$.*

Theorem 2, given below, will establish that TD-IONI systems are equivalent to a class of dynamically dissipative systems defined with respect to $\Sigma_{IONI}(s)$ and $\bar{Q}$ [termed as $(\Sigma_{IONI}, \bar{Q})$-dissipativity]. The theorem also offers a necessary and sufficient LMI condition to be satisfied by TD-IONI$_{(\delta,\varepsilon,\alpha,\beta)}$ systems. This LMI condition can be considered as a state-space characterisation for such systems and is useful for checking the strict/non-strict TD-IONI$_{(\delta,\varepsilon,\alpha,\beta)}$ properties of a given LTI system.

**Theorem 2** *Consider the cascaded systems interconnection $\bar{M}$, as shown in Fig. 4 and mathematically described in (12), where $M$ is a finite-dimensional, causal, square and initially relaxed system having a minimal state-space realisation $(A, B, C, D)$, $\lambda_i[A] \in \bar{\mathbb{C}}^- \ \forall i$ and $D = D^\top$. Define $\bar{y} = y - Du$ and $\bar{u} = \mathscr{L}^{-1}[f_s(s)I_m] \star u$ where $f_s(s) \in \mathscr{RH}_\infty$ is defined in (11) and has a minimal state-space representation $(A_f, B_f, C_f, D_f)$. Let $\delta \geq 0$, $\varepsilon \geq 0$, $\alpha \in \mathbb{N}$ and $\beta \in \mathbb{N}$. Then, the following statements are equivalent:*

I. *$M$ is $(\Sigma_{IONI}, \bar{Q})$-dissipative where $\Sigma_{IONI}$ has the real, rational transfer function matrix representation $\Sigma_{IONI}(s) = \begin{bmatrix} sI_m & 0 \\ 0 & f_s(s)I_m \\ 0 & I_m \end{bmatrix}$ and $\bar{Q} = \begin{bmatrix} -\delta I_m & 0 & I_m \\ 0 & -\varepsilon I_m & 0 \\ I_m & 0 & 0 \end{bmatrix}$;*

II. *there exists $\mathscr{P} = \mathscr{P}^\top \geq 0$ such that*

$$\prod \geq 0 \tag{13}$$

*where*

$$\prod = -\begin{bmatrix} \mathscr{P}\mathscr{A} + \mathscr{A}^\top \mathscr{P} + \delta \mathscr{C}_1^\top \mathscr{C}_1 + \varepsilon \mathscr{C}_2^\top \mathscr{C}_2 & \mathscr{P}\mathscr{B} - \mathscr{C}_1^\top + \delta \mathscr{C}_1^\top \mathscr{D}_1 + \varepsilon \mathscr{C}_2^\top \mathscr{D}_2 \\ \mathscr{B}^\top \mathscr{P} - \mathscr{C}_1 + \delta \mathscr{D}_1^\top \mathscr{C}_1 + \varepsilon \mathscr{D}_2^\top \mathscr{C}_2 & -\mathscr{D}_1 - \mathscr{D}_1^\top + \delta \mathscr{D}_1^\top \mathscr{D}_1 + \varepsilon \mathscr{D}_2^\top \mathscr{D}_2 \end{bmatrix}; \tag{14}$$

III. *$M$ is* TD-IONI$_{(\delta,\varepsilon,\alpha,\beta)}$.

*Note that the storage function in Part I and Part III can be chosen as $V(\bar{x}) = \bar{x}^\top \mathscr{P} \bar{x}$, where $\bar{x} = \begin{bmatrix} x \\ x_s \end{bmatrix} \in \mathbb{R}^{n+\hat{n}}$, with $\mathscr{P} = \mathscr{P}^\top \geq 0$ obtained from Part II.*

**Proof.** Let there exist $\delta \geq 0$, $\varepsilon \geq 0$, $\alpha \in \mathbb{N}$ and $\beta \in \mathbb{N}$.

**I ⇒ II:** The proof proceeds through the following set of statements:

$M$ is $(\Sigma_{IONI}, \bar{Q})$-dissipative with respect to $\Sigma_{IONI}(s)$
$$= \begin{bmatrix} sI_m & 0 \\ 0 & f_s(s)I_m \\ 0 & I_m \end{bmatrix} \text{ and } \bar{Q} = \begin{bmatrix} -\delta I_m & 0 & I_m \\ 0 & -\varepsilon I_m & 0 \\ I_m & 0 & 0 \end{bmatrix}$$
$\Leftrightarrow$ there exists a $C^1$ storage function $V(\bar{x}) = \bar{x}^\top \mathscr{P} \bar{x}$ with $\mathscr{P} = \mathscr{P}^\top \geq 0$ such that $z^\top \bar{Q} z \geq \dot{V}(\bar{x})$ where $\bar{x} = \begin{bmatrix} x \\ x_s \end{bmatrix}$ [using Definition 6]

$\Leftrightarrow$ there exists $\mathscr{P} = \mathscr{P}^\top \geq 0$ such that
$$\begin{bmatrix} \dot{y}^\top & \bar{u}^\top & u^\top \end{bmatrix} \begin{bmatrix} -\delta I_m & 0 & I_m \\ 0 & -\varepsilon I_m & 0 \\ I_m & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{y} \\ \bar{u} \\ u \end{bmatrix} \geq \dot{\bar{x}}^\top \mathscr{P} \bar{x} + \bar{x}^\top \mathscr{P} \dot{\bar{x}}$$
$\Leftrightarrow$ there exists $\mathscr{P} = \mathscr{P}^\top \geq 0$ such that
$$\begin{bmatrix} \bar{x}^\top & u^\top \end{bmatrix} \prod \begin{bmatrix} \bar{x} \\ u \end{bmatrix} \geq 0 \qquad \text{where } \bar{x} \in \mathbb{R}^{n+\hat{n}}$$
is evaluated along the trajectories of $\bar{M}$ subject to any admissible input $u \in \mathbb{L}_{2e}^m$ [upon expanding the term $\dot{y} = CAx + CBu$]

$\Rightarrow$ there exists $\mathscr{P} = \mathscr{P}^\top \geq 0$ such that $\prod \geq 0$ [following the necessity part of the proof of [4, Theorem 2]].

**I ⇐ II:** We have the following arguments:

There exists $\mathscr{P} = \mathscr{P}^\top \geq 0$ such that $\prod \geq 0$

$\Leftrightarrow$ there exists $\mathscr{P} = \mathscr{P}^\top \geq 0$ such that
$$\begin{bmatrix} \bar{x}^\top & u^\top \end{bmatrix} \prod \begin{bmatrix} \bar{x} \\ u \end{bmatrix} \geq 0 \text{ for all } \begin{bmatrix} \bar{x} \\ u \end{bmatrix} \in \mathbb{R}^{n+\hat{n}+m}$$
$\Leftrightarrow$ there exists $\mathscr{P} = \mathscr{P}^\top \geq 0$ such that
$2(CAx + CBu)^\top u - \delta(CAx + CBu)^\top(CAx + CBu) - \varepsilon \bar{u}^\top \bar{u} \geq \bar{x}^\top(\mathscr{P}\mathscr{A} + \mathscr{A}^\top \mathscr{P})\bar{x} + 2\bar{x}^\top \mathscr{P}\mathscr{B}u$ for all $\bar{x} \in \mathbb{R}^{n+\hat{n}}$ and all $u \in \mathbb{R}^m$

$\Rightarrow$ there exists $\mathscr{P} = \mathscr{P}^\top \geq 0$ such that a $C^1$ storage function $V(\bar{x}) = \bar{x}^\top \mathscr{P} \bar{x}$ satisfies $z^\top \bar{Q} z \geq \dot{V}(\bar{x})$ where $z = \begin{bmatrix} \dot{y}^\top & \bar{u}^\top & u^\top \end{bmatrix}^\top$ is evaluated along the trajectories of $\bar{M}$ subject to any admissible input $u \in \mathbb{L}_{2e}^m$

$\Leftrightarrow$ $M$ is $\left(\Sigma_{IONI}, \bar{Q}\right)$-dissipative [via Definitions 3 and 6].

**II ⇒ III:** We have the following set of implications.

There exists $\mathscr{P} = \mathscr{P}^\top \geq 0$ such that $\prod \geq 0$

$\Rightarrow$ there exists $\mathscr{P} = \mathscr{P}^\top \geq 0$ such that a $C^1$ storage function $V(\bar{x}) = \bar{x}^\top \mathscr{P} \bar{x}$ satisfies $2\dot{y}^\top u - \delta \dot{y}^\top \dot{y} - \varepsilon \bar{u}^\top \bar{u} \geq \dot{V}(\bar{x})$ where $\dot{y}, \bar{x}$ and $\bar{u}$ are evaluated along the trajectories of $\bar{M}$ subject to any admissible input $u \in \mathbb{L}_{2e}^m$ [following the proof of the part (I ⇐ II) on noting that $z^\top \bar{Q} z = 2\dot{y}^\top u - \delta \dot{y}^\top \dot{y} - \varepsilon \bar{u}^\top \bar{u}$]

$\Leftrightarrow$ there exist a $C^1$ storage function $V(\bar{x}) = \bar{x}^\top \mathscr{P} \bar{x}$ with $\mathscr{P} = \mathscr{P}^\top \geq 0$ such that $\bar{M}$ satisfies
$$\int_0^T \left(2\dot{y}^\top u - \delta\, \dot{y}^\top \dot{y} - \varepsilon \bar{u}^\top \bar{u}\right) dt \geq V(\bar{x}(T)) - V(\bar{x}(0))$$
for all $T \in [0, \infty)$ and any admissible $u \in \mathbb{L}_{2e}^m$
$$\Rightarrow \int_0^T \left(2\dot{y}^\top u - \delta\, \dot{y}^\top \dot{y} - \varepsilon \bar{u}^\top \bar{u}\right) dt \geq 0 \quad \text{for all } T \in [0, \infty)$$
and any admissible $u \in \mathbb{L}_{2e}^m$ [since $V(\bar{x}(T)) \geq 0$ $\forall T \in [0, \infty)$ and $V(\bar{x}(0)) = V(0) = 0$]

$\Leftrightarrow$ $M$ is TD-IONI$_{(\delta,\varepsilon,\alpha,\beta)}$ [via Definition 7].

**II ⇐ III:** This part follows directly from Theorem 3 derived in Section 4.

Combining all the above arguments, it can be assured that Part I ⇔ Part II ⇔ Part III. This completes the proof. ■

### *3.3* TD-IONI$_{(\delta,\varepsilon,\alpha,\beta)}$ *lemma*

This subsection utilises the LMI condition given in Part II of Theorem 2, referred to as TD-IONI$_{(\delta,\varepsilon,\alpha,\beta)}$ lemma from now onwards, for checking the strict/non-strict properties of a given TD-IONI$_{(\delta,\varepsilon,\alpha,\beta)}$ system depending on the definiteness of the solution matrix $\mathscr{P} = \mathscr{P}^\top \geq 0$, the values of the parameters $\delta \geq 0$, $\varepsilon \geq 0$, $\alpha \in \mathbb{N}$, $\beta \in \mathbb{N}$ and the type of stability (i.e. whether $\lambda_i[A]$ belong to $\mathbb{C}^-$ or $\bar{\mathbb{C}}^-$ for all $i$) of the system under consideration.

**Lemma 2** *(***TD-IONI$_{(\delta,\varepsilon,\alpha,\beta)}$ *lemma)* *Let $M$ be a finite-dimensional, causal, square and initially relaxed system having a minimal state-space representation $(A, B, C, D)$ where $D = D^\top$ and $\lambda_i[A] \in \bar{\mathbb{C}}^-$ $\forall i$. Let $\delta \geq 0$, $\varepsilon \geq 0$, $\alpha \in \mathbb{N}$ and $\beta \in \mathbb{N}$. Let*

$(A_f, B_f, C_f, D_f)$ *be a minimal state-space representation of* $f_s(s) \in \mathscr{RH}_\infty$*, as defined in* (11)*. Define*

$$\begin{bmatrix} \mathscr{A} & \mathscr{B} \\ \mathscr{C}_1 & \mathscr{D}_1 \\ \mathscr{C}_2 & \mathscr{D}_2 \end{bmatrix} = \left[\begin{array}{cc:c} A & 0 & B \\ 0 & I_m \otimes A_f & I_m \otimes B_f \\ \hdashline CA & 0 & CB \\ \hdashline 0 & I_m \otimes C_f & I_m \otimes D_f \end{array}\right].$$

*Then,* $M$ *is* TD-IONI$_{(\delta,\varepsilon,\alpha,\beta)}$ *if and only if there exists* $\mathscr{P} = \mathscr{P}^\top \geq 0$ *such that* $\prod \geq 0$*, where* $\prod$ *was defined in* (14)*.*

**Proof:** The proof directly follows from Theorem 2. ■

### *3.4 Numerical examples*

In this subsection, we will study several illustrative numerical examples after presenting an LMI-based state-space characterisation (i.e. the TD-IONI$_{(\delta,\varepsilon,\alpha,\beta)}$ lemma) for the full class of TD-IONI$_{(\delta,\varepsilon,\alpha,\beta)}$ systems to classify the strict/non-strict TD-IONI$_{(\delta,\varepsilon,\alpha,\beta)}$ properties, as portrayed in the Venn diagram in Fig. 3. Note that the time-domain inequality (9) can also be used to test the TD-IONI$_{(\delta,\varepsilon,\alpha,\beta)}$ properties of a given system. For convenience, it is also possible to consider the strictness parameters ($\delta$ and $\varepsilon$) separately, one at a time, as explained in the next lemma. However, for higher-order and MIMO systems, it requires a lot of manual effort. In such cases, the LMI condition (13) is the best option for checking the TD-IONI$_{(\delta,\varepsilon,\alpha,\beta)}$ properties of a given system based on its minimal state-space realisation.

**Lemma 3** *Let* $\alpha \in \mathbb{N}$*,* $\beta \in \mathbb{N}$*,* $\delta_0 > 0$ *and* $\varepsilon_0 > 0$*. Let* $M(s) \in$ TD-IONI$_{(\delta_0,0,\alpha,\beta)}$ *and* $M(s) \in$ TD-IONI$_{(0,\varepsilon_0,\alpha,\beta)}$*. Then,* $M(s) \in$ TD-IONI$_{(\delta,\varepsilon,\alpha,\beta)}$ *for all* $\delta \in \left[0, \frac{1}{2}\delta_0\right]$ *and all* $\varepsilon \in \left[0, \frac{1}{2}\varepsilon_0\right]$*.*

**Proof.** Since $\int_0^T (2\dot{\bar{y}}^\top u - \delta_0 \dot{\bar{y}}^\top \dot{\bar{y}})\,\mathrm{d}t \geq 0$ and $\int_0^T (2\dot{\bar{y}}^\top u - \varepsilon_0 \bar{u}^\top \bar{u})\,\mathrm{d}t \geq 0$ for all $T \in [0,\infty)$ and all admissible $u \in \mathbb{L}_{2e}^m$, it easily follows that $\int_0^T 2\dot{\bar{y}}^\top u\,\mathrm{d}t = \int_0^T \dot{\bar{y}}^\top u\,\mathrm{d}t + \int_0^T \dot{\bar{y}}^\top u\,\mathrm{d}t \geq \frac{1}{2}\delta_0 \int_0^T \dot{\bar{y}}^\top \dot{\bar{y}}\,\mathrm{d}t + \frac{1}{2}\varepsilon_0 \int_0^T \bar{u}^\top \bar{u}\,\mathrm{d}t \geq \delta \int_0^T \dot{\bar{y}}^\top \dot{\bar{y}}\,\mathrm{d}t + \varepsilon \int_0^T \bar{u}^\top \bar{u}\,\mathrm{d}t$, for $\delta \in [0, \frac{1}{2}\delta_0]$ and $\varepsilon \in [0, \frac{1}{2}\varepsilon_0]$. This completes the proof. ■

**Example 1** *Consider* $M_1(s) = \dfrac{1-2}{2+s}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ *with a minimal* $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{cc|cc} -2 & 0 & 4 & 0 \\ 0 & -2 & 0 & 4 \\ \hline 1.5 & 0.75 & -2 & -1 \\ 0.75 & 1.75 & -1 & -2 \end{array}\right]$*. To test the strict/non-strict* TD-IONI$_{(\delta,\varepsilon,\alpha,\beta)}$ *properties of* $M_1$*, we apply the* TD-IONI$_{(\delta,\varepsilon,\alpha,\beta)}$ *lemma and obtain a feasible solution* $\mathscr{P} = \begin{bmatrix} 0.97 & 0.39 & -0.45 & -0.02 \\ 0.39 & 0.97 & -0.02 & -0.45 \\ -0.45 & -0.02 & 0.90 & 0.05 \\ -0.02 & -0.45 & 0.05 & 0.90 \end{bmatrix} > 0$ *with* $\delta = 0.1978$ *and* $\varepsilon = 0.3447$ *when the filter function* $f_s(s)$ *is constructed with* $\alpha = 1$ *and* $\beta = 1$*. This indicates that* $M_1$ *is a* TD-VSNI *system with* $\alpha = 1$ *and* $\beta = 1$*. Hence,* $M_1$ *belongs to the stable VSNI subset (see the Venn diagram in Fig. 3).*

**Example 2** *Let* $M_2(s) = \dfrac{2s^2 + 10s + 22}{s^4 + 10s^3 + 38s^2 + 56s + 40}$ *with a minimal state-space realisation.* $M_2$ *satisfies the* TD-IONI$_{(\delta,\varepsilon,\alpha,\beta)}$ *lemma (i.e. Lemma 2) with*

$$\mathscr{P} = \begin{bmatrix} 2.61 & 2.05 & 2.59 & 0.56 & -0.61 & -0.45 \\ 2.05 & 3.02 & 3.44 & 1.32 & -0.80 & -0.80 \\ 2.58 & 3.44 & 6.21 & 2.71 & -1.21 & -1.54 \\ 0.56 & 1.32 & 2.71 & 2.92 & -0.56 & -1.65 \\ -0.61 & -0.80 & -1.21 & -0.56 & 0.61 & 0.45 \\ -0.45 & -0.80 & -1.54 & -1.65 & 0.45 & 1.32 \end{bmatrix} > 0,$$

$\delta = 1.8593$ *and* $\varepsilon = 0.1829$ *when* $\alpha = 2$ *and* $\beta = 1$*. This implies that* $M_2$ *is a stable* TD-VSNI *system with* $\alpha = 2$ *and* $\beta = 1$ *(see the Venn diagram in Fig. 3).*

**Example 3** *Let* $M_3(s) = \dfrac{s^2 + 8}{s^4 + s^3 + 25s^2 + 8s + 100}$ *with a minimal state-space realisation. Similar to the previous examples, we apply the LMI condition* (13) *to check the strict/non-strict IONI properties of* $M_3$*. We obtain* i) $\mathscr{P} = \begin{bmatrix} 4.03 & 0.10 & 4.05 & 0.10 & -0.02 \\ 0.10 & 4.59 & 0.16 & 3.45 & -0.05 \\ 4.05 & 0.16 & 6.32 & 0.15 & -0.02 \\ 0.10 & 3.45 & 0.15 & 3.44 & -0.05 \\ -0.02 & -0.05 & -0.02 & -0.05 & 0.01 \end{bmatrix} > 0$ *with* $\delta = 1.0459$ *and* $\varepsilon = 0$ *when* $(\alpha = 1, \beta = 1)$ *and* ii)

$$\mathscr{P} = \begin{bmatrix} 4.17 & 0.26 & 4.26 & 0.23 & -0.08 & -0.04 \\ 0.26 & 4.88 & 0.40 & 3.73 & -0.13 & -0.10 \\ 4.26 & 0.40 & 6.64 & 0.36 & -0.13 & -0.06 \\ 0.23 & 3.73 & 0.36 & 3.71 & -0.12 & -0.09 \\ -0.08 & -0.13 & -0.13 & -0.12 & 0.04 & 0.02 \\ -0.04 & -0.10 & -0.06 & -0.09 & 0.02 & 0.02 \end{bmatrix} > 0$$

*with* $\delta = 0.8299$ *and* $\varepsilon = 0$ *when* $(\alpha = 2, \beta = 1)$*. The end result signifies that* $M_3$ *is a stable* TD-OSNI *system (i.e.* TD-IONI$_{(\delta>0,\varepsilon=0,\alpha\in\mathbb{N},\beta\in\mathbb{N})}$ *system) that is not* TD-VSNI.

**Example 4** *Consider* $M_4(s) = \dfrac{4000(s+4)}{s^2 + 8s + 32}$ *with a* *minimal* $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{cc|c} 0 & 1 & 0 \\ -1 & -2 & 1 \\ \hline 1 & 2 & 0 \end{array}\right]$. *LMI* (13) *gives a feasible solution set* $\mathscr{P} = \begin{bmatrix} 15.14 & 3.36 & 1.65 & 4.36 \\ 3.36 & 2.15 & -1.25 & -1.54 \\ 1.65 & -1.25 & 9.98 & 10.00 \\ 4.36 & -1.54 & 10.00 & 12.34 \end{bmatrix} > 0$, $\delta = 0$ *and* $\varepsilon = 0.8086$ *to this system when* $\alpha = 1$ *and* $\beta = 2$. *However, it is verified that* (13) *does not give a feasible* $\mathscr{P} > 0$ *with both* $\delta > 0$ *and* $\varepsilon > 0$ *for any valid combinations of* $(\alpha, \beta)$. *This implies that* $M_4$ *is a stable* TD-ISNI *system but not* TD-VSNI.

**Example 5** *Take* $M_5(s) = \dfrac{2s^2 + s + 1}{(s+1)(2s+1)(s^2+2s+5)}$ *having a minimal state-space representation* $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{cccc|c} -3.5 & -2.125 & -1.063 & -0.625 & 0.5 \\ 4 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0 \\ \hline 0 & 0.5 & 0.125 & 0.25 & 0 \end{array}\right]$. *We apply the* $\text{TD-IONI}_{(\delta,\varepsilon,\alpha,\beta)}$ *lemma to this system subject to the most popular combinations of the input filter parameters* $(\alpha = 1, \beta = 1)$, $(\alpha = 2, \beta = 1)$ *and* $(\alpha = 1, \beta = 2)$. *But, none of them offered a feasible* $\mathscr{P} \geq 0$ *enforcing* $\delta > 0$ *and/or* $\varepsilon > 0$. *However,* $M_5$ *satisfies the LMI condition, given in* (13), *with* $\mathscr{P} = \begin{bmatrix} 43.76 & 25.35 & 30.07 & 24.85 & -19.88 \\ 25.35 & 17.97 & 19.17 & 16.16 & -12.43 \\ 30.07 & 19.17 & 22.74 & 18.79 & -14.91 \\ 24.85 & 16.16 & 18.79 & 15.84 & -12.43 \\ -19.88 & -12.43 & -14.91 & -12.43 & 9.94 \end{bmatrix} > 0$, $\delta = 0$ *and* $\varepsilon = 0$ *when* $(\alpha = 1, \beta = 1)$. *Same results were obtained for the other two combinations* $(\alpha = 2, \beta = 1)$ *and* $(\alpha = 1, \beta = 2)$ *[although the* $\mathscr{P}$ *matrices in these cases were numerically different]. These observations underpin that* $M_5$ *is a non-strict stable NI system (i.e.* $\text{TD-IONI}_{(\delta=0,\varepsilon=0,\alpha\in\mathbb{N},\beta\in\mathbb{N})}$*).*

**Example 6** *Following the same type of analysis as done in Examples 1–5, we find that a single integrator system* $M_6(s) = \dfrac{1}{s}$ *satisfies Lemma 2 with* $\mathscr{P} = \begin{bmatrix} 0.00 & 0.00 \\ 0.99 & 1.17 \end{bmatrix} \geq 0$, $\delta = 0.6728$ *and* $\varepsilon = 0.5485$ *when* $(\alpha = 1, \beta = 1)$. *Similarly, the system* $M_7(s) = \dfrac{s+4}{s(s+2)}$ *satisfies the* $\text{TD-IONI}_{(\delta,\varepsilon,\alpha,\beta)}$ *lemma with* $\mathscr{P} = \begin{bmatrix} 10.07 & 0.00 & -14.34 \\ 0.00 & 0.00 & 0.00 \\ -14.34 & 0.00 & 24.76 \end{bmatrix} \geq 0$, $\delta = 0.1462$ *and* $\varepsilon = 0.6263$ *when* $(\alpha = 1, \beta = 1)$. *Same observation was made when* $(\alpha = 2, \beta = 1)$ *and* $(\alpha = 1, \beta = 2)$. *Hence, we can conclude that both* $M_6$ *and* $M_7$ *belong to a subset of* TD-VSNI *systems that allows a simple pole at* $s = 0$.

**Example 7** *Finally, we choose two other types of* TD-IONI *systems: a double integrator* $M_8(s) = \dfrac{1}{s^2}$ *and a lossless NI system* $M_9(s) = \dfrac{1}{s^2+1}$. *We verify that both* $M_8$ *and* $M_9$ *fall within the non-strict* TD-IONI *systems having poles on the* $j\omega$ *axis, including the origin.*

## 4 Connections between the TD-IONI systems property and classical dissipativity

In this section, we will establish that for an initially relaxed $\text{TD-IONI}_{(\delta,\varepsilon,\alpha,\beta)}$ system with a controllable state-space, there always exists a positive semidefinite storage function $V(\bar{x})$ such that the cascaded system $\bar{M}$ (in Fig. 4) satisfies the dissipation inequality (3) with a particular time-domain supply rate $w(u, \bar{u}, \dot{\bar{y}}) = 2\dot{\bar{y}}^\top u - \delta\dot{\bar{y}}^\top\dot{\bar{y}} - \varepsilon\bar{u}^\top\bar{u}$ for some $\delta \geq 0$ and $\varepsilon \geq 0$. Here $\bar{y} = y - M(\infty)u$ is defined as an auxiliary output of $M$ and $\bar{u}$ is a filtered auxiliary input chosen as the inverse Laplace of $\bar{U}(s) = [f_s(s)I_m]U(s)$ where $U(s) = \mathscr{L}[u(t)]$ and $f_s(s) \in \mathscr{RH}_\infty$ is defined in (11).

**Theorem 3** *Let* $M$ *be a finite-dimensional, causal, square and initially relaxed system governed by the minimal state-space equations* $\dot{x} = Ax + Bu$ *and* $y = Cx + Du$, *where* $D = D^\top$ *and* $\lambda_i[A] \in \bar{\mathbb{C}}^- \ \forall i$. *Let the associated transfer function matrix be* $M(s) \in \mathscr{R}^{m\times m}$. *Define* $\bar{y} = y - Du$ *and* $\bar{u} = \mathscr{L}^{-1}[f_s(s)I_m] \star u$ *where* $f_s(s) \in \mathscr{RH}_\infty$ *is defined in* (11). *Let* $\delta \geq 0$, $\varepsilon \geq 0$, $\alpha \in \mathbb{N}$ *and* $\beta \in \mathbb{N}$. *Then, the cascaded system* $\bar{M}$ *(comprised of* $M$ *and* $\Sigma_{IONI}$*) in Fig. 4 is dissipative with respect to the supply rate* $w(u, \bar{u}, \dot{\bar{y}}) = 2\dot{\bar{y}}^\top u - \delta\dot{\bar{y}}^\top\dot{\bar{y}} - \varepsilon\bar{u}^\top\bar{u}$ *if and only if* $M$ *is a* $\text{TD-IONI}_{(\delta,\varepsilon,\alpha,\beta)}$ *system.*

**Proof:** In this proof, we will consider the combined state trajectory $\bar{x} = \begin{bmatrix} x \\ x_s \end{bmatrix} \in \mathbb{R}^{n+\hat{n}}$ where $x \in \mathbb{R}^n$ and $x_s \in \mathbb{R}^{\hat{n}}$ denote respectively the states of the system $M$ and the input filter function $f_s(s)$. Note that $x_s$ is required to be included because the filter dynamics is directly involved in the dissipation inequality.

(**Sufficiency:**) To show that the combined system $\bar{M}$ in Fig. 4 is dissipative with respect to the supply rate

$w(u,\bar{u},\dot{\bar{y}}) = 2\dot{\bar{y}}^\top u - \delta\dot{\bar{y}}^\top\dot{\bar{y}} - \varepsilon\bar{u}^\top\bar{u}$, we have to establish that there exists a storage function $V : \mathbb{R}^{n+\hat{n}} \to \mathbb{R}_{\geq 0}$ with $V(0) = 0$ such that $\bar{M}$ satisfies the dissipation inequality (3). Since the state-space is assumed to be completely controllable, there exists an admissible input $u(t)$ defined as

$$u(t) = \begin{cases} 0 & \text{when } t < t_{-1}, \\ \tilde{u}(t) & \text{when } t_{-1} \leq t \leq 0, \\ 0 & \text{when } t > 0, \end{cases}$$

which steers the system from $\bar{x}(t_{-1}) = 0$ to any $\bar{x}(0) \in \mathbb{R}^{n+\hat{n}}$. In this proof, let $y(t)$ be the output of $M$ and define $\bar{y} = y - Du$ where $D = M(\infty) = D^\top$. Now,

$$\begin{aligned}
&\int_{t_{-1}}^{0} w(u,\bar{u},\dot{\bar{y}})\,\mathrm{d}t = \int_{t_{-1}}^{0} \left(2\dot{\bar{y}}^\top u - \delta\dot{\bar{y}}^\top\dot{\bar{y}} - \varepsilon\bar{u}^\top\bar{u}\right)\mathrm{d}t \\
&= \int_{t_{-1}}^{T} \left(2\dot{\bar{y}}^\top u - \delta\dot{\bar{y}}^\top\dot{\bar{y}} - \varepsilon\bar{u}^\top\bar{u}\right)\mathrm{d}t + \delta\int_{0}^{T} \dot{\bar{y}}^\top\dot{\bar{y}}\,\mathrm{d}t \\
&\qquad + \varepsilon\int_{0}^{T} \bar{u}^\top\bar{u}\,\mathrm{d}t \quad \forall T \in [0,\infty) \\
&\qquad [\text{since } M \text{ is causal and time-invariant}] \\
&\geq \int_{t_{-1}}^{T} \left(2\dot{\bar{y}}^\top u - \delta\dot{\bar{y}}^\top\dot{\bar{y}} - \varepsilon\bar{u}^\top\bar{u}\right)\mathrm{d}t \quad \forall T \in [0,\infty) \\
&\qquad [\text{since } \delta \geq 0 \text{ and } \varepsilon \geq 0] \\
&= \int_{0}^{\bar{T}} \left(2\dot{\bar{y}}^\top u - \delta\dot{\bar{y}}^\top\dot{\bar{y}} - \varepsilon\bar{u}^\top\bar{u}\right)\mathrm{d}\tau \geq 0 \qquad \forall \bar{T} \in [0,\infty)
\end{aligned}$$

via Definition 4 and applying a change of the time variable $\tau = t - t_{-1}$ where $-\infty < t_{-1} \leq 0$ and denoting $\bar{T} = T - t_{-1}$. Hence, for arbitrary $t_{-1} \leq 0$ and $\bar{x}(t_{-1}) = 0$ we have, $\int_{t_{-1}}^{0} w(u,\bar{u},\dot{\bar{y}})\,\mathrm{d}t \geq 0$. We now construct the required supply function as $V_r(\bar{x}) = \inf_{\substack{\bar{x}^*=0\to\bar{x} \\ u(\cdot),\, t_{-1}\leq 0}} \int_{t_{-1}}^{0} w(u,\bar{u},\dot{\bar{y}})\,\mathrm{d}t \geq 0$ following [32], where origin is the point of minimum storage (i.e., $\bar{x}^* = 0$). Therefore, $V_r(\bar{x})$ can be considered as a suitable storage function candidate for $\bar{M}$ (comprised of $M$ and $\Sigma_{IONI}$).

It remains to be shown that $V_r(\bar{x})$ satisfies the dissipation inequality (3). Note that in taking the system from $\bar{x} = 0$ at $t = 0$ to $\bar{x}_1 \in \mathbb{R}^{n+\hat{n}}$ at $t = t_1$, we could first take it to $\bar{x}_0 \in \mathbb{R}^{n+\hat{n}}$ at time $t_0$ while minimizing the energy, and then take it to $\bar{x}_1$ at time $t_1$ along the path for which the dissipation inequality is to be evaluated. This is possible since $\bar{M}$ is a time-invariant system. As $V_r(\bar{x}_1)$ represents the infimum amount of energy required to reach $\bar{x}_1$ at $t = t_1$ from $\bar{x} = 0$ at $t = 0$, energy required to reach the same destination $\bar{x}_1$ from the same starting point $\bar{x} = 0$ via any other path will be greater than or equal to $V_r(\bar{x}_1)$. Therefore, $V_r(\bar{x}_0) + \int_{t_0}^{t_1} w(u,\bar{u},\dot{\bar{y}})\,\mathrm{d}t \geq V_r(\bar{x}_1)$ follows. It can hence be concluded that the cascaded interconnection $\bar{M}$ in Fig. 4, comprised of the TD-IONI$_{(\delta,\varepsilon,\alpha,\beta)}$ system $M$ and the LTI operator $\Sigma_{IONI}$, is dissipative with respect to the supply rate $w(u,\bar{u},\dot{\bar{y}}) = 2\dot{\bar{y}}^\top u - \delta\dot{\bar{y}}^\top\dot{\bar{y}} - \varepsilon\bar{u}^\top\bar{u}$.

(**Necessity:**) Let $\bar{M}$ be dissipative with respect to the supply rate $w(u,\bar{u},\dot{\bar{y}}) = 2\dot{\bar{y}}^\top u - \delta\,\dot{\bar{y}}^\top\dot{\bar{y}} - \varepsilon\bar{u}^\top\bar{u}$. Hence, there exists a storage function $V : \mathbb{R}^{n+\hat{n}} \to \mathbb{R}_{\geq 0}$ with $V(0) = 0$ such that $V(\bar{x}(0)) + \int_0^T \left(2\dot{\bar{y}}^\top u - \delta\,\dot{\bar{y}}^\top\dot{\bar{y}} - \varepsilon\bar{u}^\top\bar{u}\right)\mathrm{d}t \geq V(\bar{x}(T))$ for all $T \in [0,\infty)$ and all $u \in \mathbb{L}_{2e}^m$. The preceding inequality implies $\int_0^T \left(2\dot{\bar{y}}^\top u - \delta\,\dot{\bar{y}}^\top\dot{\bar{y}} - \varepsilon\bar{u}^\top\bar{u}\right)\mathrm{d}t \geq 0$ for all $T \in [0,\infty)$ and all admissible $u \in \mathbb{L}_{2e}^m$, which confirms that the system $M$ in Fig. 4 is TD-IONI$_{(\delta,\varepsilon,\alpha,\beta)}$ via Definition 7.

The Necessity and the Sufficiency parts together complete the proof. ■

Relying on Theorem 3, we can recommend a time-domain dissipation inequality (in Willems's framework [39]) for defining TD-IONI$_{(\delta,\varepsilon,\alpha,\beta)}$ systems involving a storage function $V : \mathbb{R}^{n+\hat{n}} \to \mathbb{R}_{\geq 0}$ satisfying

$$V(\bar{x}(0)) + \int_0^T \left(2\dot{\bar{y}}^\top u - \delta\,\dot{\bar{y}}^\top\dot{\bar{y}} - \varepsilon\bar{u}^\top\bar{u}\right)\mathrm{d}t \geq V(\bar{x}(T)) \tag{15}$$

for all $T \in [0,\infty)$ and all admissible $u \in \mathbb{L}_{2e}^m$, and where $\bar{x} = \begin{bmatrix} x \\ x_s \end{bmatrix}$ denotes the combined state trajectory of the cascaded system $\bar{M}$ shown in Fig. 4.

**Remark 3** *Theorem 2 and Theorem 3 establish the equivalence among the* TD-IONI$_{(\delta,\varepsilon,\alpha,\beta)}$ *systems property, the* TD-IONI$_{(\delta,\varepsilon,\alpha,\beta)}$ *lemma,* $(\Sigma_{IONI},\bar{Q})$*-dissipativity and time-domain dissipativity with respect to the supply rate* $w(u,\bar{u},\dot{\bar{y}})$*, where* $\bar{y} = y - Du$ *and* $\bar{u} = \mathscr{L}^{-1}[f_s(s)I_m] \star u$ *with* $f_s(s) \in \mathscr{RH}_\infty$ *being defined in* (11)*. We can now readily conclude that a finite-dimensional, causal, square and initially relaxed system* $M$ *having a minimal state-space realisation is* TD-IONI$_{(\delta,\varepsilon,\alpha,\beta)}$ $\Leftrightarrow$ *there exists* $\mathscr{P} = \mathscr{P}^\top \geq 0$ *such that* $\prod \geq 0$ *[given in* (13)*]* $\Leftrightarrow$ *the cascade combination* $\bar{M}$ *in Fig. 4, comprised of* $M$ *and* $\Sigma_{IONI}$*, is* $(\Sigma_{IONI},\bar{Q})$*-dissipative where* $\Sigma_{IONI}(s) = \begin{bmatrix} sI_m & 0 \\ 0 & f_s(s)I_m \\ 0 & I_m \end{bmatrix}$ *and* $\bar{Q} = \begin{bmatrix} -\delta I_m & 0 & I_m \\ 0 & -\varepsilon I_m & 0 \\ I_m & 0 & 0 \end{bmatrix}$ $\Leftrightarrow$ *the same* $\bar{M}$ *in Fig. 4* *is time-domain dissipative (in Willems's framework [39]) with respect to the supply rate $w(u,\bar{u},\dot{\bar{y}}) = 2\dot{\bar{y}}^\top u - \delta\,\dot{\bar{y}}^\top\dot{\bar{y}} - \varepsilon\,\bar{u}^\top\bar{u}$ and a specific storage function $V(\bar{x}) = \bar{x}^\top \mathscr{P}\bar{x}$ with $\mathscr{P} \geq 0$ being the solution of $\prod \geq 0$.*

## 5 Frequency-domain dissipative characterisation of the TD-IONI systems

This section presents another significant contribution of this paper. It develops a frequency-domain dissipative framework for characterising the full class of the TD-IONI$_{(\delta,\varepsilon,\alpha,\beta)}$ systems in contrast to the related earlier results reported in [35, 9, 10, 11] and [4] where $(Q(\omega), S(\omega), R(\omega))$-dissipativity was used only for stable IONI systems. $(Q(\omega), S(\omega), R(\omega))$-dissipativity cannot capture the marginally-stable systems because the frequency-domain dissipation inequality involves frequency-domain integrals having limits from $-\infty$ to $\infty$ and hence, restricts the system transfer function to be *stable* and the admissible inputs to be $\mathscr{L}_2$-bounded (i.e. all finite-energy signals). Another discrepancy arises between the time-domain and the frequency-domain dissipative approaches due to the fact that while the former allows $\mathbb{L}_{2e}$ signals, the latter allows only $\mathbb{L}_2$ (equivalent to $\mathscr{L}_2(j\mathbb{R})$) signals.

To circumvent the aforementioned limitations and discrepancies, in this section, we develop the idea of "shifted $(Q_a(\omega), S_a(\omega), R_a(\omega))$-dissipativity" (discussed at the end of Subsection 2.2 on page 4) instead of the conventional $(Q(\omega), S(\omega), R(\omega))$-dissipativity. Furthermore, to make both the time-domain and frequency-domain dissipative frameworks compatible, we choose to restrict the time-domain input space to $\mathbb{L}_2$. Theorem 4 proves that the class of TD-IONI systems including the ones that contain poles on the $j\omega$-axis for $\omega \in \mathbb{R}$, satisfies the "shifted $(Q_a(\omega), S_a(\omega), R_a(\omega))$-dissipative" property where $Q_a(\omega) = Q_a(\omega)^\top \in \mathbb{R}^{m\times m}$, $S_a(\omega) \in \mathbb{C}^{m\times m}$, $R_a(\omega) = R_a(\omega)^\top \in \mathbb{R}^{m\times m}$ $\forall\omega \in \mathbb{R}$ and the parameter $a$ is chosen such that $a > 0$ when the system has at least one pole (or a pole-pair) on the $j\omega$-axis, otherwise $a = 0$ (i.e. when $M(s) \in \mathscr{RH}_\infty$).

**Theorem 4** *Let $\delta \geq 0$, $\varepsilon \geq 0$, $\alpha \in \mathbb{N}$ and $\beta \in \mathbb{N}$. Let $M(s) \in \mathscr{R}^{m\times m}$ be the transfer function matrix of an initially relaxed* TD-IONI$_{(\delta,\varepsilon,\alpha,\beta)}$ *system $M$ having a minimal state-space representation $(A,B,C,D)$ with $D = M(\infty) = D^\top$. Choose $a > 0$ when $\max_i \Re\{\lambda_i[A]\} = 0$, or else $a = 0$. Then, $M$ is "shifted $(Q_a(\omega), S_a(\omega), R_a(\omega))$-dissipative" with $Q_a(\omega) = -\delta\,(a^2+\omega^2)I_m$, $S_a(\omega) = (a-j\omega)I_m + \delta\,(a^2+\omega^2)D$ and $R_a(\omega) = -a(D+D^\top) - \delta\,(a^2+\omega^2)D^\top D - \varepsilon\{f_s(a-j\omega)f_s(a+j\omega)\}I_m$ $\forall\omega \in \mathbb{R}$ where $f_s(s) \in \mathscr{RH}_\infty$ is defined in* (11).

**Proof.** We begin the proof on noting that $y = \mathscr{L}^{-1}[M(s)]\star u$, $\bar{y} = y - Du$ where $D = M(\infty) = D^\top$ and $\bar{u} = \mathscr{L}^{-1}[f_s(s)I_m]\star u$ with $u \in \mathbb{L}_2^m$ and $f_s(s) \in \mathscr{RH}_\infty$ be defined as in (11). The proof proceeds through a sequence of mathematical arguments in which improper frequency-domain integrals having limits from $-\infty$ to $\infty$ are considered being inspired by similar instances as reported in [17] and [35].

**Case I.** Let $M(s)$ contain pole(s) on the $j\omega$-axis for $\omega \in \mathbb{R}$. In this part of the proof, $Y(a+j\omega)$, $\bar{Y}(a+j\omega)$, $U(a+j\omega)$ denote respectively the Fourier Transform of the real-valued time-domain signals $e^{-at}y(t)$, $e^{-at}\bar{y}(t)$, $e^{-at}u(t)$ for all $t \geq 0$ and for a specific $a > 0$.

$M$ is TD-IONI$_{(\delta,\varepsilon,\alpha,\beta)}$

$\Leftrightarrow$ there exists $V : \mathbb{R}^{n+\hat{n}} \to \mathbb{R}_{\geq 0}$ with $V(0) = 0$ such that

$$\int_0^T \left(2\dot{\bar{y}}^\top u - \delta\,\dot{\bar{y}}^\top\dot{\bar{y}} - \varepsilon\,\bar{u}^\top\bar{u}\right)\mathrm{d}t \geq V(\bar{x}(T)) - V(\bar{x}(0))$$

for all $T \in [0,\infty)$ and all admissible inputs $u \in \mathbb{L}_2^m$ [utilising Theorem 3]

$$\Rightarrow \int_0^T \left(2\dot{\bar{y}}^\top u - \delta\,\dot{\bar{y}}^\top\dot{\bar{y}} - \varepsilon\,\bar{u}^\top\bar{u}\right)\mathrm{d}t \geq 0 \quad \text{for all } T \in [0,\infty)$$

and all admissible inputs $u \in \mathbb{L}_2^m$ [since $V(\bar{x}(T)) \geq 0$ for all $T \in [0,\infty)$ and $V(0) = 0$]

$$\Rightarrow \int_0^\infty e^{-2at}\left(2\dot{\bar{y}}^\top u - \delta\,\dot{\bar{y}}^\top\dot{\bar{y}} - \varepsilon\,\bar{u}^\top\bar{u}\right)\mathrm{d}t \geq 0$$

for all admissible inputs $u \in \mathbb{L}_2^m$ [choosing $a > 0$ when $\max_i \Re\{\lambda_i[A]\} = 0$; or else $a = 0$.]

$$\Leftrightarrow \int_0^\infty \left[2(e^{-at}\dot{\bar{y}})^\top(e^{-at}u) - \delta\,(e^{-at}\dot{\bar{y}})^\top(e^{-at}\dot{\bar{y}}) - \varepsilon\,(e^{-at}\bar{u})^\top (e^{-at}\bar{u})\right]\mathrm{d}t \geq 0 \text{ for all admissible inputs } u \in \mathbb{L}_2^m$$

$$\Leftrightarrow \frac{1}{2\pi}\int_{-\infty}^\infty (a-j\omega)\bar{Y}(a+j\omega)^* U(a+j\omega)\,\mathrm{d}\omega + \frac{1}{2\pi}\int_{-\infty}^\infty (a+j\omega)U(a+j\omega)^*\bar{Y}(a+j\omega)\,\mathrm{d}\omega - \frac{1}{2\pi}\int_{-\infty}^\infty \delta\,(a^2+\omega^2)\bar{Y}(a+j\omega)^*\bar{Y}(a+j\omega)\,\mathrm{d}\omega - \frac{1}{2\pi}\int_{-\infty}^\infty \varepsilon\,\bar{U}(a+j\omega)^*\bar{U}(a+j\omega)\,\mathrm{d}\omega \geq 0$$

for all admissible $U \in \mathscr{L}_2^m(a+j\mathbb{R})$ [using the result $(a+j\omega)Y(a+j\omega) = \mathscr{F}[e^{-at}\dot{y}(t)]$ for a specific $a > 0$ such that $Y(a+j\omega)$ exists $\forall\omega \in \mathbb{R}$ [34]]

$$\Leftrightarrow \frac{1}{2\pi}\int_{-\infty}^\infty (a-j\omega)\Big[\left[Y(a+j\omega)^* - U(a+j\omega)^*D^\top\right]\times U(a+j\omega)\Big]\mathrm{d}\omega + \frac{1}{2\pi}\int_{-\infty}^\infty (a+j\omega)\Big[U(a+j\omega)^* [Y(a+j\omega) - DU(a+j\omega)]\Big]\mathrm{d}\omega - \frac{1}{2\pi}\int_{-\infty}^\infty \Big[\delta\,(a^2+\omega^2)\,\{Y(a+j\omega)^* - U(a+j\omega)^*D^\top\}\{Y(a+j\omega) - DU(a+j\omega)\}\Big]\mathrm{d}\omega - \frac{1}{2\pi}\int_{-\infty}^\infty \Big[\varepsilon\,U(a+j\omega)^*$$

$$\{f_s(a-j\omega)f_s(a+j\omega)\}I_m\, U(a+j\omega)\Big]\mathrm{d}\omega \geq 0$$

for all admissible $U \in \mathscr{L}_2^m(a+j\mathbb{R})$

$$\Leftrightarrow \int_{-\infty}^{\infty}\Big[(a-j\omega)Y(a+j\omega)^*U(a+j\omega)+(a-j\omega) U(a+j\omega)^*Y(a+j\omega)-aU(a+j\omega)^*(D+D^\top) U(a+j\omega)-\delta\,(a^2+\omega^2)Y(a+j\omega)^*Y(a+j\omega)+ \delta\,(a^2+\omega^2)U(a+j\omega)^*D^\top Y(a+j\omega)+\delta\,(a^2+\omega^2) Y(a+j\omega)^*DU(a+j\omega)-\delta\,(a^2+\omega^2)U(a+j\omega)^* D^\top DU(a+j\omega)-\varepsilon U(a+j\omega)^*\{f_s(a-j\omega) f_s(a+j\omega)\}I_m\, U(a+j\omega)\Big]\mathrm{d}\omega \geq 0$$

for all admissible $U \in \mathscr{L}_2^m(a+j\mathbb{R})$

$$\Leftrightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty}\begin{bmatrix} Y(a+j\omega)^* & U(a+j\omega)^*\end{bmatrix} \times \begin{bmatrix} -\delta\,(a^2+\omega^2)I_m & \begin{matrix}(a-j\omega)I_m+\\ \delta\,(a^2+\omega^2)D\end{matrix} \\ \begin{matrix}(a+j\omega)I_m+\\ \delta\,(a^2+\omega^2)D^\top\end{matrix} & \begin{matrix}-a(D+D^\top)\\ -\delta\,(a^2+\omega^2)D^\top D\\ -\varepsilon\{f_s(a-j\omega)\\ f_s(a+j\omega)\}I_m\end{matrix}\end{bmatrix} \times \begin{bmatrix} Y(a+j\omega)\\ U(a+j\omega)\end{bmatrix}\mathrm{d}\omega \geq 0$$

for all admissible $U \in \mathscr{L}_2^m(a+j\mathbb{R})$

$$\Leftrightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty}\begin{bmatrix} Y(a+j\omega)^* & U(a+j\omega)^*\end{bmatrix}\begin{bmatrix} Q_a(\omega) & S_a(\omega)\\ S_a(\omega)^* & R_a(\omega)\end{bmatrix}\times \begin{bmatrix} Y(a+j\omega)\\ U(a+j\omega)\end{bmatrix}\mathrm{d}\omega \geq 0$$

for all admissible $U \in \mathscr{L}_2^m(a+j\mathbb{R})$ denoting $Q_a(j\omega) = -\delta\,(a^2+\omega^2)I_m,\ S_a(j\omega)=(a-j\omega)I_m+\delta\,(a^2+ \omega^2)D$ and $R_a(j\omega)=-a(D+D^\top)-\delta\,(a^2+\omega^2)\times D^\top D-\varepsilon\{f_s(a-j\omega)f_s(a+j\omega)\}I_m$.

**Case II.** Let $M(s)\in\mathscr{RH}_\infty^{m\times m}$. Then,

$M$ is stable TD-IONI$_{(\delta,\varepsilon,\alpha,\beta)}$

$$\Rightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty}\begin{bmatrix} Y(j\omega)^* & U(j\omega)^*\end{bmatrix}\begin{bmatrix} Q(\omega) & S(\omega)\\ S(\omega)^* & R(\omega)\end{bmatrix}\begin{bmatrix} Y(j\omega)\\ U(j\omega)\end{bmatrix} \mathrm{d}\omega \geq 0$$

for all admissible $U \in \mathscr{L}_2^m(j\mathbb{R})$ and denoting $Q(\omega)=-\delta\omega^2 I_m,\ S(\omega)=-j\omega I_m+\delta\,\omega^2 D$ and $R(\omega)=-\delta\,\omega^2D^\top D-\varepsilon\left(\dfrac{\omega^{2\beta}}{1+\omega^{2(\alpha+\beta-1)}}\right)I_m$ $\forall\omega\in\mathbb{R}$ [follows directly from Case I on setting $a=0$].

**Case I** and **Case II** together complete the proof. ■

### 5.1 *Frequency-domain dissipativity of stable* TD-IONI *systems: A necessary and sufficient result*

For stable systems, the proposed idea of "shifted $(Q_a(\omega), S_a(\omega), R_a(\omega))$-dissipativity" boils down to the conventional $(Q(\omega), S(\omega), R(\omega))$-dissipativity. This subsection shows that for stable TD-IONI$_{(\delta,\varepsilon,\alpha,\beta)}$ systems, the notion of "shifted $(Q_a(\omega), S_a(\omega), R_a(\omega))$-dissipativity" is not only a sufficient-type result, implied by the time-domain dissipative supply rate $w(u,\bar{u},\dot{\bar{y}})=2\dot{\bar{y}}^\top u-\delta\dot{\bar{y}}^\top\dot{\bar{y}}-\varepsilon\bar{u}^\top\bar{u}$, but it is also a necessary and sufficient property. To prove this claim, we will specialise the time-domain dissipation inequality (15) for stable IONI$_{(\delta,\varepsilon,\alpha,\beta)}$ systems as follows: there exists a storage function $V:\mathbb{R}^{n+\hat{n}}\to\mathbb{R}_{\geq 0}$ such that

$$V(\bar{x}(0))+\int_0^\infty\left(2\dot{\bar{y}}^\top u-\delta\,\dot{\bar{y}}^\top\dot{\bar{y}}-\varepsilon\bar{u}^\top\bar{u}\right)\mathrm{d}t \geq V(\bar{x}(\infty)) \tag{16}$$

for all admissible $u\in\mathbb{L}_2^m$, since $\dot{\bar{y}}=C\dot{x}=CAx+CBu$ now belongs to the space $\mathbb{L}_2^m$ as $x\in\mathbb{L}_2^n$ subject to $u\in\mathbb{L}_2^m$ and since $\bar{u}\in\mathbb{L}_2^m$. Note also that $\bar{x}(\infty)$ exists and is finite, since both $M(s)$ and $f_s(s)$ belong to $\mathscr{RH}_\infty$. Therefore, $V(\bar{x}(\infty))$ does also exists and is a finite quantity.

**Theorem 5** *Let $M(s)\in\mathscr{RH}_\infty^{m\times m}$ be the transfer function matrix of a finite-dimensional, causal and initially relaxed system $M$. Define $D=M(\infty)$. Let $\delta\geq 0$, $\varepsilon\geq 0$, $\alpha\in\mathbb{N}$ and $\beta\in\mathbb{N}$. Then, $M$ is a stable* TD-IONI$_{(\delta,\varepsilon,\alpha,\beta)}$ *system if and only if $M$ is $(Q(\omega), S(\omega), R(\omega))$-dissipative with $Q(\omega)=-\delta\omega^2I_m$, $S(\omega)=-j\omega I_m+\delta\omega^2D$ and $R(\omega)=-\delta\omega^2D^\top D-\varepsilon\left(\dfrac{\omega^{2\beta}}{1+\omega^{2(\alpha+\beta-1)}}\right)I_m\ \forall\omega\in\mathbb{R}$.*

**Proof.** First note that $M(s)$ belongs to the stable TD-IONI$_{(\delta,\varepsilon,\alpha,\beta)}$ class implies that $M(s)$ is stable NI according to Definition 1, which in turn implies $D=M(\infty)=D^\top$ [26]. Let $f_s(s)$ be defined as in (11), $y=\mathscr{L}^{-1}[M(s)]\star u$, $\bar{y}=y-Du$ and $\bar{u}=\mathscr{L}^{-1}[f_s(s)I_m]\star u$ with $u\in\mathbb{L}_2^m$. Then,

$M(s)$ is stable TD-IONI$_{(\delta,\varepsilon,\alpha,\beta)}$

$$\Leftrightarrow \int_0^\infty\left(2\dot{\bar{y}}^\top u-\delta\,\dot{\bar{y}}^\top\dot{\bar{y}}-\varepsilon\,\bar{u}^\top\bar{u}\right)\mathrm{d}t\geq 0 \quad \forall u\in\mathbb{L}_2^m$$

[via (16) and since $V(\bar{x}(\infty))=V(0)$ as $\bar{x}(\infty)=0$]

$$\Leftrightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty}\Big[(j\omega\bar{Y}(j\omega))^*U(j\omega)+U(j\omega)^*(j\omega\bar{Y}(j\omega))- \delta\,(j\omega\bar{Y}(j\omega))^*(j\omega\bar{Y}(j\omega))-\varepsilon\,\bar{U}(j\omega)^*\bar{U}(j\omega)\Big]\mathrm{d}\omega\geq 0$$

$\forall U\in\mathscr{L}_2^m(j\mathbb{R})$ [by applying Parseval's theorem [7]]

$$\Leftrightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty}\Big[Y(j\omega)^*(-\delta\omega^2I_m)Y(j\omega)+Y(j\omega)^*(-j\omega I_m+$$

$\delta\omega^2 D)U(j\omega) + U(j\omega)^*(j\omega I_m + \delta\omega^2 D^\top)Y(j\omega) + U(j\omega)^*$
$\left\{-\delta\omega^2 D^\top D - \varepsilon\left(\frac{\omega^{2\beta}}{1+\omega^{2(\alpha+\beta-1)}}\right) I_m\right\} U(j\omega)\bigg] \mathrm{d}\omega \geq 0$
$\forall U \in \mathscr{L}_2^m(j\mathbb{R})$ [substituting $\bar{Y}(j\omega) = Y(j\omega) - DU(j\omega)$ and $\bar{U}(j\omega) = (f_s(j\omega)I_m)U(j\omega)$]

$$\Leftrightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty}\Big[Y(j\omega)^*Q(\omega)Y(j\omega) + Y(j\omega)^*S(\omega)U(j\omega) + U(j\omega)^*S(\omega)^*Y(j\omega) + U(j\omega)^*R(\omega)U(j\omega)\Big]\mathrm{d}\omega \geq 0 \quad \forall U \in \mathscr{L}_2^m(j\mathbb{R})$$

$\Leftrightarrow M$ is $(Q(\omega), S(\omega), R(\omega))$-dissipative.

This completes the proof. ■

**Remark 4** *The "shifted $(Q_a(\omega), S_a(\omega), R_a(\omega))$-dissipativity" cannot explicitly recover the point-wise frequency-domain condition $j\omega[M(j\omega) - M(j\omega)^*] - \delta\,\omega^2\bar{M}(j\omega)^*\bar{M}(j\omega) - \varepsilon\left(\frac{\omega^{2\beta}}{1+\omega^{2(\alpha+\beta-1)}}\right)I_m \geq 0\ \forall\omega \in \mathbb{R}$ because the frequencies at which $M(j\omega)$ does not exist cannot be simply discarded from within the frequency-domain integration having the range from $-\infty$ to $+\infty$. This issue has been there in the NI literature right from the beginning, but it was not explored and investigated earlier. To bypass this limitation of the frequency-domain definition, in the current paper, we have proposed a stand-alone time-domain definition for IONI systems without relying on the classical point-wise frequency-domain inequality. However, for asymptotically stable IONI systems, it recovers the frequency-domain definition - as discussed in Case II of Theorem 4.*

## 6 Asymptotic stability of a TD-ONI systems interconnection without poles at the origin

In this section, we restrict the class of TD-IONI$_{(\delta,\varepsilon,\alpha,\beta)}$ systems into the TD-ONI [5] (i.e. TD-IONI$_{(\delta\geq 0,\varepsilon=0,\alpha,\beta)}$) subset without poles at the origin, since such systems exhibit interesting closed-loop stability properties when connected in a positive feedback loop. Theorem 6 will derive the internal asymptotic stability conditions for a positive feedback interconnection (see Fig. 1) of a TD-ONI system $M$ without poles at the origin and a stable TD-OSNI system $N$. Let $M$ and $N$ be described by the following state-space equations:

$$M: \begin{cases} \dot{x}_1 = A_1x_1 + B_1u_1, \quad x_{1,0} = x_1(0); \\ y_1 = C_1x_1 + D_1u_1; \end{cases}$$

and

$$N: \begin{cases} \dot{x}_2 = A_2x_2 + B_2u_2, \quad x_{2,0} = x_2(0); \\ y_2 = C_2x_2 + D_2u_2; \end{cases}$$

where $D_1 = D_1^\top$, $D_2 = D_2^\top$, $\det[A_1] \neq 0$ and $A_2$ is Hurwitz. Now, by specialising the dissipative characterisation of TD-IONI$_{(\delta,\varepsilon,\alpha,\beta)}$ systems to TD-ONI systems without poles at the origin, we can say that there exist two positive definite storage functions $V_1(x_1)$ and $V_2(x_2)$ such that $M$ satisfies

$$2\dot{y}_1^\top u_1 - \delta_1\dot{y}_1^\top\dot{y}_1 \geq \dot{V}_1(x_1) \quad (17)$$

for some $\delta_1 \geq 0$ and $N$ satisfies

$$2\dot{y}_2^\top u_2 - \delta_2\dot{y}_2^\top\dot{y}_2 \geq \dot{V}_2(x_2) \quad (18)$$

for some $\delta_2 > 0$.

The following technical lemma is an essential prerequisite to prove the internal asymptotic stability of an unforced TD-ONI interconnection without poles at the origin. Lemma 4 provides the criteria to be imposed on the TD-ONI systems under consideration to ensure that the feedback interconnection (Fig. 1) does not have any pole on the $j\omega$-axis for any $\omega \in (0, \infty)$.

**Lemma 4** *[24] Let $M(s)$ be a (not necessarily stable)* TD-ONI *system without poles at the origin and $N(s)$ be a stable* TD-OSNI *system. Let $[I - M(s)N(s)]$ have full normal rank. Let $\Omega = \{\omega \in (0,\infty) : s = j\omega$ is not a pole of $M(s)\}$ and*

$$j\,[N(j\omega_0) - N(j\omega_0)^*] > 0 \qquad \forall\omega_0 \in (0,\infty)\backslash\Omega. \quad (19)$$

*Finally, let there exist no $\omega \in \Omega$ such that* $\det[M(j\omega) - M(j\omega)^*] = 0$ *and* $\det[N(j\omega) - N(j\omega)^*] = 0$*. Then, $[I - M(s)N(s)]$ does not have any transmission zero on the $j\omega$-axis for any $\omega \in (0, \infty)$.*

Theorem 6 derives a set of sufficient conditions to ensure the closed-loop asymptotic stability of the origin of an unforced positive feedback interconnection (see Fig. 1) of two TD-ONI systems without any poles at the origin. In contrast to the conventional approach for proving the closed-loop stability results of interconnected NI and SNI/OSNI systems, as followed in [26, 41, 25] and [24], this paper utilises a completely different approach relying on the dissipative characterisation of the TD-ONI systems and the Lyapunov stability concept. Theorem 6 covers the stability proof for an NI-OSNI interconnection (refer to Case I of the proof of Theorem 6) as well as an NI-SNI interconnection (refer to Case II). In the context of proving the stability of an NI-SNI interconnection using Lyapunov approach, we want to refer to the proof of [15, Theorem 1], which followed a similar approach to us but some of the steps were conceptually incorrect [in (12), given in page 3427 of [15], $V(0) = V(x_0) = V(x(t=0)) = 0$ is taken to show

[5] Note that for TD-ONI systems, that is TD-IONI$_{(\delta\geq 0,\varepsilon=0,\alpha,\beta)}$, the parameters $\alpha$ and $\beta$ become irrelevant since $\varepsilon = 0$ has been imposed. Accordingly, the TD-OSNI subset is also characterised with $\varepsilon = 0$ in this section.

$\tilde{y}_2 = 0$; however $V(x_0)$ cannot be zero since the proof needs non-zero initial condition]. Our proof corrects that part. Besides, the proposed theorem (Theorem 6) also removes the restrictions $D_2 \geq 0$ and $D_1 D_2 = 0$ used in [15, Theorem 1] to prove the closed-loop stability of an NI-SNI interconnection. It therefore has the advantage of capturing two bi-proper systems (for which $D_1 D_2 \neq 0$) together in an interconnection.

Before presenting the theorem, we provide another technical lemma that will be used in showing the positive definiteness of the matrix $\mathscr{T}$ in (20) [appearing in the step (22)] required to establish the positive definiteness of the closed-loop Lyapunov function $V_{cl}(x_1, x_2)$ in the proof of Theorem 6.

**Lemma 5** *Let $M(s)$ and $N(s)$ have minimal state-space realisations $(A_1, B_1, C_1, D_1)$ and $(A_2, B_2, C_2, D_2)$ respectively, where $D_1 = D_1^\top$, $D_2 = D_2^\top$, $\det[A_1] \neq 0$ and $A_2$ is Hurwitz. Let also $P_1 = P_1^\top > 0$ and $P_2 = P_2^\top > 0$. Denote $V = I - D_2 D_1$ and $U = I - D_1 D_2$. Then, the matrix*

$$\mathscr{T} = \begin{bmatrix} P_1 - C_1^\top D_2 U^{-1} C_1 & -C_1^\top V^{-1} C_2 \\ -C_2^\top U^{-1} C_1 & P_2 - C_2^\top U^{-1} D_1 C_2 \end{bmatrix} > 0 \tag{20}$$

*if and only if*

$$\begin{cases} \det[I - M(\infty)N(\infty)] \neq 0, \\ \lambda_{\max}\left[(I - M(\infty)N(\infty))^{-1}(M(\infty)N(0) - I)\right] < 0, \\ \lambda_{\max}\left[(I - N(0)M(\infty))^{-1}(N(0)M(0) - I)\right] < 0. \end{cases} \tag{21}$$

**Proof.** The proof proceeds along a sequence of if and only if matrix manipulations, as shown in the proof of [25, Theorem 9], which took the inspiration from [26, Theorem 5]. ■

**Theorem 6** *Let $M(s)$ be a (not necessarily stable)* TD-ONI *system without poles at the origin and $N(s)$ be a stable* TD-OSNI *or an SNI system. Let $\Omega = \{\omega \in (0, \infty) : s = j\omega$ is not a pole of $M(s)\}$ and $j\,[N(j\omega_0) - N(j\omega_0)^*] > 0$ $\forall \omega_0 \in (0, \infty)\backslash\Omega$. Suppose there exists no $\omega \in \Omega$ such that $\det[M(j\omega) - M(j\omega)^*] = 0$ and $\det[N(j\omega) - N(j\omega)^*] = 0$. Suppose further that* (21) *holds. Then, the unforced positive feedback interconnection of $M(s)$ and $N(s)$, shown in Fig. 1, is internally asymptotically stable.*

**Proof:** Let there exist two storage functions $V_1 = x_1^\top P_1 x_1$ with $P_1 = P_1^\top > 0$ and $V_2 = x_2^\top P_2 x_2$ with $P_2 = P_2^\top > 0$ such that $M$ and $N$ satisfy (17) and (18) respectively. Let $M$ and $N$ have minimal state-space realisations $(A_1, B_1, C_1, D_1)$ and $(A_2, B_2, C_2, D_2)$, respectively, where $D_1 = D_1^\top$, $D_2 = D_2^\top$, $\det[A_1] \neq 0$ and $A_2$ is Hurwitz. In the following proof, we will use the shorthand $V = I - D_2 D_1$ and $U = I - D_1 D_2$.

**Case I** – When $M(s)$ has poles on the imaginary axis and $N(s)$ is stable TD-OSNI:

Suppose $M(s)$ has $K \in \mathbb{N}$ non-repeated pole-pairs on the $j\omega$-axis. The assumption $j\,[N(j\omega_0) - N(j\omega_0)^*] > 0$ $\forall \omega_0 \in (0, \infty)\backslash\Omega$ implies $\det[N(j\omega_0)] \neq 0$ via Lemma 4 which in turn implies that for all $\omega_0 \in (0, \infty)\backslash\Omega$, $s = j\omega_0$ is not a transmission zero of $N(s)$. This hence prevents any pole-zero cancellation of $M(s)N(s)$ at $s = j\omega_0$ for all $\omega_0 \in (0, \infty)\backslash\Omega$ since $N(s)$ has no poles nor transmission zeros at $s = j\omega_0$ for all $\omega_0 \in (0, \infty)\backslash\Omega$. For the rest of the frequencies $\omega \in \Omega$, no pole-zero cancellation occurs since $N(s) \in \mathscr{RH}_\infty^{m \times m}$ and $M(s)$ does not have any pole at $s = j\omega$ for all $\omega \in \Omega$. Furthermore, for all $\{s \in \mathbb{C} : \Re[s] > 0\} \cup \{0\}$, no pole-zero cancellation can occur in $M(s)N(s)$ as $N(s) \in \mathscr{RH}_\infty$ and $M(s)$ has no poles in $\{s \in \mathbb{C} : \Re[s] > 0\} \cup \{0\}$. Hence, $M(s)N(s)$ has no pole-zero cancellation in the entire closed right-half plane.

Now, a specific storage function $V_{cl}(x_1, x_2)$ is defined for the closed-loop system and expanded as shown below:

$$\begin{aligned}
& V_{cl}(x_1, x_2) \\
&= V_1(x_1) + V_2(x_2) - y_1^\top y_2 - \bar{y}_1^\top \bar{y}_2 + y_1^\top D_2 D_1 y_2 \\
&= x_1^\top P_1 x_1 + x_2^\top P_2 x_2 - y_1^\top V y_2 - \bar{y}_1^\top \bar{y}_2 \\
&= x_1^\top P_1 x_1 + x_2^\top P_2 x_2 - \Big[ x_1^\top C_1^\top U^{-\top} C_2 x_2 + \\
&\qquad x_2^\top C_2^\top D_1^\top U^{-\top} C_2 x_2 + x_1^\top C_1^\top U^{-\top} D_2 C_1 x_1 + \\
&\qquad x_2^\top C_2^\top D_1^\top U^{-\top} D_2 C_1 x_2 \Big] - x_1^\top C_1^\top C_2 x_2 \\
&= x_1^\top P_1 x_1 + x_2^\top P_2 x_2 - x_1^\top C_1^\top U^{-\top} C_2 x_2 - \\
&\qquad x_2^\top C_2^\top D_1^\top U^{-\top} C_2 x_2 - x_1^\top C_1^\top U^{-\top} D_2 C_1 x_1 - \\
&\qquad x_1^\top C_1^\top [I + D_2^\top U^{-1} D_1] C_2 x_2 \\
&= x_1^\top P_1 x_1 + x_2^\top P_2 x_2 - x_1^\top C_1^\top U^{-\top} C_2 x_2 - \\
&\qquad x_2^\top C_2^\top D_1^\top U^{-\top} C_2 x_2 - x_1^\top C_1^\top U^{-\top} D_2 C_1 x_1 - \\
&\qquad x_1^\top C_1^\top \big[(I - D_2 D_1)^{-1}(I - D_2 D_1) + \\
&\qquad\qquad D_2 (I - D_1 D_2)^{-1} D_1\big] C_2 x_2 \\
&= x_1^\top P_1 x_1 + x_2^\top P_2 x_2 - x_1^\top C_1^\top U^{-\top} C_2 x_2 - \\
&\qquad x_2^\top C_2^\top D_1^\top U^{-\top} C_2 x_2 - x_1^\top C_1^\top U^{-\top} D_2 C_1 x_1 - \\
&\qquad x_1^\top C_1^\top V^{-1} C_2 x_2 \\
&\qquad [\text{since } D_2 (I - D_1 D_2)^{-1} D_1 = (I - D_1 D_2)^{-1} D_2 D_1] \\
&= \begin{bmatrix} x_1^\top & x_2^\top \end{bmatrix} \begin{bmatrix} P_1 - C_1^\top D_2 U^{-1} C_1 & -C_1^\top V^{-1} C_2 \\ -C_2^\top U^{-1} C_1 & P_2 - C_2^\top U^{-1} D_1 C_2 \end{bmatrix} \times \\
&\qquad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} > 0 \qquad \forall \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \neq 0
\end{aligned} \tag{22}$$

due to satisfying (21) and relying on Lemma 5. Therefore, $V_{cl}(x_1, x_2)$ qualifies to be a Lyapunov function for the closed-loop system. The time-derivative of

$V_{cl}(x_1,x_2)$ is computed below:

$$\begin{aligned}
&\dot{V}_{cl}(x_1,x_2)\\
&= \dot{V}_1(x_1)+\dot{V}_2(x_2)-\dot{y}_1^\top y_2 - y_1^\top \dot{y}_2 - \dot{\bar{y}}_1^\top \bar{y}_2 - \\
&\quad \bar{y}_1^\top \dot{\bar{y}}_2 + \dot{y}_1^\top D_2 D_1 y_2 + y_1^\top D_2 D_1 \dot{y}_2\\
&= \dot{V}_1(x_1)+\dot{V}_2(x_2)-\Big(\dot{\bar{y}}_1^\top u_1 + \dot{y}_2^\top D_1 y_2 + \dot{\bar{y}}_2^\top u_2 + \dot{y}_1^\top D_2 y_1\\
&\quad + \dot{\bar{y}}_1^\top u_1 - \dot{y}_1^\top D_2 y_1 + \dot{y}_2^\top D_1 D_2 y_1 + \dot{\bar{y}}_2^\top u_2 - \dot{y}_2^\top D_1 y_2\\
&\quad + \dot{y}_1^\top D_2 D_1 y_2\Big) + \dot{y}_1^\top D_2 D_1 y_2 + \dot{y}_2^\top D_1 D_2 y_1\\
&= \dot{V}_1(x_1)+\dot{V}_2(x_2) - 2\dot{\bar{y}}_1^\top u_1 - 2\dot{\bar{y}}_2^\top u_2\\
&= \Big(\dot{V}_1(x_1) - 2\dot{\bar{y}}_1^\top u_1\Big) + \Big(\dot{V}_2(x_2) - 2\dot{\bar{y}}_2^\top u_2\Big)\\
&\le \; -\delta_1 \dot{\bar{y}}_1^\top \dot{\bar{y}}_1 - \delta_2 \dot{\bar{y}}_2^\top \dot{\bar{y}}_2 \qquad \text{[via (17) and (18)].} \qquad (23)
\end{aligned}$$

Therefore $\dot{V}_{cl}(x_1,x_2) \le 0$ since $\delta_1 \ge 0$ and $\delta_2 > 0$. This implies boundedness of the states $x_1(t)$ and $x_2(t)$ for all $t \ge 0$. In order to establish the asymptotic convergence of $x_1(t)$ and $x_2(t)$ towards the origin, we will show that no poles of the closed-loop system lie on the $j\omega$-axis, which means that the closed-loop system matrix

$$A_{cl} = \begin{bmatrix} A_1 & B_1C_2 \\ 0 & A_2 \end{bmatrix} + \begin{bmatrix} B_1D_2 \\ B_2 \end{bmatrix} (I - D_1D_2)^{-1} \begin{bmatrix} C_1 & D_1C_2 \end{bmatrix}$$

is Hurwitz. Lemma 4 is then invoked to ensure that $[I - M(s)N(s)]^{-1}$ does not contain any pole on the $j\omega$-axis for any $\omega \in (0,\infty)$. Subsequently, the condition $\det[N(0)M(0) - I] \neq 0$, implied from (21), ensures that $[I - M(s)N(s)]^{-1}$ does not have any pole at $s = 0$ via [42, Lemma 3.38] on noting that $[I - M(s)N(s)]$ does not have any pole at $s = 0$. Combining all the preceding arguments, it can be concluded that the unforced closed-loop system is internally asymptotically stable.

**Case II** – When $M(s)$ has poles on the imaginary axis and $N(s)$ is SNI:

Note that for the SNI systems that are also TD-OSNI, the proof remains the same as in Case I. For the rest of the SNI systems, we proceed as follows. Since SNI systems are asymptotically stable and cannot contain any purely imaginary zeros for $\omega \in (0,\infty)$, no pole-zero cancellation of the loop transfer function $M(s)N(s)$ can occur in the entire closed right-half plane. The Lyapunov inequality, derived in (23), still holds when $N(s)$ is SNI and implies $\dot{V}_{cl}(x_1,x_2) \le 0$ since $\delta_1 = \delta_2 = 0$. This implies the boundedness of the states. Then, [25, Lemma 6] is exploited to prove that $[I - M(s)N(s)]^{-1}$ does not contain any pole on the $j\omega$-axis for any $\omega \in (0,\infty)$. Finally, the set of conditions (21) guarantees that $[I - M(s)N(s)]^{-1}$ does not have any pole at $s = 0$, as explained in the last part of Case I. These arguments jointly establish that the unforced positive feedback interconnection of a TD-ONI system without poles at the origin and an SNI system is internally asymptotically stable.

**Case III** – When $M(s)$ is stable TD-ONI and $N(s)$ is either stable TD-OSNI or SNI:

Since in this case, both $M(s)$ and $N(s)$ belong to $\mathscr{RH}_\infty^{m\times m}$, no pole-zero cancellation of the loop transfer function $M(s)N(s)$ can occur in the entire closed right-half plane. The rest of the proof follows from Case I and Case II and it re-establishes the sufficiency parts of proofs of [6, Theorem 1] and [25, Theorem 5] respectively.

**Case I**, **Case II** and **Case III** complement each other to prove the theorem. ■

The following corollary is an immediate consequence of Theorem 6 under the additional constraints on the gains of the systems at infinite frequency. It offers an appealing and more elegant 'DC loop gain' condition for checking the internal stability of a TD-ONI interconnection without poles at the origin.

**Corollary 1** *Let $M(s)$ be a (not necessarily stable)* TD-ONI *system without poles at the origin and $N(s)$ be a stable* TD-OSNI *or an SNI system. Let $\Omega = \{\omega \in (0,\infty) : s = j\omega$ is not a pole of $M(s)\}$ and $j\,[N(j\omega_0) - N(j\omega_0)^*] > 0\ \forall \omega_0 \in (0,\infty)\backslash\Omega$. Suppose there exists no $\omega \in \Omega$ such that $\det[M(j\omega) - M(j\omega)^*] = 0$ and $\det[N(j\omega) - N(j\omega)^*] = 0$. Suppose further that either $N(\infty) \ge 0$ and $M(\infty)N(\infty) = 0$, or else $M(\infty) = 0$, and $\lambda_{\max}\big[N(0)M(0)\big] < 1$. Then, the unforced positive feedback interconnection of $M(s)$ and $N(s)$, shown in Fig. 1], is asymptotically stable.*

**Proof.** The proof readily follows from Theorem 6 subject to the additional constraints that either $N(\infty) \ge 0$ and $M(\infty)N(\infty) = 0$, or else $M(\infty) = 0$. ■

**Example 8** *Consider a positive feedback interconnection of* $M(s) = \dfrac{s^2+4s+12}{5(s+2)(s^2+4)}\begin{bmatrix} 5 & -2 \\ -2 & 5 \end{bmatrix}$ *being a* TD-ONI *system without poles at the origin and* $N(s) = \dfrac{0.25s^2+3.75}{s^4+0.25s^3+25s^2+3.75s+83.33}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ *being a stable* TD-OSNI *system. It is checked that both $[M(s) - M(s)^*]$ and $[N(s) - N(s)^*]$ have full normal rank, $M(s)N(s)$ does not have any pole-zero cancellation in the entire closed right-half plane, $j\,[N(j\omega) - N(j\omega)^*] > 0$ at $\omega = 2$ rad/s, $j\,[M(j\omega) - M(j\omega)^*] > 0$ at $\omega = \sqrt{15}$ rad/s and there does not exist any $\omega_z \in (0,\infty)$ such that $\det[M(\omega_z) - M(\omega_z)^*] = 0$ and $\det[N(\omega_z) - N(\omega_z)^*] = 0$. Thus, $M(s)$ and $N(s)$ satisfy all the assumptions of*

*Theorem 6. Finally, we test the DC loop gain condition* $\lambda_{\max}[N(0)M(0)] = \lambda_{\max}\begin{bmatrix} 0.2295 & -0.1485 \\ -0.1485 & 0.2295 \end{bmatrix} = 0.3780 < 1$. *Therefore, the unforced positive feedback interconnection of $M(s)$ and $N(s)$ is guaranteed to be asymptotically stable via Theorem 6.*

## 7 Conclusions

This paper introduces the notion of Time-Domain Input-Output Negative Imaginary systems [denoted by TD-IONI$_{(\delta,\varepsilon,\alpha,\beta)}$], which unifies all the existing versions of the input and/or output negative imaginary systems, including those with poles on the $j\omega$-axis [35, 10, 6, 4, 5]. TD-IONI$_{(\delta,\varepsilon,\alpha,\beta)}$ systems have been defined in the spirit of a new time-domain dissipative supply rate $w(u, \bar{u}, \dot{\bar{y}})$ instead of relying on conventional frequency-domain definitions. A new dynamic-dissipative framework, termed as $(\Sigma_{IONI}, \bar{Q})$-dissipativity, is developed for characterising and classifying the TD-IONI$_{(\delta,\varepsilon,\alpha,\beta)}$ systems, that leads to a necessary and sufficient LMI condition (referred to as the TD-IONI$_{(\delta,\varepsilon,\alpha,\beta)}$ lemma). The proposed lemma does not impose any *a priori* restrictions (such as stability, minimality, full normal rank constraint, etc. – commonly used in the NI literature) on the system and thereby, it captures the earlier results. This paper explores the fundamental relationship amongst TD-IONI$_{(\delta,\varepsilon,\alpha,\beta)}$ systems, dynamic dissipativity and classical dissipativity (in the sense of Willems [39]). Subsequently, a new frequency-domain dissipative supply rate, termed as the "shifted $(Q_a(\omega), S_a(\omega), R_a(\omega))$-dissipativity", is proposed to characterise the whole class of the TD-IONI$_{(\delta,\varepsilon,\alpha,\beta)}$ systems (i.e. allowing $j\omega$-axis poles) in contrast to the conventional $(Q(\omega), S(\omega), R(\omega))$-dissipativity, which applies to only stable IONI/ONI systems [35, 10, 4]. It is also shown that for stable TD-IONI$_{(\delta,\varepsilon,\alpha,\beta)}$ systems, the "shifted $(Q_a(\omega), S_a(\omega), R_a(\omega))$-dissipativity" reduces to $(Q(\omega), S(\omega), R(\omega))$-dissipativity. Finally, the paper offers a closed-loop stability theorem for an unforced positive feedback interconnection containing a TD-ONI system (without poles at the origin) and a stable TD-OSNI or an SNI system. In a future scope, the $(Q_a(\omega), S_a(\omega), R_a(\omega))$-dissipative framework proposed here can be further explored to establish a strong link between NI theory and IQC theory relying on the ideas given in [22].

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