Brief paper

Robust output consensus of homogeneous multi-agent systems with negative imaginary dynamics☆

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ARTICLE INFO

ABSTRACT

This paper addresses the robust output consensus problem for multiple homogeneous negative imaginary (NI) systems subject to $\mathcal{L}_2$ external disturbances and model uncertainties. The principal aim of this paper is to extend and overcome limitations in previous studies which had required restrictive assumptions to be imposed and thereby derive necessary and sufficient conditions for robust output consensus of networked NI systems when such assumptions do not hold. In doing so, the recently published robust NI stability results are utilised. An advantage of the derived conditions is that they depend on the dc and infinite frequency gains of the systems as well as the network graph but not on the precise dynamics of the systems. Furthermore, it is shown that the derived conditions specialise to the earlier results in literature either by imposing the same assumptions or by imposing different ones which had not been known previously. It is also shown that the derived conditions simplify in the single-input single-output case and an intuition about these results is provided. Finally, examples are given which demonstrate the effectiveness of the proposed results over earlier results when earlier assumptions are not met.

1. Introduction

In recent years, there has been an increased emphasis on the study of negative imaginary (NI) systems. Much of the literature on NI systems has emphasised on robust stability analysis such as the early work in Lanzon and Petersen (2008) and Xiong, Petersen, and Lanzon (2010), and more recent work in Lanzon and Chen (2017) and Mabrok, Kallapur, Petersen, and Lanzon (2014); controller synthesis by reformulating the problem into a bounded real framework (Song, Lanzon, Patra, & Petersen, 2010), via state feedback (Mabrok, Kallapur, Petersen, & Lanzon, 2015; Song, Lanzon, Patra, & Petersen, 2012), via output feedback (Xiong, Lam, & Petersen, 2016), and via a data driven approach (Mabrok & Petersen, 2016); and applications to multi-agent systems (MAS) (Chen, Lanzon, & Petersen, 2017; Skeik & Lanzon, 2019; Wang, Lanzon, & Petersen, 2015a, 2015b). This paper contributes to the existing literature on cooperative control of multiple NI systems. (Robust) cooperative control of multiple NI systems is motivated by applications where an individual NI system cannot achieve a desired collective behaviour on its own. Consensus, where agents cooperate to reach an agreement, is one of the most important and desirable collective behaviours due to the potential real-world applications it may have Chen, Lu, Yu, and Hill (2013). Consensus of MAS has been studied widely by many researchers. In terms of agents dynamics both homogeneous and heterogeneous dynamics have been considered and in terms of the shared information both state and output information have been considered such as, for example, Knorn, Chen, and Middleton (2016), Li, Soh, and Xie (2017), Olfati-Saber and Murray (2004) and Wang et al. (2015a, 2015b). In this paper, we address the robust output consensus problem for multiple homogeneous NI systems. First, we focus here on homogeneous NI dynamics since the null space property of the Laplacian matrix, by which a collective behaviour is governed by, only exists for homogeneous dynamics. Furthermore, although Wang et al. (2015b) overcome this issue and consider heterogeneous NI dynamics it comes at the expense of providing some robust output consensus conditions that are sufficient but not necessary. Therefore, by considering homogeneous NI dynamics, we are able here to obtain necessary and sufficient conditions for robust output consensus. Second, we use here relative output measurements as it is practically more significant since full state information is not always available.

In Li, Duan, Chen, and Huang (2010), Li, Liu, Lin, and Ren (2011) and Li et al. (2017) consensus problems for homogeneous MAS using relative output measurements were addressed. However, unlike Li et al. (2010, 2011) and Li et al. (2017) which use state space techniques and observer-based consensus protocols where protocol design involves the solution of Riccati equations or/and linear matrix inequalities, the robust output consensus conditions we propose here are much simpler since they depend on the dc and infinite frequency gains of the systems as well as the network graph but not on the precise dynamics of the systems. Meanwhile, most closely related to our work is Wang et al. (2015a) and the motivations for this current study come from the importance of consensus of MAS in real-world applications (Chen et al., 2013), the many practical systems that can be modelled as NI systems (Lanzon & Petersen, 2008; Mabrok et al., 2014), and the establishment of the general internal stability results in Lanzon and Chen (2017) by which it is possible to extend the work of Wang et al. (2015a). While the work of Wang et al. (2015a) has successfully considered the robust output consensus problem for networked homogeneous NI systems, it has certain limitations in terms of the imposed assumptions. A principal limitation of Wang et al. (2015a) is that for NI systems with no poles at the origin, two assumptions at infinite frequency need to hold before the robust output consensus condition can be considered, while for systems with poles at the origin, the NI systems are limited to being strictly proper, matrix factorisation is required, and null space conditions need to be satisfied before the robust output consensus conditions can be considered. As a result, it not possible to determine robust output consensus for networked NI systems with the results of Wang et al. (2015a) when such assumptions do not hold. In this paper, we build on and extend the work of Wang et al. (2015a). Similar to Wang et al. (2015a) we address the robust output consensus problem as an internal stability problem for networked NI systems subject to external disturbances and model uncertainties but different from Wang et al. (2015a) we use the generalised internal stability results in Lanzon and Chen (2017) to do so rather than those in Lanzon and Petersen (2008), Mabrok et al. (2014) and Xiong et al. (2010) thereby extending the results of Wang et al. (2015a). Therefore, the advantages of this current work over (Wang et al., 2015a) and the main contributions of this paper to existing knowledge are summarised as follows: (i) we relax the assumptions imposed in Wang et al. (2015a) thereby derive robust output consensus conditions which are not restricted; (ii) one distinct advantage that unfolds in our work is that not only do the derived conditions specialise to those in Wang et al. (2015a) by imposing the same two assumptions at infinite frequency but also specialise to those in Wang et al. (2015a) by imposing different assumptions which were unknown in Wang et al. (2015a); (iii) the derived conditions simplify in the single-input single-output (SISO) case providing several insights which are not easily captured in the multi-input multi-output (MIMO) case (for SISO NI systems with no poles at origin) and are less sensitive to the network graph that models the interconnection of the systems (for SISO NI systems with poles at origin); and (iv) we show that consensus for some networked NI systems including a network of robotic arms cannot be determined by the results in Wang et al. (2015a) but can easily be concluded via the results of this paper.

Notation: Let $\mathbb{R}^{m \times n}$ denote the set of $m \times n$ real matrices. Given a matrix A, A^T and A^* are the transpose and the complex conjugate transpose of A respectively. $\bar{\lambda}(A)$ denotes the largest eigenvalue of A (when the matrix A has only real eigenvalues). $\Re[\cdot]$ is the real part of a complex number. I_N is the identity matrix of dimension $N \times N$ and 1_N is an $N \times 1$ vector with entries 1. $A \otimes B$ denotes the Kronecker product of matrices A and B. $\text{diag}(A_i)$ represents a block-diagonal matrix with matrices A_i for all $i \in \{1, \ldots, N\}$ on

the main diagonal. $\mathfrak{L}_2^n[0, \infty)$ denotes the n-dimensional square integrable function space. $[P, K]$ represents a positive feedback interconnection between systems P and K.

2. Preliminaries

2.1. Negative imaginary systems

Negative imaginary systems are defined as follows.

Definition 1 (*Mabrok et al., 2014*). A square, real, rational, proper transfer function matrix $P(s)$ is said to be negative imaginary if

(1) $P(s)$ has no poles in $\Re[s] > 0$;
(2) $j[P(j\omega) - P(j\omega)^*] \geq 0$ for all $\omega \in (0, \infty)$ except values of ω where $j\omega$ is a pole of $P(s)$;
(3) if $j\omega_0$ with $\omega_0 \in (0, \infty)$ is a pole of $P(s)$, then it is a simple pole and the residue matrix $K_0 = \lim_{s \to j\omega_0}(s - j\omega_0)jP(s)$ is Hermitian and positive semidefinite;
(4) if $s = 0$ is a pole of $P(s)$, then $\lim_{s \to 0} s^k P(s) = 0 \; \forall k \geq 3$ and $\lim_{s \to 0} s^2 P(s)$ is Hermitian and positive semidefinite.

It is important to stress that the definition of NI systems was extended to non-rational systems in Ferrante, Lanzon, and Ntogramatzidis (2016) and Ferrante and Ntogramatzidis (2013). However, we here restrict our work to NI systems with rational transfer functions.

Strictly negative imaginary (SNI) systems are defined as follows.

Definition 2 (*Lanzon & Petersen, 2008*). A square, real, rational, proper transfer function matrix $K(s)$ is said to be strictly negative imaginary if

(1) $K(s)$ has no poles in $\Re[s] \geq 0$;
(2) $j[K(j\omega) - K(j\omega)^*] > 0$ for all $\omega \in (0, \infty)$.

2.2. Graph theory

An undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ consists of a nonempty finite vertex set $\mathcal{V} = \{v_1, v_2, \ldots, v_N\}$ and an edge set $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ of unordered pairs of vertices, called edges. An edge in $\mathcal{G}$ is denoted by (v_i, v_j). If $(v_i, v_j) \in \mathcal{E}$, then vertices (i.e., agents) v_i and v_j are adjacent (or neighbours) and can obtain information from each other. The set of neighbours of vertex v_i is defined as $\mathcal{N}_i = \{v_j \in \mathcal{V} : (v_j, v_i) \in \mathcal{E}\}$. Self edges are not allowed, that is, $(v_i, v_i) \notin \mathcal{E}$. A path in a graph from v_i to v_j is a sequence of edges of the form $(v_i, v_{i+1}), (v_{i+1}, v_{i+2}), \ldots, (v_{j-1}, v_j)$. An undirected graph is connected if there is an undirected path between every pair of distinct vertices. The adjacency matrix $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N \times N}$ of $\mathcal{G}$ is defined as $a_{ij} = a_{ji} = 1$ if $(v_i, v_j) \in \mathcal{E}$, 0 otherwise. The Laplacian matrix $\mathcal{L} = [l_{ij}] \in \mathbb{R}^{N \times N}$ of $\mathcal{G}$ is defined as $l_{ij} = -a_{ij}$, for $i \neq j$ and $l_{ii} = \sum_{j=1}^{N} a_{ij}$ for all $i \in \{1, \ldots, N\}$. It is well known that $\mathcal{L}$ is positive semidefinite when the graph is undirected. Furthermore, for undirected graphs, zero is a simple eigenvalue of $\mathcal{L}$ and the associated eigenvector is 1_N if and only if the undirected graph is connected (Olfati-Saber, Fax, & Murray, 2007; Ren, Beard, & Atkins, 2007). Let μ_i be the ith eigenvalue of an $\mathcal{L}$ associated with an undirected and connected graph. Then the eigenvalues of $\mathcal{L}$ can be arranged as

$$0 = \mu_1 < \mu_2 \leq \mu_3 \leq \ldots \leq \mu_N, \qquad (1)$$

and throughout this paper we will denote $\bar{\lambda}(\mathcal{L})$ as the largest eigenvalue of $\mathcal{L}$ and $\underline{\lambda}_2(\mathcal{L})$ as the second smallest eigenvalue of $\mathcal{L}$, that is $\bar{\lambda}(\mathcal{L}) = \mu_N$, and $\underline{\lambda}_2(\mathcal{L}) = \mu_2$.

3. Problem description

Consider a network of N homogeneous NI systems with external disturbances acting on each system. The dynamics of the ith NI system are described as

$$y_i = d_{o_i} + P(s)\left(d_{in_i} + u_i\right) \quad \forall i \in \{1, \ldots, N\} \tag{2}$$

where $P(s)$ is an $n \times n$ transfer function matrix of the ith NI system, u_i, y_i, d_{in_i} and d_{o_i} are all vector signals with "n" elements and d_{in_i} and d_{o_i} are also energy-bounded in an H_2 (or in the time domain $\mathfrak{L}_2[0, \infty)$) sense. The signals u_i, y_i, d_{in_i} and d_{o_i} denote control input, output of the ith NI system, input and output disturbances respectively. It is assumed that relative output measurements with respect to neighbouring agents are available to each system. The network graph which models the information exchange among the systems is assumed fixed and satisfies the following assumption:

Assumption 3. The network graph $\mathcal{G}$ is undirected and connected.

Following Wang et al. (2015a), the distributed control protocol for the ith NI system is given by

$$
\begin{aligned}
u_i &= K(s)z_i, \\
z_i &= \sum_{j=1}^{N} a_{ij}(y_i - y_j), \quad \forall i \in \{1, \ldots, N\}
\end{aligned}
\tag{3}
$$

where $K(s)$ is the transfer function matrix of an SNI feedback controller, z_i represents the signal of relative measurements of neighbouring agents with respect to system i and a_{ij} denotes the elements of the adjacency matrix associated with the network graph $\mathcal{G}$. The collective network dynamics can thus be written as

$$y = d_o + (I_N \otimes P(s))(d_{in} + u), \tag{4}$$

and

$$
\begin{aligned}
u &= (I_N \otimes K(s))z, \\
z &= (\mathcal{L} \otimes I_n)y,
\end{aligned}
\tag{5}
$$

where $z = \left[z_1^T, \ldots, z_N^T\right]^T$, $y = \left[y_1^T, \ldots, y_N^T\right]^T$, $u = \left[u_1^T, \ldots, u_N^T\right]^T$, $d_{in} = \left[d_{in_1}^T, \ldots, d_{in_N}^T\right]^T$ and $d_o = \left[d_{o_1}^T, \ldots, d_{o_N}^T\right]^T$ are all vector signals with "nN" elements and d_{in} and d_o are also energy-bounded in an H_2 (or in the time domain $\mathfrak{L}_2[0, \infty)$) sense. $\mathcal{L} \in \mathbb{R}^{N \times N}$ is the Laplacian matrix associated with the network graph $\mathcal{G}$. A block diagram of the closed loop networked MAS is depicted in Fig. 1. This figure represents the block diagram of the real physical system with real disturbances. In this paper we address the robust output consensus problem for networks of NI systems as an internal stability problem. To this end, via block diagram algebra, it is possible to move the block $(\mathcal{L} \otimes I_n)$ in Fig. 1 right past the summing junction which leads to $(\mathcal{L} \otimes I_n)d_o$ being the disturbance acting on signal z. Let $w_o = (\mathcal{L} \otimes I_n)d_o$. w_o is a subset of the disturbances acting on signal y in Fig. 1 due to $\mathcal{L}$ being rank deficient. Let $\bar{P}(s) = (\mathcal{L} \otimes I_n)(I_N \otimes P(s))$ denote the transfer function matrix from u_p to z and let $\bar{K}(s) = I_N \otimes K(s)$ denote the transfer function matrix from z to u. According to Wang et al. (2015a, Lemma 3), $\bar{P}(s)$ is NI if and only if $P(s)$ is NI with $\mathcal{G}$ satisfying Assumption 3. Similarly, $\bar{K}(s)$ is SNI since $K(s)$ is SNI. Then, the internal stability framework we consider, for addressing the output consensus problem, is given in Fig. 2. That is, we address the output consensus problem as an internal stability problem for the interconnection $[\bar{P}(s), \bar{K}(s)]$ where the plant $\bar{P}(s)$ is black-boxed thus we are being silent about the signal y_p.
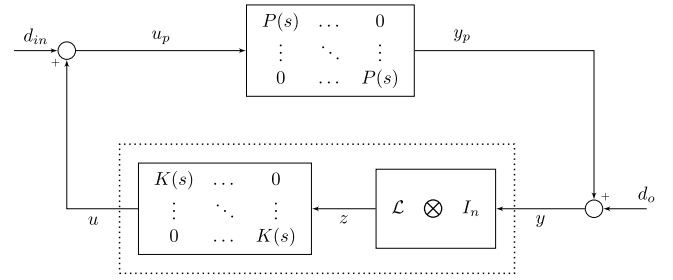

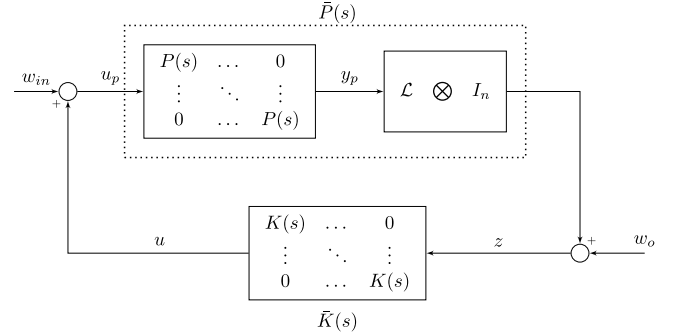**Fig. 1.** Real physical system with real disturbances.

Fig. 2. Internal stability framework.

Remark 4. Internal stability of the interconnection $[\bar{P}, \bar{K}]$ guarantees that for all bounded inputs (w_{in}, w_o), the outputs (u_p, z) are bounded. (see e.g. Zhou, Doyle, & Glover, 1996). Then, internal stability on the signals u_p and z is equivalent to consensus on signal y in Fig. 1 via properties of $\mathcal{L}$ and via rank deficiency in the matrix $\mathcal{L} \otimes I_n$. On the other hand, internal stability of the interconnection $[\bar{P}, \bar{K}]$ does not imply asymptotic stability of the state space description; since $\bar{P}(s)$ is unobservable. Thus, the nature of the output signal y (or y_p) is depended on the block $(I_N \otimes P(s))$ and hence the final convergence trajectory of the output will depend on the dynamics of $P(s)$ as will be seen in the examples provided in Section 5 (see also Skeik & Lanzon, 2018, Sec.V for more details regarding convergence analysis).

The following remark discusses how model uncertainties are captured in this framework.

Remark 5. Model uncertainties are captured in this framework by noting that any additive NI perturbations to a nominal NI system result in an NI perturbed system. Other forms of feedback uncertainties are also possible that preserve NI properties (see e.g. Chen et al., 2017; Ferrante et al., 2016). Hence, $P(s)$ is regarded interchangeably as a nominal or perturbed plant as long as it fulfils the robust output consensus conditions (see Skeik & Lanzon, 2018, Remark. 1 for more details).

The robust output consensus problem we consider is defined as follows.

Definition 6 (*Wang et al., 2015a, 2015b*). For a family of NI plant dynamics and for all $\mathfrak{L}_2[0, \infty)$ disturbances acting on the plant input and/or output, robust output consensus is said to be achieved with distributed control protocol (3) for a network of NI systems if there exists $\epsilon_i(t) \in \mathfrak{L}_2[0, \infty)$ $\forall i \in \{1, \ldots, N\}$ such that $y_i(t) \to y_{ss}(t) + \epsilon_i(t)$ $\forall i \in \{1, \ldots, N\}$, where $y_{ss}(t)$ is the final convergence trajectory. Note that $\epsilon_i(t) = 0$ $\forall t$ and $\forall i \in \{1, \ldots, N\}$ when there are no external disturbances.

Our objective is to derive conditions for robust output consensus of a network of homogeneous NI systems under $\mathfrak{L}_2$ external disturbances and model uncertainty by using the general internal stability results in Lanzon and Chen (2017).

4. Robust output consensus

4.1. Networked homogeneous NI systems with no poles at the origin

The following theorem gives conditions under which robust output consensus is achieved for networked NI systems with no poles at the origin.

Theorem 7. *Consider a network of homogeneous NI systems $P(s)$ without poles at the origin, a network graph $\mathcal{G}$ that satisfies Assumption 3 and an SNI feedback controller $K(s)$ for each NI agent. Let μ_i for all $i \in \{1, \ldots, N\}$ be the eigenvalues of the Laplacian matrix $\mathcal{L}$ associated with $\mathcal{G}$ ordered as stated in (1). Then, the following three statements are equivalent:*

(a) robust output consensus is achieved via control protocol (5) for networked system (4) as shown in Fig. 1 (or in a distributed manner (3) for each system (2)) under any external disturbances $d_{in}, d_o \in \mathfrak{L}_2^{nN}[0, \infty)$ and model uncertainty that retains the NI property of the perturbed system $P(s)$;

(b) the set of conditions $I_n - \mu_i P(\infty)K(\infty)$ is nonsingular, $\bar{\lambda}[[I_n - \mu_i P(\infty)K(\infty)]^{-1}[\mu_i P(\infty)K(0) - I_n]] < 0$, and $\bar{\lambda}[[I_n - \mu_i K(0)P(\infty)]^{-1}[\mu_i K(0)P(0) - I_n]] < 0$, are satisfied for all $i \in \{2, \ldots, N\}$;

(c) the set of conditions $I_n - \mu_i P(\infty)K(\infty)$ is nonsingular, $\bar{\lambda}[[\mu_i P(0)K(\infty) - I_n][I_n - \mu_i P(\infty)K(\infty)]^{-1}] < 0$, and $\bar{\lambda}[[\mu_i K(0)P(0) - I_n][I_n - \mu_i K(\infty)P(0)]^{-1}] < 0$, are satisfied for all $i \in \{2, \ldots, N\}$.

Proof. We begin by proving the equivalence of conditions (a) and (b). Let $\bar{P}(s) = \mathcal{L} \otimes P(s)$ and $\bar{K}(s) = I_N \otimes K(s)$. Now $\bar{P}(s)$ is NI by Wang et al. (2015a, Lemma 3) and has no poles at the origin since $P(s)$ has no poles at origin. Also, $\bar{K}(s)$ is SNI since $K(s)$ is SNI. Via Remark 4 and as in the proof of Wang et al. (2015a, Th. 1), the internal stability of $[\bar{P}(s), \bar{K}(s)]$ in Fig. 2 implies output consensus (Fig. 1) when $d_{in} = d_o = 0$, by noting that $z \to 0 \Leftrightarrow y \to 1_N \otimes y_{ss}$ since Assumption 3 holds. According to Lanzon and Chen (2017, Th. 9), $[\bar{P}(s), \bar{K}(s)]$ is internally stable if and only if

$I_{Nn} - \bar{P}(\infty)\bar{K}(\infty)$ is nonsingular,

$\bar{\lambda}[[I_{Nn} - \bar{P}(\infty)\bar{K}(\infty)]^{-1}[\bar{P}(\infty)\bar{K}(0) - I_{Nn}]] < 0$, and

$\bar{\lambda}[[I_{Nn} - \bar{K}(0)\bar{P}(\infty)]^{-1}[\bar{K}(0)\bar{P}(0) - I_{Nn}]] < 0$.

Now $\mathcal{L}$ is a real symmetric matrix due to Assumption 3. Thus, $\mathcal{L}$ can be written as $\mathcal{L} = U\Lambda U^T$ where U is an orthogonal matrix and Λ is a diagonal matrix with eigenvalues of $\mathcal{L}$ on the diagonal. Then,

$I_{Nn} - \bar{P}(\infty)\bar{K}(\infty)$

$= I_{Nn} - (\mathcal{L} \otimes P(\infty))(I_N \otimes K(\infty))$

$= I_{Nn} - (\mathcal{L} \otimes P(\infty)K(\infty))$

$= I_{Nn} - (U\Lambda U^T \otimes P(\infty)K(\infty))$

$= (U \otimes I_n)[I_{Nn} - (\Lambda \otimes P(\infty)K(\infty))](U^T \otimes I_n)$

$= (U \otimes I_n)\mathrm{diag}(I_n - \mu_i P(\infty)K(\infty))(U^T \otimes I_n)$

$$\forall i \in \{1, 2, \ldots, N\}.$$

So,

$I_{Nn} - \bar{P}(\infty)\bar{K}(\infty)$ is nonsingular

$\Leftrightarrow I_n - \mu_i P(\infty)K(\infty) \ \forall i \in \{2, \ldots, N\}$ is nonsingular

(due to the fact that U and U^T are nonsingular

matrices and for $\mu_1 = 0$, I_n is nonsingular),

Furthermore,

$\bar{\lambda}[[I_{Nn} - \bar{P}(\infty)\bar{K}(\infty)]^{-1}[\bar{P}(\infty)\bar{K}(0) - I_{Nn}]] < 0$

$\Leftrightarrow \bar{\lambda}[[I_{Nn} - (\mathcal{L} \otimes P(\infty)K(\infty))]^{-1}$
$\qquad\qquad \times [(\mathcal{L} \otimes P(\infty)K(0)) - I_{Nn}]] < 0$

$\Leftrightarrow \bar{\lambda}[[I_{Nn} - (U\Lambda U^T \otimes P(\infty)K(\infty))]^{-1}$
$\qquad\qquad \times [(U\Lambda U^T \otimes P(\infty)K(0)) - I_{Nn}]] < 0$

$\Leftrightarrow \bar{\lambda}[(U \otimes I_n)[I_{Nn} - (\Lambda \otimes P(\infty)K(\infty))]^{-1}(U^T \otimes I_n)$
$\quad \times (U \otimes I_n)[(\Lambda \otimes P(\infty)K(0)) - I_{Nn}](U^T \otimes I_n)] < 0$

$\Leftrightarrow \bar{\lambda}[[I_{Nn} - (\Lambda \otimes P(\infty)K(\infty))]^{-1}$
$\qquad\qquad \times [(\Lambda \otimes P(\infty)K(0)) - I_{Nn}]] < 0$

$\Leftrightarrow \max_{i=1,\ldots,N} \bar{\lambda}[[I_n - \mu_i P(\infty)K(\infty)]^{-1}$
$\qquad\qquad\qquad \times [\mu_i P(\infty)K(0) - I_n]] < 0$

(since the matrix in the previous step is block diagonal)

$\Leftrightarrow \bar{\lambda}[[I_n - \mu_i P(\infty)K(\infty)]^{-1}[\mu_i P(\infty)K(0) - I_n]] < 0$
$$\forall i \in \{2, \ldots, N\}$$

(since for $\mu_1 = 0$, the condition is trivially fulfilled)

and

$\bar{\lambda}[[I_{Nn} - \bar{K}(0)\bar{P}(\infty)]^{-1}[\bar{K}(0)\bar{P}(0) - I_{Nn}]] < 0$

$\Leftrightarrow \bar{\lambda}[[I_n - \mu_i K(0)P(\infty)]^{-1}[\mu_i K(0)P(0) - I_n]] < 0$
$$\forall i \in \{2, \ldots, N\}$$

by following similar steps as the steps taken in the second condition (see also the steps of proof Skeik & Lanzon, 2018, Th.1). The proof for robust output consensus under external disturbances and model uncertainties then follows similarly to that in the proof of Wang et al. (2015a, Th.1) (see also Remark 5 for model uncertainties). The equivalence of conditions (a) and (c) can be proved in a similar manner by applying Lanzon and Chen (2017, Th. 14) instead of Lanzon and Chen (2017, Th. 9). ∎

Remark 8. The first and second conditions within conditions (b) of Theorem 7 guarantee that the matrix $I_n - \mu_i K(0)P(\infty)$ in the third condition is nonsingular $\forall i \in \{2, \ldots, N\}$.

Remark 9. Unlike Wang et al. (2015a), both sets of conditions (b) and (c) of Theorem 7 include the nonzero eigenvalues of the Laplacian matrix $\mathcal{L}$. Thus, it can be concluded that the nonzero eigenvalues of $\mathcal{L}$ play a central role in achieving output consensus for networks of NI systems when the assumptions $P(\infty)K(\infty) = 0$ and $K(\infty) \geq 0$ of Wang et al. (2015a) are relaxed.

The following corollary shows that the conditions of Theorem 7 not only specialise to that in Wang et al. (2015a) by imposing the same two assumptions at infinite frequency but also specialise to that in Wang et al. (2015a) by imposing different assumptions which were not known previously in Wang et al. (2015a).

Corollary 10. *Let the hypotheses of Theorem 7 hold and furthermore let either (i) $P(\infty)K(\infty) = 0$ and $K(\infty) \geq 0$, or (ii) $P(\infty)K(\infty) = 0$ and $P(0) > 0$, or (iii) $P(\infty) = 0$ hold. Then, robust output consensus is achieved via control protocol (5) for networked system (4) as shown in Fig. 1 (or in a distributed manner (3) for each system (2)) under any external disturbances $d_{in}, d_o \in \mathfrak{L}_2^{nN}[0, \infty)$ and model uncertainty that retains the NI property of the perturbed system $P(s)$ if and only if*

$$\bar{\lambda}[P(0)K(0)] < \frac{1}{\bar{\lambda}(\mathcal{L})}.$$

Proof. For the case that (i) holds, the result is a direct consequence of the equivalence of conditions (a) and (b) of Theorem 7 and the lines of the proof here follow those of Lanzon and Chen (2017, Cor. 12) with the difference of having the eigenvalues of $\mathcal{L}$ included here. Furthermore, $\bar{\lambda}[P(0)K(0)] < 1/\mu_i \ \forall i \in \{2,\dots,N\}$ $\Leftrightarrow \bar{\lambda}[P(0)K(0)] < 1/\bar{\lambda}(\mathcal{L})$ since (1) holds by Assumption 3. Also, $\bar{\lambda}[P(0)K(0)] < 1/\bar{\lambda}(\mathcal{L}) \Leftrightarrow \mu_i P(0) < K(0)^{-1} \ \forall i \Rightarrow \mu_i P(\infty) < K(0)^{-1}$ (since $P(\infty) \leq P(0)$ via Xiong et al., 2010, Cor. 3) and $\mu_i > 0 \ \forall i \in \{2,\dots,N\}$ and $\mu_1 = 0$ via Assumption 3. For the case that (ii) holds, the result is a direct consequence of the equivalence of conditions (a) and (c) of Theorem 7 and by following similar steps as in case (i) but by following similar steps as in Lanzon and Chen (2017, Cor. 15) instead. For the case that (iii) holds, the result is a direct consequence of the equivalence of conditions (a) and (b) of Theorem 7 with $P(\infty) = 0$ and by (1). $\blacksquare$

4.1.1. SISO specialisation: no poles at the origin

The following theorem shows that when the NI systems are SISO, the robust output consensus conditions of Theorem 7 can be simplified as follows.

Theorem 11. *Consider a network of homogeneous SISO NI systems $P(s)$ without poles at the origin, a network graph $\mathcal{G}$ that satisfies Assumption 3 and an SNI feedback controller $K(s)$ for each NI agent. Let μ_i for all $i \in \{1,\dots,N\}$ be the eigenvalues of the Laplacian matrix $\mathcal{L}$ associated with $\mathcal{G}$ ordered as stated in (1). Then, robust output consensus is achieved via control protocol (5) for networked system (4) as shown in Fig. 1 (or in a distributed manner (3) for each system (2)) under any external disturbances $d_{in}, d_o \in \mathfrak{L}_2^{nN}[0,\infty)$ and model uncertainty that retains the NI property of the perturbed system $P(s)$ if and only if any of the following five conditions holds:*

(1) $P(0)K(0) < 1/\bar{\lambda}(\mathcal{L})$ and $P(\infty)K(\infty) < 1/\bar{\lambda}(\mathcal{L})$;
(2) $K(\infty) > 0$ and $P(\infty)K(\infty) > 1/\underline{\lambda}_2(\mathcal{L})$;
(3) $K(0) < 0$ and $P(0)K(0) > 1/\underline{\lambda}_2(\mathcal{L})$;
(4) $K(\infty) > 0$ and there exists $i \in \{i \in \{2,\dots,N-1\}: \mu_i \neq \mu_{i+1}\}$ such that $P(0)K(0) < 1/\mu_i$ and $P(\infty)K(\infty) > 1/\mu_{i+1}$;
(5) $K(0) < 0$ and there exists $i \in \{i \in \{2,\dots,N-1\}: \mu_i \neq \mu_{i+1}\}$ such that $P(0)K(0) > 1/\mu_{i+1}$ and $P(\infty)K(\infty) < 1/\mu_i$.

Proof. Since $n = 1$, conditions (b) of Theorem 7 become $P(\infty)K(\infty) \neq 1/\mu_i$, $\frac{\mu_i P(\infty)K(0)-1}{1-\mu_i P(\infty)K(\infty)} < 0$ and $\frac{\mu_i K(0)P(0)-1}{1-\mu_i K(0)P(\infty)} < 0 \ \forall i \in \{2,\dots,N\}$. Furthermore, since (1) holds, these three conditions reduce to either conditions (i), (ii), or (iii).

  (i) $P(0)K(0) < 1/\bar{\lambda}(\mathcal{L})$, $P(\infty)K(\infty) < 1/\bar{\lambda}(\mathcal{L})$ and $P(\infty)K(0) < 1/\bar{\lambda}(\mathcal{L})$;
  (ii) $P(0)K(0) > 1/\underline{\lambda}_2(\mathcal{L})$, $P(\infty)K(\infty) > 1/\underline{\lambda}_2(\mathcal{L})$ and $P(\infty)K(0) > 1/\underline{\lambda}_2(\mathcal{L})$;
  (iii) There exists $i \in \{i \in \{2,\dots,N-1\}: \mu_i \neq \mu_{i+1}\}$ such that $1/\mu_{i+1} < P(0)K(0) < 1/\mu_i$, $1/\mu_{i+1} < P(\infty)K(\infty) < 1/\mu_i$ and $1/\mu_{i+1} < P(\infty)K(0) < 1/\mu_i$.

Likewise, conditions (c) of Theorem 7 with (1) lead to either conditions (I), (II), or (III).

  (I) $P(0)K(0) < 1/\bar{\lambda}(\mathcal{L})$, $P(\infty)K(\infty) < 1/\bar{\lambda}(\mathcal{L})$ and $P(0)K(\infty) < 1/\bar{\lambda}(\mathcal{L})$;
  (II) $P(0)K(0) > 1/\underline{\lambda}_2(\mathcal{L})$, $P(\infty)K(\infty) > 1/\underline{\lambda}_2(\mathcal{L})$ and $P(0)K(\infty) > 1/\underline{\lambda}_2(\mathcal{L})$;
  (III) There exists $i \in \{i \in \{2,\dots,N-1\}: \mu_i \neq \mu_{i+1}\}$ such that $1/\mu_{i+1} < P(0)K(0) < 1/\mu_i$, $1/\mu_{i+1} < P(\infty)K(\infty) < 1/\mu_i$ and $1/\mu_{i+1} < P(0)K(\infty) < 1/\mu_i$.

($\Leftarrow$) Condition (1) implies both conditions (i) and (I) via (Lanzon & Chen, 2017, Lemma 5). Condition (2) together with using proper-
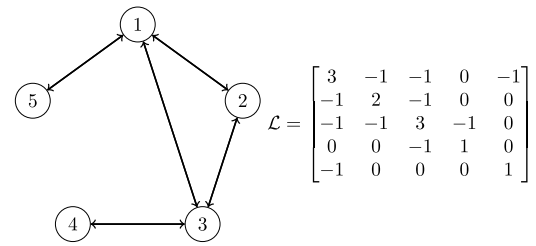

Fig. 3. Network graph and associated Laplacian matrix.

ties of Xiong et al. (2010, Cor. 3) imply $P(0) \geq P(\infty) > 0$ and $K(0) > K(\infty) > 0$ respectively. Then we get $P(0)K(0) \geq P(\infty)K(0)$, $P(0)K(\infty) \geq P(\infty)K(\infty) > 1/\underline{\lambda}_2(\mathcal{L})$, $P(0)K(0) > P(0)K(\infty)$, and $P(\infty)K(0) > P(\infty)K(\infty) > 1/\underline{\lambda}_2(\mathcal{L})$ which together imply conditions (ii) and (II). Likewise, condition (3) together with using properties of Xiong et al. (2010, Cor. 3) imply $P(\infty) \leq P(0) < 0$ and $K(\infty) < K(0) < 0$ respectively. Then we get $1/\underline{\lambda}_2(\mathcal{L}) < P(0)K(0) < P(0)K(\infty)$, $P(\infty)K(0) < P(\infty)K(\infty)$, $1/\underline{\lambda}_2(\mathcal{L}) < P(0)K(0) \leq P(\infty)K(0)$ and $P(0)K(\infty) \leq P(\infty)K(\infty)$ which together imply conditions (ii) and (II). Condition (4) together with using properties of Xiong et al. (2010, Cor. 3) imply $P(0) \geq P(\infty) > 0$ and $K(0) > K(\infty) > 0$ respectively. Then we get $1/\mu_i > P(0)K(0) \geq P(\infty)K(0)$, $P(0)K(\infty) \geq P(\infty)K(\infty) > 1/\mu_{i+1}$, $1/\mu_i > P(0)K(0) > P(0)K(\infty)$ and $P(\infty)K(0) > P(\infty)K(\infty) > 1/\mu_{i+1}$ which together give conditions (iii) and (III). Likewise, condition (5) together with using properties of Xiong et al. (2010, Cor. 3) imply $P(\infty) \leq P(0) < 0$ and $K(\infty) < K(0) < 0$ respectively. Then we get $P(0)K(\infty) > P(0)K(0) > 1/\mu_{i+1}$, $1/\mu_i > P(\infty)K(\infty) > P(\infty)K(0)$, $P(\infty)K(0) > P(0)K(0) > 1/\mu_{i+1}$ and $1/\mu_i > P(\infty)K(\infty) > P(0)K(\infty)$ which together imply conditions (iii) and (III).

($\Rightarrow$) Both conditions (i) and (I) reduce to condition (1). Now consider the five cases as in the proof of Lanzon and Chen (2017, Th. 17) which are $0 < K(\infty) < K(0)$, $0 = K(\infty) < K(0)$, $K(\infty) < 0 < K(0)$, $K(\infty) < K(0) = 0$ and $K(\infty) < K(0) < 0$. Only the first and last cases are allowed by conditions (ii), (iii), (II) and (III) as the three middle cases violate them. Consequently, it is easy to see that condition (ii) [resp. (II)] implies either condition (2) or (3) while condition (iii) [resp. (III)] implies either condition (4) or (5). $\blacksquare$

The following example shows the effectiveness of Theorem 11.

Example 12. Given five homogeneous SISO NI systems each with transfer function $P(s) = \frac{1}{s^2+5} + 2$. We consider the connection of these NI systems over the network topology shown in Fig. 3. The associated Laplacian matrix $\mathcal{L}$ is also shown in Fig. 3. The nonzero eigenvalues of $\mathcal{L}$ arranged as in (1) are $\{0.6972, 1.3820, 3.6180, 4.3028\}$. Consider distributed control protocol (3) with the following SNI feedback controller $K(s) = \frac{1}{s+5} + d$ where d is a tuning parameter and $d \neq 0$. We use Theorem 11 to study the effect of tuning parameter d on achieving robust output consensus. It is important to note that since $P(\infty)K(\infty) = 2d \neq 0$, the results in Wang et al. (2015a, Th. 1) cannot be used to determine whether robust output consensus of the networked NI systems can be achieved or not. The values of d are chosen as 0.2, 0.4, 0.6, 4, and -4. It is easy to check that for $d = 0.2$ and $d = 0.6$, conditions (1)–(5) of Theorem 11 fail to hold. Thus, we conclude that robust output consensus is not achieved with these values. For $d = 0.4$, condition (4) is satisfied; for $d = 4$, condition (2) is satisfied; and for $d = -4$, condition (1) is satisfied. Thus, we conclude that robust output consensus is achieved with these values. However, it is important

to note that Condition (4) of Theorem 11 involves the knowledge of all nonzero eigenvalues (hence more sensitive to the network graph) whereas conditions (1) and (2) of Theorem 11 depend only on the knowledge of the largest and second smallest eigenvalues of $\mathcal{L}$ respectively (hence less sensitive to network graph).

Remark 13. Although SNI controller synthesis for performance is not explicitly covered in this paper, Example 12 gives an indication how an SNI controller to each NI system in the SISO case can be selected to reduce the effect of the network graph which is not apparent in the MIMO case. It can be deduced from Theorem 11 and Example 12 that it is preferable to select the SNI controller in protocol (3) to satisfy either one of the first three conditions in Theorem 11 and avoid satisfying the last two conditions of Theorem 11 in order to minimise the effect of the network graph on robust output consensus since an estimate for the second smallest and largest eigenvalues of $\mathcal{L}$ would only be needed. Furthermore, unlike the MIMO case, a Nyquist plot interpretation can be drawn for the SISO case in a similar manner as in Lanzon and Chen (2017) but with the difference that the crucial point here is no longer $+1$. Theorem 11 indicates that robust output consensus is achieved via condition (1) when the Nyquist plot of $P(s)K(s)$ starts and ends to the left of $1/\bar{\lambda}(L)$. Robust output consensus is achieved via condition (4) when the Nyquist plot of $P(s)K(s)$ starts to the left of $1/\mu_i$ and ends to the right of $1/\mu_{i+1}$ for an $i \in \{i \in \{2, \ldots, N-1\}: \mu_i \neq \mu_{i+1}\}$ and additionally $P(0)$, $P(\infty)$, $K(0)$ and $K(\infty)$ all have positive signs. The interpretation of the remaining conditions can be easily drawn in a similar manner.

4.2. Networked homogeneous NI systems with poles at the origin

The following theorem gives conditions under which robust output consensus is achieved for networked NI systems with possible poles at the origin.

Theorem 14. *Consider a network of homogeneous NI systems $P(s)$, a network graph $\mathcal{G}$ that satisfies Assumption 3, and an SNI feedback controller $K(s)$ for each NI agent. Let μ_i for all $i \in \{1, \ldots, N\}$ be the eigenvalues of the Laplacian matrix $\mathcal{L}$ associated with $\mathcal{G}$ ordered as stated in (1). Let $\Psi < 0$ be such that $\bar{\lambda}[P(\infty)\Psi] < 1/\bar{\lambda}(\mathcal{L})$. Then, the following three conditions are equivalent:*

(a) robust output consensus is achieved via control protocol (5) for networked system (4) as shown in Fig. 1 (or in a distributed manner (3) for each system (2)) under any external disturbances $d_{in}, d_o \in \mathfrak{L}_2[0, \infty)$ and model uncertainty that retains the NI property of the perturbed system $P(s)$;

(b) the set of conditions
$I_n - \mu_i P(\infty)K(\infty)$ *is nonsingular,*
$\bar{\lambda}[[I_n - \mu_i P(\infty)K(\infty)]^{-1}[\mu_i P(\infty)K(0) - I_n]] < 0$, *and*
$\bar{\lambda}[\lim_{s \to 0}[[I_n - \mu_i \Psi P(\infty)][I_n - \mu_i K(s)P(\infty)]^{-1}$
$\qquad \times [\mu_i K(s)P(s) - I_n][I_n - \mu_i \Psi P(s)]^{-1}]] < 0$,
are satisfied for all $i \in \{2, \ldots, N\}$;

(c) the set of conditions
$I_n - \mu_i P(\infty)K(\infty)$ *is nonsingular,*
$\bar{\lambda}[\lim_{s \to 0}[[I_n - \mu_i P(s)\Psi]^{-1}[\mu_i P(s)K(\infty) - I_n]$
$\quad \times [I_n - \mu_i P(\infty)K(\infty)]^{-1}[I_n - \mu_i P(\infty)\Psi]]] < 0$, *and*
$\bar{\lambda}[\lim_{s \to 0}[[\mu_i K(s)P(s) - I_n][I_n - \mu_i K(\infty)P(s)]^{-1}]] < 0$,
are satisfied for all $i \in \{2, \ldots, N\}$.

Proof. We begin by proving the equivalence of conditions (a) and (b). Recall that $\bar{P}(s) = \mathcal{L} \otimes P(s)$ in Fig. 2 is NI and now has poles at the origin since $P(s)$ has poles at the origin and $\bar{K}(s) = I_N \otimes K(s)$ in Fig. 2 is SNI since $K(s)$ is SNI. Also, recall that the internal stability of $[\bar{P}(s), \bar{K}(s)]$ implies output consensus when $d_{in} = d_o = 0$. Thus, we shall prove the internal stability of

$[\bar{P}(s), \bar{K}(s)]$ and then the proof for robust output consensus runs as before. Let $\bar{\Psi} = (I_N \otimes \Psi)$. We have $\bar{\Psi} < 0$ if and only if $\Psi < 0$. Also, $\bar{\lambda}[\bar{P}(\infty)\bar{\Psi}] = \bar{\lambda}[\mathcal{L} \otimes P(\infty)\Psi] < 1$ if and only if $\bar{\lambda}[P(\infty)\Psi] < 1/\bar{\lambda}(\mathcal{L})$. Hence by applying Lanzon and Chen (2017, Th. 24), $[\bar{P}(s), \bar{K}(s)]$ is internally stable if and only if

$I_{Nn} - \bar{P}(\infty)\bar{K}(\infty)$ is nonsingular,
$\bar{\lambda}[[I_{Nn} - \bar{P}(\infty)\bar{K}(\infty)]^{-1}[\bar{P}(\infty)\bar{K}(0) - I_{Nn}]] < 0$, and
$\bar{\lambda}[\lim_{s \to 0}[[I_{Nn} - \bar{\Psi}\bar{P}(\infty)][I_{Nn} - \bar{K}(s)\bar{P}(\infty)]^{-1}$
$\qquad \times [\bar{K}(s)\bar{P}(s) - I_{Nn}][I_{Nn} - \bar{\Psi}\bar{P}(s)]^{-1}]] < 0$.

Recall that $\mathcal{L}$ is a real symmetric matrix due to Assumption 3. Thus, by applying the same transformation as in Theorem 7 we arrive at conditions (b) of this theorem. It is not difficult to verify that the equivalence of conditions (a) and (c) can be proved in a similar manner by applying Lanzon and Chen (2017, Th. 26) rather than Lanzon and Chen (2017, Th. 24).

It is important to show that the limits in Theorem 14 are finite $\forall i \in \{2, \ldots, N\}$. To this end, we begin by stating a modified version of Lanzon and Chen (2017, Lemma 28).

Lemma 15. *Let the hypotheses of Theorem 14 hold and furthermore consider $I_n - \mu_i P(\infty)K(\infty)$ nonsingular and $\bar{\lambda}[\lim_{s \to 0}[[I_n - \mu_i P(s)\Psi]^{-1}[\mu_i P(s)K(\infty) - I_n][I_n - \mu_i P(\infty)K(\infty)]^{-1}[I_n - \mu_i P(\infty)\Psi]]] < 0 \ \forall i \in \{2, \ldots, N\}$. Then, $\lim_{s \to 0}[[I_n - \mu_i \Psi P(s)][I_n - \mu_i K(\infty)P(s)]^{-1}] \ \forall i \in \{2, \ldots, N\}$ is finite and nonsingular.*

Proof. The proof is omitted since it follows similar arguments as in the proof of Lanzon and Chen (2017, Lemma 28).

Remark 16. For the limit in conditions (b) of Theorem 14: Since $\mu_i P(s)$ is NI $\forall i \in \{2, \ldots, N\}$, $\Psi < 0$ and $\bar{\lambda}[\mu_i P(\infty)\Psi] < 1$ $\forall i \in \{2, \ldots, N\}$, then Lemma 20 of Lanzon and Chen (2017) can be employed to show that both $[I_n - \mu_i \Psi P(s)]^{-1}$ and $\mu_i P(s)[I_n - \mu_i \Psi P(s)]^{-1}$ have no poles at the origin $\forall i \in \{2, \ldots, N\}$. Hence, $\lim_{s \to 0}[[\mu_i K(s)P(s) - I_n][I_n - \mu_i \Psi P(s)]^{-1}]$ is finite $\forall i \in \{2, \ldots, N\}$. For the limits in conditions (c) of Theorem 14: The limit in the second condition is finite $\forall i \in \{2, \ldots, N\}$ since $\lim_{s \to 0}[[I_n - \mu_i P(s)\Psi]^{-1}[\mu_i P(s)K(\infty) - I_n]]$ is finite $\forall i \in \{2, \ldots, N\}$ by Lanzon and Chen (2017, Lemma 20) while the limit in the third condition is finite $\forall i \in \{2, \ldots, N\}$ by Lanzon and Chen (2017, Lemma 20) and Lemma 15 since $\lim_{s \to 0}[[\mu_i K(s)P(s) - I_n][I_n - \mu_i K(\infty)P(s)]^{-1}] = \lim_{s \to 0}[[\mu_i K(s)P(s) - I_n][I_n - \mu_i \Psi P(s)]^{-1}[I_n - \mu_i \Psi P(s)][I_n - \mu_i K(\infty)P(s)]^{-1}] \ \forall i \in \{2, \ldots, N\}$.

When the networked NI systems have a single or double pole at the origin in all directions, the robust output consensus conditions will neither depend on the nonzero eigenvalues of $\mathcal{L}$ nor on the matrix Ψ as follows.

Corollary 17. *Consider a network of homogeneous strictly proper NI systems $P(s)$, a network graph $\mathcal{G}$ that satisfies Assumption 3, and an SNI feedback controller $K(s)$ for each NI agent. Assume one of the following conditions holds:*

(1) $\lim_{s \to 0} s^2 P(s)$ is nonsingular;
(2) $\lim_{s \to 0} s^2 P(s) = 0$ and $\lim_{s \to 0} sP(s)$ is nonsingular.

Then, robust output consensus is achieved via control protocol (5) for networked system (4) as shown in Fig. 1 (or in a distributed manner (3) for each system (2)) under any external disturbances $d_{in}, d_o \in \mathfrak{L}_2[0, \infty)$ and model uncertainty that retains the NI property of the perturbed system $P(s)$ if and only if $K(0) < 0$.

Proof. Since $P(\infty) = 0$, conditions (b) in Theorem 14 reduce to

$$\bar{\lambda}[\lim_{s \to 0}[[\mu_i K(s)P(s) - I_n][I_n - \mu_i \Psi P(s)]^{-1}]] < 0 \qquad (6)$$

while conditions (c) in Theorem 14 reduce to

$$\bar{\lambda}[\lim_{s\to 0}[[I_n - \mu_i P(s)\Psi]^{-1}[\mu_i P(s)K(\infty) - I_n]]] < 0, \text{ and}$$

$$\bar{\lambda}[\lim_{s\to 0}[[\mu_i K(s)P(s) - I_n][I_n - \mu_i K(\infty)P(s)]^{-1}]] < 0 \qquad (7)$$

$\forall i \in \{2, \ldots, N\}$. The proof is then similar to the proof of Lanzon and Chen (2017, Cor. 32) but when applied either to condition (6) or (7).

Remark 18. A procedure was provided in Lanzon and Chen (2017) for selecting Ψ by first decomposing $P(\infty)$ as $P(\infty) = Q\Lambda Q^T$, where Q is an orthogonal matrix and Λ is diagonal, then selecting Λ_1 as a diagonal matrix with negative elements such that $\Lambda - \Lambda_1 > 0$ so that $\Psi = Q\Lambda_1^{-1}Q^T$ fulfils the required condition $\bar{\lambda}[P(\infty)\Psi] < 1$ as this is equivalent to $P(\infty) - \Psi^{-1} > 0$. Since the nonzero eigenvalues of $\mathcal{L}$ play an important role in the robust output consensus conditions, we hence need to select Λ_1 such that $\Lambda - \frac{1}{\bar{\lambda}(\mathcal{L})}\Lambda_1 > 0$ since the condition $\bar{\lambda}[P(\infty)\Psi] < 1$ is replaced here by $\bar{\lambda}[P(\infty)\Psi] < 1/\bar{\lambda}(\mathcal{L})$.

4.2.1. SISO specialisation: must have pole(s) at the origin

The following corollary shows that when the NI systems are SISO and must either have a single or double pole at the origin, the robust output consensus conditions of Theorem 14 can be simplified as follows.

Corollary 19. *Consider a network of homogeneous SISO NI systems $P(s)$ with $s = 0$ being a single or double pole of $P(s)$, a network graph $\mathcal{G}$ that satisfies Assumption 3 and an SNI feedback controller $K(s)$ for each NI agent. Then, robust output consensus is achieved via control protocol (5) for networked system (4) as shown in Fig. 1 (or in a distributed manner (3) for each system (2)) under any external disturbances $d_{in}, d_o \in \mathcal{L}_2^{nN}[0, \infty)$ and model uncertainty that retains the NI property of the perturbed system $P(s)$ if and only if either one of the following conditions holds:*

(1) $P(\infty)K(\infty) < 1/\bar{\lambda}(\mathcal{L})$ and $K(0) < 0$;
(2) $P(\infty)K(\infty) > 1/\underline{\lambda}_2(\mathcal{L})$ and $K(\infty) > 0$.

Proof. The two conditions in this corollary can be obtained either via conditions (b) or conditions (c) of Theorem 14. We give the proof via conditions (c) of Theorem 14. It is easy to verify that the conditions can also be obtained via conditions (b) of Theorem 14 and thus omitted. Since $n = 1$, conditions (c) of Theorem 14 become $P(\infty)K(\infty) \neq 1/\mu_i \forall i \in \{2, \ldots, N\}$, $\frac{K(\infty)}{1-\mu_i K(\infty)P(\infty)} < 0$ $\forall i \in \{2, \ldots, N\}$, and $\frac{K(0)}{K(\infty)} > 0$ after writing $P(s)$ in its Laurent series form and taking the limit at $s \to 0$. Consider the three cases for the first condition: $P(\infty)K(\infty) > 1/\mu_i \forall i \in \{2, \ldots, N\}$, $P(\infty)K(\infty) < 1/\mu_i \forall i \in \{2, \ldots, N\}$ and there exists $i \in \{i \in \{2, \ldots, N-1\}: \mu_i \neq \mu_{i+1}\}$ such that $1/\mu_{i+1} < P(\infty)K(\infty) < 1/\mu_i$. The third case is violated by the second condition. Hence we consider the first two cases. These two cases with (1) and the NI property $K(0) > K(\infty)$ lead to the three conditions being reduced to either condition (1) or (2) in this corollary.

5. Illustrative examples

In this section, we give two examples to demonstrate the effectiveness of the robust output consensus results proposed in this paper. In each example, four NI systems are considered. The network topology that models the interaction among the NI systems and its associated Laplacian matrix are given in Fig. 4.
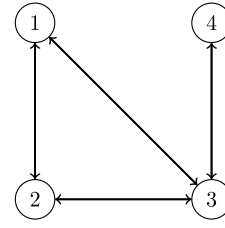
$$\mathcal{L} = \begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

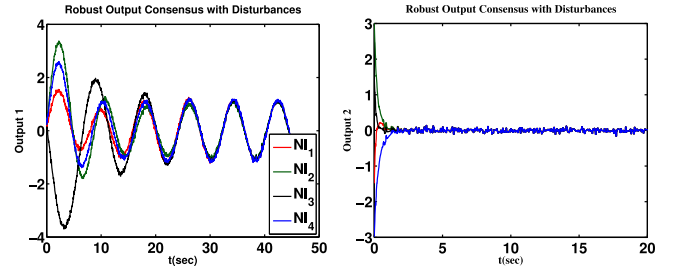**Fig. 4.** Network graph and associated Laplacian matrix.

Fig. 5. Robust output consensus for networked NI systems.

5.1. Without poles at the origin

The transfer function matrix of the four NI systems and the SNI feedback controller for each NI agent are

$$P(s) = \begin{bmatrix} \frac{2s^2+2.2}{s^2+0.6} & 0 \\ 0 & \frac{1}{s+1} \end{bmatrix}, \quad K(s) = \begin{bmatrix} \frac{1}{s^2+15s+20} & 0 \\ 0 & \frac{-2s-9}{s+5} \end{bmatrix},$$

respectively. Although $P(\infty)K(\infty) = 0$, $K(\infty) \not\geq 0$. Thus, the results in Wang et al. (2015a, Th. 1) cannot be used to determine robust output consensus of the networked NI systems. On the other hand, since $P(0) > 0$ we conclude via Corollary 10 (case (ii)) that robust output consensus is achieved for the NI systems since $\bar{\lambda}[P(0)K(0)] = 0.18 < (1/\bar{\lambda}(\mathcal{L})) = 0.24$ as shown in Fig. 5 with external disturbances acting on the systems.

5.2. With poles at the origin

Here we consider four homogeneous flexible robotic arms. Fig. 6 shows the model of the ith robotic arm. The arm is modelled by slewing beam with co-located piezoelectric actuator and sensor and is driven by a motor pinned to one of its ends (Pota & Alberts, 1995). Thus, the ith robotic arm has two inputs (V_{ai}, τ_i) which represent voltage and torque applied to the piezoelectric actuator and motor respectively, and two outputs (V_{si}, θ_i) which represent voltage sensed by the piezoelectric sensor and the motor hub angle respectively $\forall i \in \{1, \ldots, 4\}$. Whereas the robotic arm has an infinite dimensional model, for purpose of control design a finite dimensional model can be approximated. A finite dimensional model $P_i(s) = P(s) \forall i \in \{1, \ldots, 4\}$ for the flexible robotic arms, taking the first resonant mode into account (see Mabrok et al., 2014 for more details), is obtained as

$$P(s) = \begin{bmatrix} P_{\tau,\theta}(s) & P_{V_a,\theta}(s) \\ P_{\tau,V_s}(s) & P_{V_a,V_s}(s) \end{bmatrix} = \begin{bmatrix} \frac{3.231s^2+1.618}{s^2(s^2+3.4^2)} & \frac{3.5573\times10^{-4}}{s^2+3.4^2} \\ \frac{3.5573\times10^{-4}}{s^2+3.4^2} & \frac{2.35}{s^2+3.4^2} \end{bmatrix}. \qquad (8)$$

It can be verified by Definition 1 that (8) is NI with two poles at the origin. Consider the SNI controller in Mabrok et al. (2014)

$$K(s) = \begin{bmatrix} \frac{-4.29s^2-231.5s-5.11.9}{s^2+62.13s+232.4} & \frac{15s-247.5}{s^2+62.13s+232.4} \\ \frac{15s-247.5}{s^2+62.13s+232.4} & \frac{-2.22s^2-117.9s-162}{s^2+62.13s+232.4} \end{bmatrix}.$$
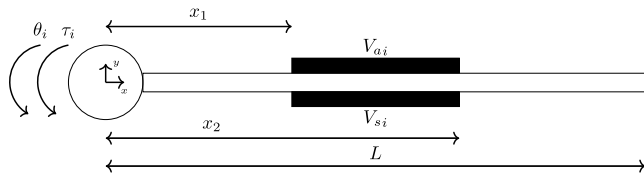
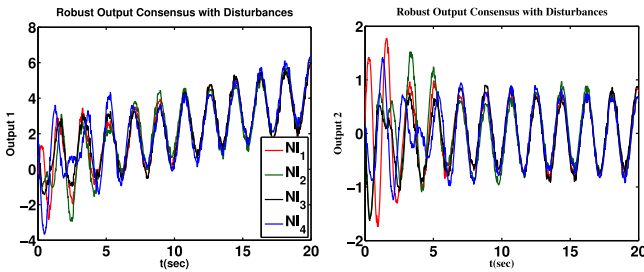**Fig. 6.** Schematic diagram of a slewing beam equivalent to ith robotic arm (Mabrok et al., 2014).

Fig. 7. Robust output consensus for networked robotic arms.

Although $P(s)$ has poles at the origin and is strictly proper, the results in Wang et al. (2015a) cannot be used to determine robust output consensus for the robotic arms since $\mathbb{N}(P_2) \not\subseteq \mathbb{N}(P_0)$ where $\mathbb{N}$ denotes the null space and P_0, P_2 are the coefficients in the Laurent series expansion of $P(s)$ around the zero. On the other hand, robust output consensus for the robotic arms can be easily concluded via Theorem 14 of this paper. In fact, we need only check condition (6) since $P(\infty) = 0$. Now since it is possible to select $\Psi = K(0) < 0$ (see Lanzon & Chen, 2017), we check that (6) is satisfied $\forall i \in \{2,3,4\}$, i.e. $\bar{\lambda}[\lim_{s\to 0}[[\mu_i K(s)P(s) - I_n][I_n - \mu_i \Psi P(s)]^{-1}]] = -1 < 0$ $\forall i \in \{2,3,4\}$. Observe that with this choice of Ψ, we were able to easily determine robust output consensus without knowledge of the eigenvalues of $\mathcal{L}$. Note that Wang et al. (2015b, Th. 15), which captures robust output consensus for the heterogeneous case, is much more complicated to use since the conditions are checked for the augmented networked plant and controller which increase in dimension by increasing the number of connected agents. Moreover, matrix factorisation is needed and additional conditions need to be satisfied, such as non-singularity and sign semidefiniteness, for specific matrices before being able to determine whether output consensus is achieved or not. Simulation results, using the finite dimensional model (8), are shown in Fig. 7. The initial conditions have been arbitrarily chosen as $[1, 0.1, 2, 0.2, 3, 0.3]^T$, $[-1, 0.1, -2, 0.2, -3, 0.3]^T$, $[2.5, 0.5, -2.5, 0.6, -3.2, 0.1]^T$, $[2, 0.3, -5.5, 0.4, -1, 0.2]^T$ for the four NI systems respectively, and $[0, 0]^T$ for all four SNI controllers. It can be seen from Fig. 7 that robust output consensus is achieved with external disturbances.

6. Conclusion

Necessary and sufficient conditions for robust output consensus were proposed for multiple homogeneous NI systems which are subject to $\mathfrak{L}_2$ external disturbances and model uncertainties by utilising the recently published robust NI stability results. Advantages of the proposed results over existing results in literature were discussed. It was shown that the derived conditions specialise to those in earlier literature by either imposing the same assumptions at infinite frequency or by imposing different ones which had not been known previously. It was also shown

that the derived conditions simplify in the SISO case. The results were enhanced by several examples such that the effectiveness of the proposed results over earlier results were apparent when the assumptions of earlier results do not hold.

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