Notes on Producer Theory

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Producer theory

Reference: Jehle and Reny, *Advanced Microeconomic Theory*, 3rd ed., Pearson 2011: Ch. 3.

The second important actor in economics is the firm (producer).

We begin with aspects of production and costs that are common to all firms. Then we consider the behavior of competitive firms, a very special but important class of firms.

A firm (producer) carries out the production process transforming inputs into outputs. To do that, the firm employs a certain technology.

If the firm produces a single product from many inputs, its technology can be represented by a production function.

Producer theory

A production function $f : \mathbb{R}^n_+ \rightarrow \mathbb{R}_+$ describes for each vector of inputs $x = (x_1, \ldots, x_n)$ the amount of output $q = f(x)$ that can be produced.

For any fixed level of output $\bar{q}$, the set of input vectors producing $\bar{q}$,

$$\{x \in \mathbb{R}^n_+ : f(x) = \bar{q}\},$$

is called the $\bar{q}$-level isoquant. An isoquant is just a level set of f.

When the production function f is differentiable, its partial derivative $\frac{\partial f(x)}{\partial x_i}$ is called the marginal product of input i.

The marginal product of input i, denoted by $MP_i(x)$, indicates the rate at which output changes per additional unit of input i employed.

Producer theory

The marginal product is a measure of returns to variable proportions (RVP); i.e., of how output varies as the proportions in which inputs are used change.

On the contrary, returns to scale (RS) measures how output responds when *all* inputs are varied in the *same proportion*; i.e., when the entire "scale" of operation is increased or decreased proportionally.

- RVP concern how output changes along $\bar{x}_2$, keeping x_2 constant and varying x_1.
- RS concern how output changes along OA, varying x_2 and x_1 at the same time and maintaining the proportion $x_2/x_1 = \alpha$.

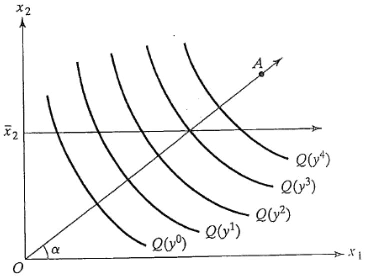

Figure 1: RS & RVP.

Producer theory

A production function $f : \mathbb{R}^n_+ \to \mathbb{R}_+$ has the property of:

1. Constant returns to scale if for all $t > 0$ and $x \in \mathbb{R}^n_+$, $f(tx) = tf(x)$;
2. Increasing returns to scale if for all $t > 1$ and $x \in \mathbb{R}^n_+$, $f(tx) > tf(x)$;
3. Decreasing returns to scale if for all $t > 1$ and $x \in \mathbb{R}^n_+$, $f(tx) < tf(x)$.

If the production function is homogenous, returns to scale can be associated with the degree of homogeneity.

N.B. Recall that a production function $f : \mathbb{R}^n_+ \to \mathbb{R}_+$ is homogeneous of degree k if for all $\lambda > 0$ and $x \in \mathbb{R}^n_+$, $f(\lambda x) = \lambda^k f(x)$; (e.g. $f(x_1, x_2) = x_1^\alpha x_2^\beta$ is homogeneous of degree $k = \alpha + \beta$).

- If the production function is homogeneous of degree $k > 1$ ($k < 1$), it must exhibit increasing (decreasing) RS; the converse need not hold.
- If the production function is homogeneous of degree $k = 1$, it has constant RS, and viceversa.

Producer theory

As we did in consumer theory with the MRS, it is possible to investigate here the rate at which one input can be substituted by another without changing the amount of output produced.

This rate is given by the marginal rate of technical substitution (MRTS).

When $n = 2$, $MRTS_{12}(x)$ is obtained by totally differentiating $f(x_1, x_2) = \bar{q}$:

$$d\bar{q} = \frac{\partial f(x_1, x_2)}{\partial x_1} dx_1 + \frac{\partial f(x_1, x_2)}{\partial x_2} dx_2 = 0.$$

$$\Rightarrow \quad \left.\frac{dx_2}{dx_1}\right|_{d\bar{q}=0} = -\frac{\frac{\partial f(x_1, x_2)}{\partial x_1}}{\frac{\partial f(x_1, x_2)}{\partial x_2}} = -\frac{MP_1(x_1, x_2)}{MP_2(x_1, x_2)} = MRTS_{12}(x_1, x_2).$$

The $MRTS_{12}(x)$ is the slope at x of the isoquant passing through $\bar{q}$.

Producer theory

More generally, when $n > 2$ for any two inputs $i \neq j$,

$$MRTS_{ij}(x) = -\frac{MP_i(x)}{MP_j(x)}.$$

In the two input case, the production function $f(x_1, x_2)$ exhibits a diminishing MRTS if for any q, the absolute value of MRTS, $|MRTS_{12}|$, diminishes as x_1 increases and x_2 is restricted by the isoquant $f(x_1, x_2) = q$.

- A diminishing MRTS is consistent with increasing marginal productivities;
- A diminishing MRTS implies that the slope of the isoquant *in absolute value* is decreasing (i.e. that the isoquants are convex).

The MRTS is one *local* measure of substitutability between inputs in producing a given level of output.

Producer theory

The elasticity of substitution of input j for input i is defined as the % change in the proportions x_j/x_i associated with a one % change in the $MRTS_{ij}$, holding all other inputs and the level of output constant.

For a production function $f(x)$, the elasticity of substitution of input j for input i at $x^0 \in \mathbb{R}_{++}$ is defined as

$$\sigma_{ij}(x^0) = \left(\frac{d \ln MRTS_{ij}(x(r))}{d \ln r} \Big|_{r=x_j^0/x_i^0} \right)^{-1}, \qquad (1)$$

where $x(r)$ is the unique vector of inputs $x = (x_1, \ldots, x_n)$ such that (i) $x_j/x_i = r$, (ii) $x_k = x_k^0$ for all $k \neq i, j$, and (iii) $f(x) = f(x^0)$.

The elasticity of substitution $\sigma_{ij}(x^0)$ is a measure of the curvature of the $i - j$ isoquant through x^0 at x^0.

Producer theory

When the production function f is quasi-concave, $\sigma_{ij} \geq 0$; (convex isoquants imply that $\uparrow (x_2/x_1) \Rightarrow \uparrow |MRTS_{12}|$).

The closer σ_{ij} is to zero, the more difficult is the substitution between inputs j and i; the larger it is, the easier is the substitution between them.

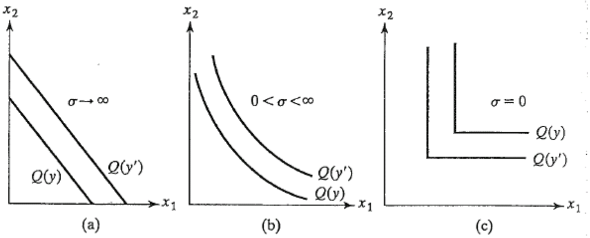

Figure 2: Elasticity of substitution.

In Fig 2, panel (a) represents perfect substitutability; panel (c) no substitutability (fixed proportions); and panel (b) an intermediate case.

Producer theory

Consider a firm (producer) that looks for the optimal demand of each input x_i, $i = 1, \ldots, n$, to minimize the cost of producing q units of output, given the prevailing technology $f(\cdot)$ and the input prices $p = (p_1, \ldots, p_n) \gg 0$.

The cost minimization problem (CMP) of the firm takes the form

$$\min_{x_1,\ldots,x_n} \; p_1x_1 + \ldots + p_nx_n \;\text{ subject to }\; f(x_1,\ldots,x_n) \geq q. \tag{2}$$

The objective function is linear in the decision variables $x_1, \ldots, x_n$.

Hence, if the production function $f(\cdot)$ exhibits "some kind of concavity" and there is an interior solution, (2) can be solved using the Lagrange method.

The solution, denoted by $x_i^c(p,q)$, determines the conditional demand for input $i = 1, \ldots, n$, (conditional on the output level q).

Producer theory

Unfortunately, not all production functions are concave.

For instance, the Cobb-Douglas production function $f(x_1, x_2) = k\, x_1^{\alpha}\, x_2^{\beta}$, with $k,\ \alpha,\ \beta > 0$, has a Hessian matrix

$$H_f(x_1, x_2) = \begin{pmatrix} k\,\alpha\,(\alpha-1)\,x_1^{\alpha-2}\,x_2^{\beta} & k\,\alpha\,\beta\,x_1^{\alpha-1}\,x_2^{\beta-1} \\ k\,\alpha\,\beta\,x_1^{\alpha-1}\,x_2^{\beta-1} & k\,\beta\,(\beta-1)\,x_1^{\alpha}\,x_2^{\beta-2} \end{pmatrix}.$$

The elements of the main diagonal are non-positive if $\alpha \leq 1$ and $\beta \leq 1$.

The determinant $|H_f(x_1, x_2)| = k^2\, x_1^{2(\alpha-1)}\, x_2^{2(\beta-1)}\,(1-\alpha-\beta)\,\alpha\beta$.

Thus, $|H_f(x_1, x_2)| \geq 0$ if and only if $(1-\alpha-\beta) \geq 0$ (recall $\alpha, \beta > 0$).

That is, the Cobb-Douglas production function $f(x_1, x_2) = k\, x_1^{\alpha}\, x_2^{\beta}$ is concave on $\mathbb{R}^2_+$ if and only if $\alpha + \beta \leq 1$.

Producer theory

However, as happens in consumer theory, a solution for (2) requires less than concavity on f.

It is enough, for instance, if the production function is indirectly concave.

A production function $f(\cdot)$ is indirectly concave if it is a strictly increasing transformation of a concave function $F(\cdot)$, so that for all $x \in \mathbb{R}^n_+$, $f(x) = m(F(x))$, with $m'(r) > 0$ for all $r \in \mathbb{R}$.

Clearly, concavity implies indirect concavity, but the converse is not true.

Indeed, all Cobb-Douglas production functions are indirectly concave on $\mathbb{R}^n_+$, but we proved not all of them are concave.

N.B. In the two input case, $f(x_1, x_2) = k\, x_1^\alpha\, x_2^\beta$ can be rewritten as,

$$f(x_1, x_2) = \exp[\ln(k) + \alpha \ln(x_1) + \beta \ln(x_2)]. \tag{3}$$

which is a strictly increasing transformation of the concave function $\ln(k) + \alpha \ln(x_1) + \beta \ln(x_2)$. Hence, f is indirectly concave.

Producer theory

Going back to the CMP stated in (2), if the production function is indirectly concave (or quasi-concave) and the constraint qualification is satisfied, then FOCs are necessary and sufficient for the existence of a interior solution.

The Lagrange function corresponding to (2) is,

$$L(x, \lambda) = -p \cdot x + \lambda(f(x) - q).$$

Assuming strictly positive input prices $p \gg 0$ and an interior solution $x^c \in \mathbb{R}^n_{++}$, the first-order Kuhn-Tucker conditions are:

- $\frac{\partial L(x^c,\lambda)}{\partial x_i} = -p_i + \lambda \cdot \frac{\partial f(x^c)}{\partial x_i} = 0, i = 1, \ldots, n;$
- $\frac{\partial L(x^c,\lambda)}{\partial \lambda} = f(x^c) - q \geq 0;$
- $\lambda \geq 0$ and $\lambda \cdot [f(x^c) - q] = 0.$

Producer theory

If the production function f is strictly increasing at x^c, (i.e., if $MP_i(x^c) > 0,\ \forall i$), the constraint is binding at the solution and $\lambda > 0$.

This implies that at the equilibrium point x^c, the slope of the isoquant $f(x) = q$, given by $MRTS_{ij}(x^c)$, equals the slope of the iso-cost curve $p \cdot x = \bar{c}$ passing through x^c, which is the relative price $\frac{p_i}{p_j}$.

That is, for all i, j, with $i \neq j$, we have that

$$|MRTS_{ij}(x^c)| = \frac{MP_i(x^c)}{MP_j(x^c)} = \frac{p_i}{p_j}. \tag{4}$$

Notice the (formal) similarity between (4) and the optimality condition of consumer theory, namely

$$|MRS_{ij}(x^*)| = \frac{MU_i(x^*)}{MU_j(x^*)} = \frac{p_i}{p_j}. \tag{5}$$

Producer theory

If $x^c(p,q) = (x_1^c(p,q), \dots, x_n^c(p,q))$ solves (2), then the minimum cost of producing q, given the market prices p and the available technology f, is

$$\begin{aligned} C(p,q) &= \min_{x \in \mathbb{R}_+^n} \{p \cdot x : f(x) \geq q\}, \\ &= p \cdot x^c(p,q). \end{aligned} \tag{6}$$

The similarities pointed out before between consumer theory and producer theory are exact when we compare the cost and the expenditure functions:

$$\begin{aligned} C(p,q) &= \min_{x \in \mathbb{R}_+^n} \{p \cdot x : f(x) \geq q\}, \\ E(p,w) &= \min_{x \in \mathbb{R}_+^n} \{p \cdot x : u(x) \geq w\}. \end{aligned}$$

Mathematically, these two functions are identical!

Producer theory

If the production function f is continuous, strictly increasing, and strictly quasi-concave, and $p \gg 0$, the cost function satisfies the following properties:

1. $C(p,q)$ is strictly increasing in q;
2. $C(p,q)$ is increasing in p;
3. $C(p,q)$ is homogeneous of degree one in p;
4. $C(p,q)$ is concave in p;
5. $C(p,q)$ is differentiable in p and $\frac{\partial C(p,q)}{\partial p_i} = x_i^c(p,q)$.

The proofs of these properties are analogous to the proofs given for the expenditure function.

Producer theory

As solution to the firm's cost minimization problem, the conditional input demands possess certain general properties as well.

These are analogous to the properties of the Hicksian demands.

Suppose that f is continuous, strictly increasing, and strictly quasi-concave, and that $C(p, q)$ is twice continuously differentiable:

- $x^c(p, q)$ is homogenous of degree zero in p;
- The substitution matrix, i.e., the $n \times n$-matrix of first-order partial derivatives of the conditional inputs demands,

$$\begin{pmatrix} \frac{\partial x_1^c(p,q)}{\partial p_1} & \cdots & \frac{\partial x_1^c(p,q)}{\partial p_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_n^c(p,q)}{\partial p_1} & \cdots & \frac{\partial x_n^c(p,q)}{\partial p_n} \end{pmatrix},$$

  is symmetric and negative semi-definite.

Producer theory

Therefore, by definition of negative semi-definiteness, the elements of the diagonal are non-positive; i.e., for all $i = 1, \ldots, n$,

$$\frac{\partial x_i^c(p,q)}{\partial p_i} = \frac{\partial^2 C(p,q)}{\partial p_i^2} \leq 0. \tag{7}$$

That means, the conditional input demands cannot have a positive slope!

After a fall in p_1, the firm substitutes x_2, whose relative price has increased, by the relatively cheaper input x_1 (law of demand).

The substitution effect is driven by the assumed nature of the technology, namely, by the convexity of the isoquant curves.

Producer theory

Finally, let's determine the optimal output of the competitive firm to maximize its profits, which amounts to solving

$$\max_{q \geq 0} \; p_q \cdot q - C(p, q), \tag{8}$$

where p_q is the market price at which the firm sell each unit of q.

If $q^* > 0$ is the solution of (8), then the firm must satisfy the FOC

$$p_q - \frac{\partial C(p, q^*)}{\partial q} = 0. \tag{9}$$

That is, the optimal output q^* is chosen in such a way that the output price equals the marginal cost at q^*!

SOC requires that the marginal cost be nondecreasing at q^*, i.e.,

$$\frac{\partial^2 C(p, q^*)}{\partial q^2} \geq 0.$$

Producer theory

The optimal output q^* depends on (p_q, p). By changing "this data", we get the output supply function of the competitive firm, denoted by $q(p_q, p)$.

Replacing $q(p_q, p)$ into the condition input demands, we get the unconditional input demands $x(p_q, p) = x^c(p, q(p_q, p))$.

Finally, recall that by the Envelope theorem, $\frac{\partial C(p,q)}{\partial q} = \lambda$.

Moreover, by the FOC, $\lambda = p_i / MP_i(x)$.

Therefore, using (9), it follows that for all $i = 1, \ldots, n$,

$$p_q - \frac{p_i}{MP_i(x)} = 0 \quad \Leftrightarrow \quad p_q \cdot MP_i(x) = p_i. \tag{10}$$

In words, (10) says that the competitive firm employs additional units of input i until its marginal revenue product, $p_q \cdot MP_i(x)$, equals its unit cost, p_i.

Producer theory

An important implication of (10) is that, if the production function exhibits constant returns to scale, the remuneration of the production factors (inputs), $\sum_{i=1}^{n} p_i \cdot x_i$, exhausts total revenue, $p_q \cdot f(x)$; i.e.,

$$\sum_{i=1}^{n} p_i \cdot x_i = p_q \cdot f(x). \tag{11}$$

This means the competitive firm makes in the long-run zero profits.

The proof of this important result rests on Euler's theorem.

Suppose the production function $f : \mathbb{R}^n_+ \to \mathbb{R}_+$ is homogeneous of degree k, so that for all $\lambda > 0$ and all $x \in \mathbb{R}^n_+$,

$$f(\lambda x) = \lambda^k f(x). \tag{12}$$

Producer theory

Differentiating (12) with respect to λ,

$$\frac{\partial f(\lambda x)}{\partial \lambda x_1} \cdot x_1 + \ldots + \frac{\partial f(\lambda x)}{\partial \lambda x_n} \cdot x_n = k \cdot \lambda^{k-1} f(x). \tag{13}$$

Since (13) holds for every $\lambda > 0$, it holds in particular for $\lambda = 1$. Hence,

$$\sum_{i=1}^{n} \frac{\partial f(x)}{\partial x_i} \cdot x_i = k \cdot f(x). \tag{14}$$

The expression in (14) is known as Euler's theorem.

Remember that according with (10),

$$\frac{\partial f(x)}{\partial x_i} = MP_i(x) = \frac{p_i}{p_q}. \tag{15}$$

Producer theory

Therefore, substituting (15) into (14), we have that

$$\sum_{i=1}^{n} p_i \cdot x_i = k \cdot p_q \cdot f(x).$$

Thus, if the production function f has constant returns to scale (i.e., if $k = 1$), then we have that in the equilibrium of the competitive firm:

$$\sum_{i=1}^{n} p_i \cdot x_i = p_q \cdot f(x),$$

which is exactly the expression in (11).

Producer theory

The maximum-value function of the profit function depends on input and output prices and is defined as follows:

$$\pi(p_q, p) = \max_{(x,q)\in\mathbb{R}_+^{n+1}} \left\{ p_q\, q - \sum_{i=1}^{n} p_i\, x_i \,:\, f(x_1, \ldots, x_n) \geq q \right\}. \qquad (16)$$

It is easy to see that $\pi(p_q, p)$ is well defined only if the production function doesn't exhibit increasing returns to scale.

On the contrary, suppose f has increasing RS, and let x^* and $q^* = f(x^*)$ maximize π at prices p_q and $p = (p_1, \ldots, p_n)$.

Producer theory

With increasing returns,

$$\text{for all } t > 1,\ f(tx^*) > tf(x^*).$$

Multiplying by p_q and subtracting $p\,(tx^*)$ both sides,

$$\text{for all } t > 1,\ p_q f(tx^*) - p\,(tx^*) > t[p_q f(x^*) - p\,x^*].$$

Since $t > 1$ and π is bounded below by 0 (because $f(0) = 0$), it follows from the last inequality that

$$p_q f(tx^*) - p\,(tx^*) > p_q f(x^*) - p\,x^*,$$

contradicting that x^* and $q^* = f(x^*)$ maximizes π.

Producer theory

When the profit function is well defined, it possesses several properties that should be by now quite familiar.

If the production function f is continuous, strictly increasing, and strictly concave, and $p_q > 0$ and $p \gg 0$, the maximum-value function of the profit function $\pi(p_q, p)$ satisfies the following properties:

1. $\pi(p_q, p)$ is increasing in p_q;
2. $\pi(p_q, p)$ is decreasing in p;
3. $\pi(p_q, p)$ is homogeneous of degree one in (p_q, p);
4. $\pi(p_q, p)$ is convex in (p_q, p);
5. $\pi(p_q, p)$ is differentiable in (p_q, p) and (Hotelling's lemma)

$$\frac{\partial \pi(p_q, p)}{\partial p_q} = q(p_q, p) \text{ and } -\frac{\partial \pi(p_q, p)}{\partial p_i} = x_i(p_q, p) \quad i = 1, \ldots, n.$$

Producer theory

In particular, the fact that $\pi(p_q, p)$ is convex in (p_q, p) implies that the Hessian of $\pi(\cdot)$ is positive semi-definite.

Therefore, all of the elements in the main diagonal are nonnegative, i.e.,

$$\frac{\partial^2 \pi(p_q, p)}{\partial p_q^2} = \frac{\partial q(p_q, p)}{\partial p_q} \geq 0,$$

and

$$-\frac{\partial^2 \pi(p_q, p)}{\partial p_i^2} = \frac{\partial x_i(p_q, p)}{\partial p_i} \leq 0 \quad \text{for all } i = 1, \ldots, n.$$

In words, the output supply $q(p_q, p)$ is increasing in the product price p_q, and the input demands $x_i(p_q, p)$ are decreasing in their own input price p_i.