Notes on General Equilibrium

Alejandro Saporiti

General equilibrium

Reference: Jehle and Reny, *Advanced Microeconomic Theory*, 3rd ed., Pearson 2011: Ch. 5.

Behind the superficial chaos of countless market transactions by selfish individuals, Adam Smith (1776) saw a harmonizing force (the invisible hand) operating in a competitive economy.

Smith believed that force guides individuals to coordinate their choices, i.e. their consumption and productions plans, in such a way that all markets in the economy are brought into balance simultaneously.

He also believed that the resulting equilibrium possesses socially desirable properties, in the sense that it maximizes social welfare through no conscious collective intention of its members.

Does this vision of the competitive markets possess any substance?

Exchange economy

To answer that question, consider first a society without organized markets.

- There are $i = 1, \ldots, I$ individuals and $j = 1, \ldots, N$ consumption goods.
- Each individual i is endowed by nature with
  - A certain (nonnegative) amount $e^i = (e^i_1, \ldots, e^i_N) \in \mathbb{R}^N_+$ of the consumption goods, and
  - A preference relation $\succsim^i$ over $\mathbb{R}^N_+$ represented by the utility function $u^i : \mathbb{R}^N_+ \to \mathbb{R}$.
- The utility function u^i is continuous, increasing and quasiconcave.
- Agents can 'eat' their initial endowment or engage in trade with others.
- Exchange is voluntary and private ownership is respected.

Exchange economy

- Consumer i's consumption bundle is denoted $x^i = (x^i_1, \ldots, x^i_N) \in \mathbb{R}^N_+$.
- An allocation is a vector of consumption bundles $x = (x^1, \ldots, x^I)$.
- Let $e = (e^1, \ldots, e^I)$ and $u = (u^1, \ldots, u^I)$. The set of feasible allocations in this pure exchange economy $\mathcal{E} = (e, u)$ is given by

$$F(e) = \left\{ x \in \mathbb{R}^{I \times N}_+ : \sum_{i=1}^{I} x^i_j = \sum_{i=1}^{I} e^i_j \ \ \forall j = 1, \ldots, N \right\}.$$

How does trade take place in this exchange economy? Where does this system of voluntary exchanges might come to rest?

Bilateral exchange

To simplify the analysis, let's focus on a 2×2 economy, i.e., an economy with two consumers, A and B, and two goods, 1 and 2.

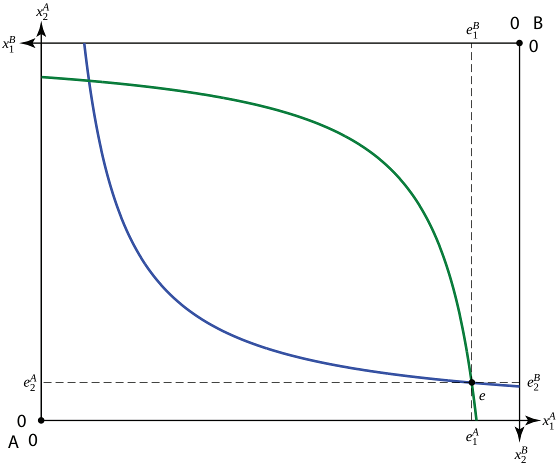
The main advantage of the 2×2 economy is that it can be graphically described through the so called Edgeworth box.

The Edgeworth box has height $e_2^A + e_2^B$ and width $e_1^A + e_1^B$, and every point x represents a feasible allocation, where for all $j = 1, 2$,

$$x_j^A + x_j^B = e_j^A + e_j^B.$$

The box offers a complete picture of every feasible distribution of the existing commodities between the consumers.

The Edgeworth box

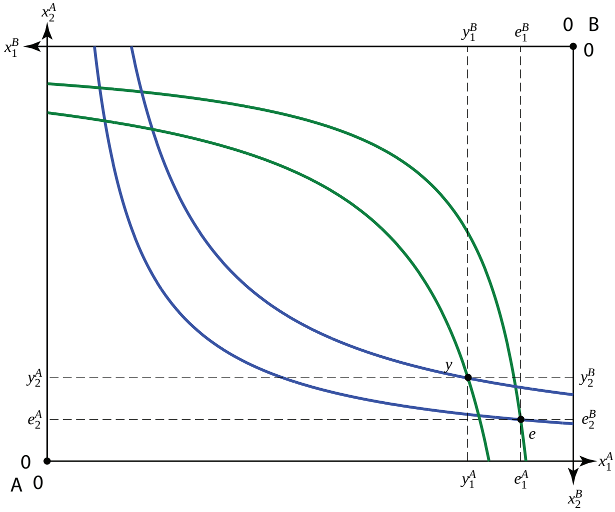


A and *B*'s initial endowments are given by (e_1^A, e_2^A) and (e_1^B, e_2^B).

All bundles inside the indifference lenticular are preferred to *e* by both *A* and *B*.

To achieve these gains, consumers must exchange part of their initial endowments.

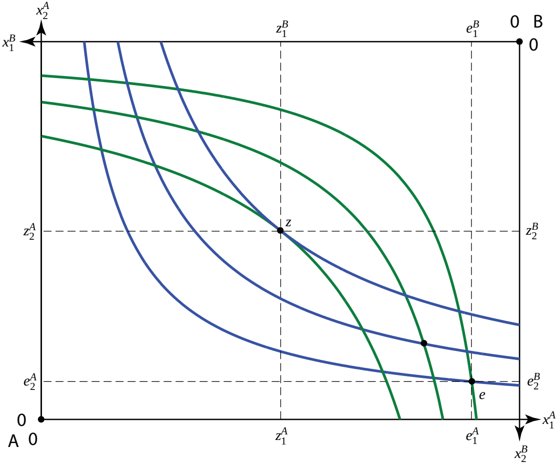
The Edgeworth box



For instance, y is preferred by both A and B to their initial endowment.

But y doesn't exhaust the possibilities for beneficial trade between A and B.

The Edgeworth box



Only at a point like z there are no more mutually beneficial trades between A and B.

The ‘no worse than’ (upper counter) sets touch only at z.

Pareto efficiency

We would expect rational agents will trade until all possibilities for mutually beneficial trade are exhausted. Such allocations are said to be Pareto efficient.

Definition 1 (Pareto efficiency in consumption)

A feasible allocation $z \in F(e)$ is said to be Pareto efficient (PE) if there is no other feasible allocation $\hat{z} \in F(e)$ such that $u^i(\hat{z}^i) \geq u^i(z^i)$ for all $i = 1, \ldots, I$, with strict inequality for some i.

Geometrically in the Edgeworth box, PE allocations are points $x = (x^A, x^B)$ at which consumers' indifference curves are tangent; i.e., points at which the marginal rates of substitution are all equal:

$$MRS^A_{1,2}(x^A_1, x^A_2) = MRS^B_{1,2}(x^B_1, x^B_2). \tag{1}$$

Pareto efficiency

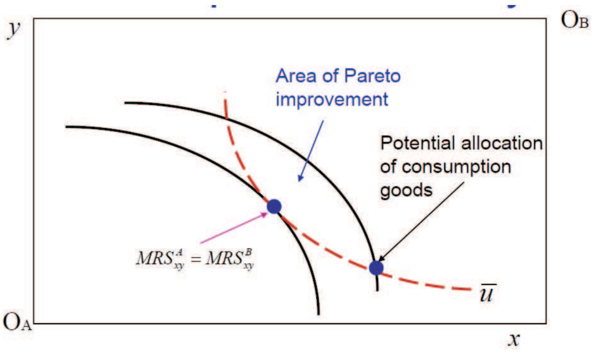


Figure 1: Pareto efficiency in consumption.

Definition 2 (Contract curve)

The set *CC* of all feasible and Pareto efficient allocations is called the Pareto set or contract curve.

Contract curve and the core

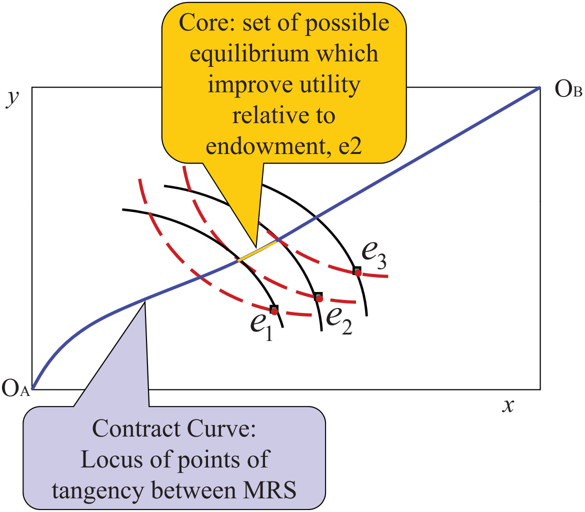


Figure 2: Contract curve and the core.

The "equilibrium" of the exchange process is going to be a point on the subset of the contract curve determined by consumers' indifference curves through *e*. This set is called the core of the economy.

However, we don't know where exactly the agents will end up.

Generalizing the core

Definition 3 (Blocking coalitions)

A set of agents S blocks $x \in F(e)$ if there exists an allocation y such that (i) $\sum_{i \in S} y^i = \sum_{i \in S} e^i$, and (ii) $u^i(y^i) \geq u^i(e^i)$ for all $i \in S$, with strict inequality for some i.

Definition 4 (The core)

The core of an exchange economy $\mathcal{E} = (e, u)$, denote $C(\mathcal{E})$, is the set of all unblocked feasible allocations.

Apart from the the fact that the core might be "pretty big" (see Fig. 2) and therefore lack any prediction power, the amount of information needed to arrange mutually beneficial trade and converge to $C(\mathcal{E})$ might also be huge!

The next step is to examine how market economies deal with these problems.

Competitive markets

Let's now add to our primitive economy $\mathcal{E} = (e, u)$ of pure exchange a few other assumptions:

- Markets exist for all goods.
- Agents can freely participate in markets without cost.
- "Standard" consumer theory assumptions:
  - Preferences (strictly) monotone & represented by utility function;
  - Some "convexity" if needed;
  - All agents are price-takers;
  - Finite number of perfectly divisible goods;
  - Linear prices;
  - Perfect information about goods and prices.
- All agents face the same prices.

Walrasian equilibrium

Given prices $p \gg 0$, denote by $y^i(p) = \sum_j p_j e^i_j$ consumer i's initial income, and by $B^i(p) = \{x^i \in \mathbb{R}^L_+ : p \cdot x^i \leq y^i(p)\}$ the budget set for i.

Definition 5 (Walrasian equilibrium)

A price vector $p = (p_1, \ldots, p_N)$ and an allocation $x = (x^1, \ldots, x^I)$ are said to be a Walrasian or competitive equilibrium (WE) iff

(i) For all $i = 1, \ldots, I$, $x^i \in \arg\max_{x \in B^i(p)} u^i(x)$; i.e., x^i is consumer i's Walrasian demand at prices p and income $y^i(p)$; and

(ii) Markets clear at p; i.e, for all $j = 1, \ldots, N$, the excess demand function

$$z_j(p) \equiv \sum_{i=1}^{I} x^i_j - \sum_{i=1}^{I} e^i_j = 0.$$

Walrasian equilibrium

In words, a WE is a set of prices such that

(i) Each agent chooses his most-preferred affordable bundle; and

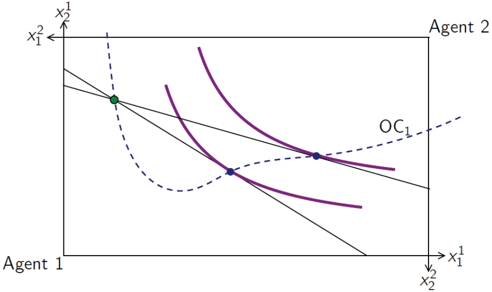
(ii) Consumers' choices are compatible among each other, in the sense that total demand equals total supply in every market.

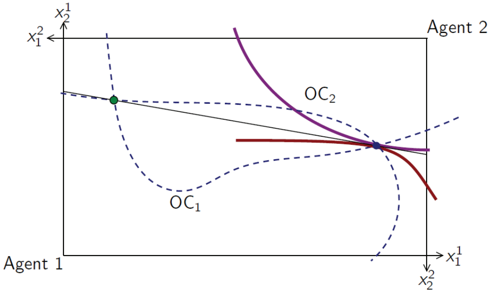
Actually, this definition of a WE is stronger than necessary: It turns out that if the aggregate excess demand for $N-1$ goods is zero, then the excess demand for the remaining good must be zero too.

This follows from a property of the excess demand function called Walras' law: the value of the aggregate excess demand is zero for *all* possible prices (see Exercise 3.1):

$$\sum_{j=1}^{N} p_j \, z_j(p_1, \ldots, p_N) = 0.$$

Walrasian equilibrium





The offer curve traces out Walrasian demand as prices change.

Walrasian equilibria are at the intersection of the offer curves.

The intersection and thereby the Walrasian equilibrium need not be unique.

Equilibrium existence

For arbitrary prices (p_1, p_2) there is no guarantee that supply will equal demand, i.e., there is no guarantee that (ii) is satisfied.

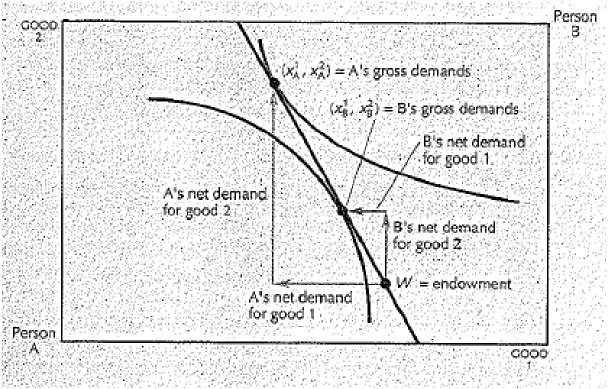


Figure 3: No equilibrium.

Equilibrium existence

How do we know that there exists a set of prices such that (i) and (ii) are simultaneously satisfied?

This is known as the question of the existence of a competitive equilibrium.

Early economists thought that equilibrium prices would always exist because the system has $N-1$ independent (excess demand) equations (by Walras' law) and $N-1$ independent prices.

- There are in fact N prices, but we are free to choose one of the prices and set it equal to a constant, namely, equal to 1 (recall Walrasian demands are homogeneous of degree zero in prices).
- This price plays the role of numeraire, and all other prices are measured relative to it.

Equilibrium existence

However, counting the number of equations and unknowns is not sufficient to prove that a WE exists.

Wald (1936) was the first to points out Walras' error by offering a simple counterexample: $x^2 + y^2 = 0$ and $x^2 - y^2 = 1$.

The key issue is to ensure that the aggregate excess demand function is continuous (less than that is actually necessary).

That means that a small change in the prices shouldn't result in a big jump in the quantity demanded.

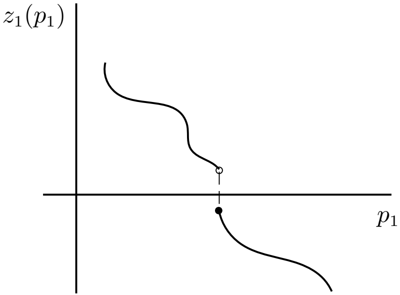


Figure 4: Two goods.

Excess demand

Under what conditions will the aggregate excess demand functions be continuous?

Definition 6 (Excess demand)

The excess demand of agent i is $z^i(p) = x^i(p, y^i(p)) - e^i$, where $x^i(p, y^i(p))$ is i's Walrasian demand at prices p and income $y^i(p) = p \cdot e^i$. The aggregate excess demand is $z(p) = \sum_{i=1}^{I} z^i(p)$.

Note that a WE is a price vector $p^* \in \mathbb{R}^N_+$ such that $z(p^*) = \mathbf{0}$.

Some of the properties of the aggregate excess demand are the following.

Properties of the excess demand

Suppose u^i is continuous, increasing, and concave, and $e^i \gg 0$ for all i.

- $z(\cdot)$ is continuous.
  - Continuity of u^i implies continuity of Walrasian demand $x^i(p, y^i(p))$.
- $z(\cdot)$ homogenous of degree zero (only relative prices matter).
  - $x^i(p, y^i(p))$ homogeneous of degree zero in p.
  - $z^i(p) = x^i(p, y^i(p)) - e^i$ homogeneous of degree zero in p.
  - $\sum_i z^i(p)$ homogeneous of degree zero in p.
  - Therefore, we can normalize one price $\Rightarrow$ $N - 1$ unknowns.
- $z(\cdot)$ verifies Walras' law; i.e., $p \cdot z(p) = 0 \; \forall p$.
  - u^i increasing implies $p \cdot x^i(p, y^i(p)) = p \cdot e^i$.
  - $p \cdot z^i(p) = p \cdot [x^i(p, y^i(p)) - e^i] = 0$.
  - $p \cdot \sum_i z^i(p) = 0$.
  - Therefore, we have $N - 1$ excess demand equations.

Equilibrium existence

Theorem 1 (Existence of WE)

Suppose for all $i = 1, \ldots, I$,

1. $u^i(\cdot)$ *is continuous;*
2. $u^i(\cdot)$ *is increasing; i.e.,* $u^i(\hat{x}) > u^i(x)$ *for any* $\hat{x} \gg x$*;*
3. $u^i(\cdot)$ *is concave;*
4. $e^i \gg 0$*; i.e., agent i has at least a little bit of every good.*

   Then there exists a price system $p^* \in \mathbb{R}^N_+$ *such that* $z(p^*) = \mathbf{0}$.

Wald (1936) offered the first correct proof of equilibrium existence, but for a restrictive class of preferences (separable & decreasing MU). McKenzie (1954) and Arrow & Debreu (1954) were the first to provide general proofs (based on fixed point theorems).

The fascinating story behind the proof of equilibrium existence is very well described in a recent paper by Roy Weintraub, J of Econ Perspectives, 2011.

Equilibrium existence: two-good case

Consider a two-good economy. We can find a WE whenever $z_1(p_1, 1) = 0$.

- $z_1(p_1, 1)$ continuous in p_1.
- $\lim_{p_1 \to +0} z_1(p_1, 1) = \infty$ because u^i is increasing and $e_2^i > 0$ (every i has some money to spend on x_1 even if $p_1 \to^+ 0$).
- $\lim_{p_1 \to \infty} z_1(p_1, 1) < 0$ because u^i is increasing and $e_1^i > 0$ (every i spends money in both goods and value of e_1^i becomes huge relative to value of x_1^i).

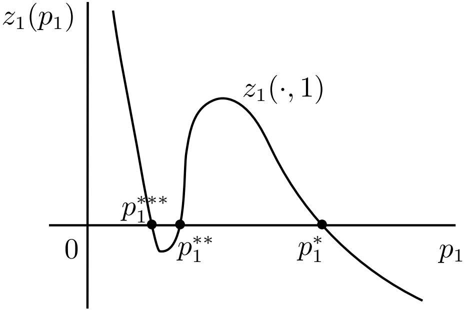


Figure 5: Two goods.

By the intermediate value theorem, there exists p_1^* such that $z_1(p_1^*, 1) = 0$.

Uniqueness

As Figs 5 and 6 indicate, the Walrasian equilibrium need not be unique.

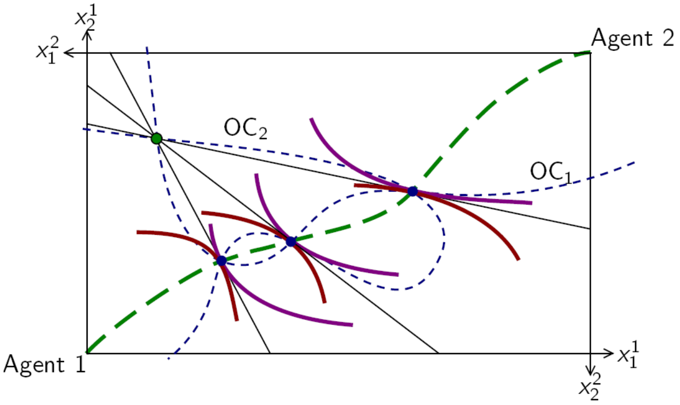
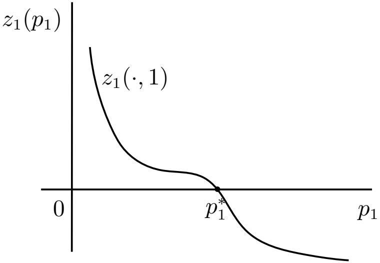


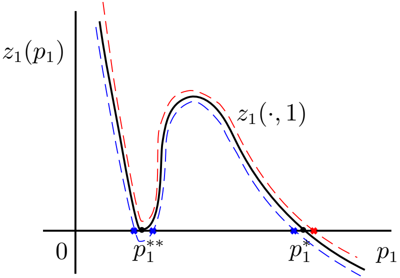
Figure 6: Multiplicity of WE.

Uniqueness

There could be one Walrasian equilibrium.

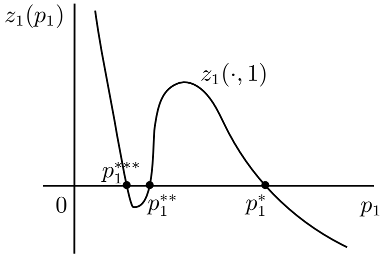


There could be two WE (though this is "non-generic").

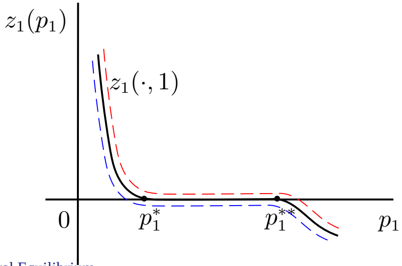


Uniqueness

There could be three Walrasian equilibria.



There could be infinite WE (though this is again "non-generic").



Uniqueness

It seems (and can be formally shown) that:

- WE are globally non-unique (generically).
- WE are locally unique (generically).
- There are a finite number of WE (generically).
- There are an odd number of WE (generically).

Equilibrium and the core

Do competitive markets exhaust all of the gains from trade?

We said that a Walrasian equilibrium requires individual optimality, which translates into the well known condition

$$-MRS^i_{\ell,k}(x^i) = \frac{p_\ell}{p_k} \text{ for all } \ell \neq k \text{ and all } i = 1, \ldots, I. \tag{2}$$

Hence, since all individuals face the same prices, (2) implies that

$$MRS^i_{\ell,k}(x^i) = MRS^j_{\ell,k}(x^j) \text{ for all } \ell \neq k \text{ and all } i \neq j. \tag{3}$$

Walrasian equilibrium involves tangency between consumers' indifference curves through their demanded bundles, as illustrated in Fig. 6.

Thus, for an exchange economy $\mathcal{E} = (u^i, e^i)_{i=1}^I$ satisfying the assumptions of Theorem 1, every Walrasian equilibrium allocation is in the core.

Equilibrium and efficiency

If we denote by $W(e)$ the set of Walrasian equilibrium allocations $x(p^*) = (x^1(p^*, y^1(p^*)), \ldots, x^I(p^*, y^I(p^*)))$, then,

Theorem 2

For an exchange economy $\mathcal{E} = (u^i, e^i)_{i=1}^I$ satisfying the assumptions of Theorem 1, $W(e) \subset C(e)$.

And we also get the following two results:

Corollary 1 (Nonemptiness of the core)

For an exchange economy $\mathcal{E} = (u^i, e^i)_{i=1}^I$ satisfying the assumptions of Theorem 1, $C(e) \neq \emptyset$.

Theorem 3 (First Welfare Theorem (FWT))

Under the hypotheses of Theorem 1, every Walrasian equilibrium allocation $x(p^*)$ is Pareto efficient.

Equilibrium and efficiency

The FWT guarantees that a competitive economy will exhaust all of the gains from trade: a market mechanism, with each agent seeking to maximize his own utility, results in a PE allocation. That was Adam Smith's conjecture!

FWT says nothing about the distribution of economic benefits. That is, a Walrasian equilibrium allocation might not be a 'fair' or desirable allocation.

Having said that, note that in a market economy Pareto efficiency is achieved without demanding much information: each consumer needs to know only his own preferences and endowments and the market prices.

The fact that competitive markets economize on the use of information in the way just described is a strong argument in favor of using them to allocate scarce resources.

Efficiency and equilibrium

What about the converse of FWT?

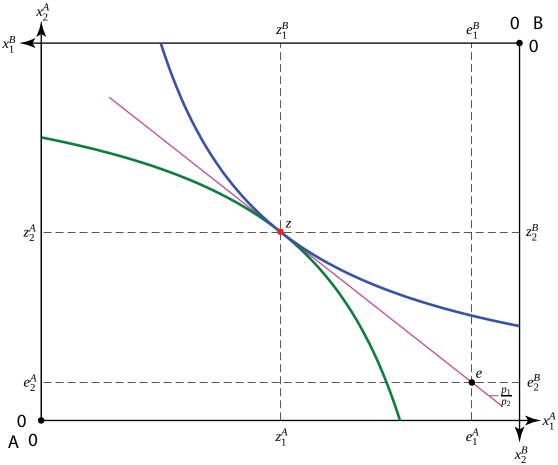
That is, given a Pareto efficient allocation, can we find prices such that those prices and the resulting allocation constitute a Walrasian equilibrium?

It turns out that, under certain conditions, the answer is yes.

The intuition is the following:

- Pick any PE allocation, say z;
- The indifference curves are tangent at z;
- Draw a straight line representing the common slope;
- Suppose now the initial endowments are reallocate such that the straight line denotes the budget constraint;
- The individual's demands associated to that budget line constitute a WE that coincides with the initial PE allocation z.

Efficiency and equilibrium



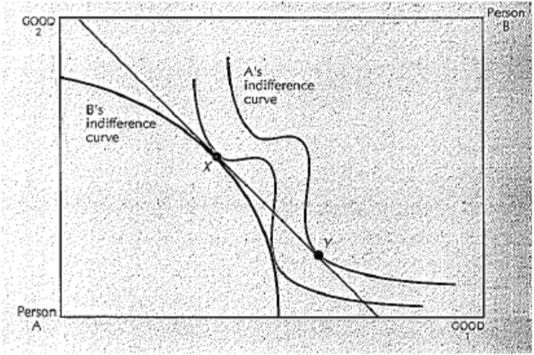
There exists a straight 'budget' line separating the two 'upper counter' sets at z.

The line connects points e and z and has a slope of $-p_1/p_2$.

Efficiency and equilibrium

Is it always possible the construction of such budget line?

Unfortunately, the answer is no.



In the graph, X is PE, but at the budget line that is tangent to the indifference curves at X, agent A demands Y and agent B demands X, so demand doesn't equal supply at these prices.

Efficiency and equilibrium

This observation gives us the Second Welfare Theorem (SWT).

Theorem 4 (Second Welfare Theorem (SWT))

Suppose an exchange economy $\mathcal{E} = (u^i, e^i)_{i=1}^I$ *satisfies the assumptions of Theorem 1. If* $z \in F(e)$ *is Pareto efficient, then* z *is a Walrasian equilibrium allocation for some Walrasian equilibrium prices* p^* *after redistribution of initial endowments to any allocation* $e^* \in F(e)$*, such that* $p^* \cdot e^{*i} = p^* \cdot z^i$.

SWT says that if all agents have convex preferences, there exists a set of prices such that every PE allocation is a WE for an appropriate redistribution of the initial endowments.

SWT implies the problems of distribution and efficiency can be separated.

Whatever PE allocation we wish to achieve can be implemented by the market mechanism, i.e., markets are distributionally neutral.

Efficiency and equilibrium

Prices play two roles in the market system:

1. Allocate role: they indicate (signal) the relative scarcity of the goods.
2. Distributive role: they determine the value of the initial endowments, thereby how much of the different goods each agent can affort.

SWT tells us these two roles can be separated: endowments can be redistributed, and then prices can be used to indicate relative scarcity.

In fact, what is needed is to transfer the purchasing power of the physical endowments, which can be done using nondistortionary (lump-sum) taxes that don't depend on economic agents' choices.

General equilibrium with production

Reference: Jehle and Reny, *Advanced Microeconomic Theory*, 3rd ed., Pearson 2011: Ch. 5.

The important properties of competitive markets saw before continue to hold.

However, production brings with it new matters:

- Firms' profits must be distributed back to the consumers who own the firms.
- Distinction between inputs and outputs become obscure when we view the production size of the economy as a whole (an input for one firm might be an output for another).

Economy

The economy is made of (i) $j = 1, \ldots, J$ firms; (ii) $i = 1, \ldots, I$ consumers; and (iii) $k = 1, \ldots, N$ goods.

Each firm j possesses a production possibility set Y^j.

Assumption 1

- $\mathbf{0} \in Y^j \subseteq \mathbb{R}^N$.
  - Profits are bounded from below by zero.
- Y^j is closed and bounded.
  - Single-value and continues output supply and input demand functions.
- Y^j strongly convex: for all y, $y' \in Y^j$, there exists $\bar{y} \in Y^j$ such that $\bar{y} \geq \alpha y + (1 - \alpha) y'$ for all $\alpha \in (0, 1)$.
  - Rules out constant and increasing returns to scale; ensures profit maximizer is unique.

Producers

A production plan for firm j is a vector $y^j \in Y^j$, with the convention that $y^j_k < 0$ (resp. $y^j_k > 0$) if good k is an input (resp. an output) for j.

Given a nonnegative price vector $p \in \mathbb{R}^N_+$, each firm j solves the profit maximization problem (PMP)

$$\max_{y^j \in Y^j} p \cdot y^j. \tag{4}$$

- The objective function is continuous.
- The constraint set is bounded and closed.
- By the Weierstrass Theorem, a maximum always exists.

For all $p \in \mathbb{R}^N_+$, let $\Pi^j(p) \equiv \max_{y^j \in Y^j} p \cdot y^j$ be firm's j profit function.

Consumers

Each individual $i = 1, \ldots, I$ is endowed by nature with

- A certain (nonnegative) amount $e^i = (e^i_1, \ldots, e^i_N) \in \mathbb{R}^N_+$ of each good;
- A continuous, increasing and quasiconcave utility function $u^i : \mathbb{R}^N_+ \to \mathbb{R}$;
- A fraction (share) $0 \leq \theta^{ij} \leq 1$ of firm j's profits, with

$$\sum_{i=1}^{I} \theta^{ij} = 1 \text{ for all } j.$$

For any $p \geq 0$, let $m^i(p) = p \cdot e^i + \sum_{j=1}^{J} \theta^{ij}\, \Pi^j(p)$ be consumer i's income. In this economy with production and private ownership of firms, consumer i's utility maximization problem (UMP) is

$$\max_{x^i \in B^i(p)} u^i(x^i), \tag{5}$$

where $B^i(p) \equiv \{x^i \in \mathbb{R}^N_+ : p \cdot x^i \leq m^i(p)\}$ denotes consumer i's budget set.

Equilibrium

Denote by $y^j(p)$ and $x^i(p, m^i(p))$ the solutions of (4) and (5), respectively.

The aggregate excess demand function in market k is

$$z_k(p) = \sum_{i=1}^{I} x_k^i(p, m^i(p)) - \sum_{j=1}^{J} y_k^j(p) - \sum_{i=1}^{I} e_k^i, \tag{6}$$

and the aggregate excess demand vector is

$$z(p) = (z_1(p), \ldots, z_N(p)). \tag{7}$$

A Walrasian equilibrium for the economy $\mathcal{E} = (u^i, e^i, \theta^{ij}, Y^j)$ is a price vector $p^* \in \mathbb{R}_+^N$ such that $z(p^*) = \mathbf{0}$.

Theorem 5 (Equilibrium existence with production)

Consider the economy $\mathcal{E} = (u^i, e^i, \theta^{ij}, Y^j)$, where each u^i and Y^j satisfy the assumptions made above and $y + \sum_{i=1}^{I} e^i \gg 0$ for some aggregate production vector $y \in \sum_{j=1}^{J} Y^j$. Then there exists a price system $p^* \in \mathbb{R}_+^N$ for $\mathcal{E}$ such that $z(p^*) = \mathbf{0}$.

Welfare

An allocation $(x,y) = (x^1,\dots,x^I,y^1,\dots,y^J)$ is feasible if $x^i \in \mathbb{R}^N_+$ for all i, $y^j \in Y^j$ for all j, and

$$\sum_{i=1}^{I} x^i = \sum_{i=1}^{I} e^i + \sum_{j=1}^{J} y^j. \qquad (8)$$

Definition 7 (Pareto efficiency with production)

A feasible allocation (x,y) is said to be Pareto efficient (PE) if there is no other feasible allocation $(\hat{x},\hat{y})$ such that $u^i(\hat{x}^i) \geq u^i(x^i)$ for all $i = 1,\dots,I$, with strict inequality for some i.

Theorem 6 (First Welfare Theorem with production)

If each u^i is strictly increasing on $\mathbb{R}^n_+$, then every Walrasian equilibrium allocation (x,y) is Pareto efficient.

Welfare

Theorem 7 (Second Welfare Theorem with production)

Suppose the economy $\mathcal{E} = (u^i, e^i, \theta^{ij}, Y^j)$ satisfies the following assumptions:

1. *u^i is a continuous, increasing and quasiconcave;*
2. *Y^j verifies Assumption 1;*
3. *$\sum_{i=1}^{I} e^i + \sum_{j=1}^{J} y^j \gg 0$ for some $(y^1, \ldots, y^J)$.*

If $(\hat{x}, \hat{y})$ is a Pareto efficient allocation, then there are (i) income transfers $T^1, \ldots, T^I$, with $\sum_{i=1}^{I} T^i = 0$, and (ii) a price vector $\hat{p} \in \mathbb{R}^N_+$ such that:

- *For all $i = 1, \ldots, I$, $\hat{x}^i$ maximizes $u^i(x^i)$ subject to $\hat{p} \cdot x^i \leq m^i(\hat{p}) + T^i$;*
- *For all $j = 1, \ldots, J$, $\hat{y}^j$ maximizes $\hat{p} \cdot y^j$ subject to $y^j \in Y^j$.*