Strategic-Form Games

References:


A game in strategic or normal form has three elements:

- Finite set $\mathcal{N} = \{1, \ldots, N\}$ of players, with generic element $i \in \mathcal{N}$;
- Pure strategy sets $S_i$ for each player $i \in \mathcal{N}$, with generic element $s_i$;
- The payoff functions $u_i(\cdot)$ for each player $i \in \mathcal{N}$, that give $i$’s vNM utility $u_i(s)$ for each $s \in S \equiv \prod_{i \in \mathcal{N}} S_i$.

Usually, the structure of the strategic game $G = (S_i, u_i)_{i \in \mathcal{N}}$ is assumed to be common knowledge.
Nash Equilibrium

A mixed strategy for player $i$ is a probability distribution $\sigma_i$ over $S_i$.

Player $i$’s (expected) payoff to profile $(\sigma_i, \sigma_{-i}) \in \Sigma \equiv \prod_{i=1}^{N} \Sigma_i$ is

$$U_i(\sigma_i, \sigma_{-i}) = \sum_{s \in S} \left( \prod_{j=1}^{N} \sigma_j(s_j) \right) u_i(s).$$  

(1)

Definition 1 (NE)

A mixed strategy profile $\sigma^* = (\sigma_i^*, \sigma_{-i}^*) \in \Sigma$ is a Nash Equilibrium for $G = (S_i, u_i)_{i \in \mathcal{N}}$ if for all player $i \in \mathcal{N}$,

$$U_i(\sigma_i^*, \sigma_{-i}^*) \geq U_i(s_i, \sigma_{-i}^*) \text{ for all } s_i \in S_i.$$  

(2)

A pure strategy Nash equilibrium is a profile of pure strategies that satisfies the same conditions (recall pure strategies are degenerated mixed strategies).
Nash Equilibrium Existence I: Finite games

Theorem 1 (Nash 1950)

Every finite strategy-form game $G = (S_i, u_i)_{i \in \mathcal{N}}$ has a mixed strategy equilibrium.

The idea of the proof is to apply Kakutani’s fixed point theorem (FPT) the the players’ reaction correspondences.

Player $i$’s reaction correspondence, $r_i$, maps each strategy profile $\sigma$ to the set of mixed strategies that maximize player $i$’s conditional payoff $U_i(\cdot, \sigma_{-i})$.

Let $r: \Sigma \rightrightarrows \Sigma$ be the Cartesian product of the $r_i$, such that $r(\sigma) = \prod_{i=1}^N r_i(\sigma)$ for all $\sigma \in \Sigma$.

A fixed point of $r(\cdot)$ is a profile $\sigma \in \Sigma$ such that $\sigma \in r(\sigma)$ (alternatively, it is a profile $\sigma \in \Sigma$ such that $\sigma_i \in r_i(\sigma_i, \sigma_{-i})$ for all $i \in \mathcal{N}$).
Nash Equilibrium Existence I: Finite games

From Kakutani’s FPT, \( r : \Sigma \rightarrow \Sigma \) has a fixed point if:

1. \( \Sigma \) is a compact, convex, and nonempty subset of a (finite-dimensional) Euclidean space;
2. \( r(\sigma) \) is nonempty for all \( \sigma \in \Sigma \);
3. \( r(\sigma) \) is convex for all \( \sigma \in \Sigma \);
4. \( r(\cdot) \) has a closed graph, in the sense that for all convergent sequences \((\sigma^n, \hat{\sigma}^n) \rightarrow (\sigma, \hat{\sigma})\) with \( \hat{\sigma}^n \in r(\sigma^n) \), it follows that \( \hat{\sigma} \in r(\sigma) \).

N.B. Property (4) is usually referred to as u.h.c.: \( r(\cdot) \) is u.h.c. if for any \( \sigma^* \) and for any open set \( V \supseteq r(\sigma^*) \), there exists \( \delta > 0 \) such that for all \( \sigma \in B_\delta(\sigma^*) \) we have that \( r(\sigma) \subseteq V \).

N.B. Recall that a subset \( X \) of a Euclidean space is compact if any sequence in \( X \) has a subsequence that converges to a limit point in \( X \).
Nash Equilibrium Existence I: Finite games

Cond. (1) is easy, since each $\Sigma_i$ is a simplex of dimension $|S_i| - 1$.

For cond. (2), note that each conditional payoff $U_i(\cdot, \sigma_{-i})$ is linear (and therefore continuous) in $\sigma_i$; and since continuous functions on a compact set attain a maxima, $r_i(\cdot)$ is nonempty and cond. (2) is satisfied.

For cond. (3), fix any $\sigma \in \Sigma$, and consider any two profiles $\sigma', \sigma'' \in r(\sigma)$ and any scalar $\alpha \in (0, 1)$. We wish to show that $\alpha \sigma' + (1 - \alpha) \sigma'' \in r(\sigma)$.

By definition, for each player $i$,

$$U_i(\alpha \sigma'_i + (1 - \alpha) \sigma''_i, \sigma_{-i}) = \sum_{s \in S} \left[ \alpha \sigma'_i(s_i) + (1 - \alpha) \sigma''_i(s_i) \right] \left( \prod_{j \neq i} \sigma_j(s_j) \right) u_i(s)$$

$$= \alpha \sum_{s \in S} \sigma'_i(s_i) \left( \prod_{j \neq i} \sigma_j(s_j) \right) u_i(s) + (1 - \alpha) \sum_{s \in S} \sigma''_i(s_i) \left( \prod_{j \neq i} \sigma_j(s_j) \right) u_i(s) \tag{\text{= } U_i(\sigma'_i, \sigma_{-i}) \text{ and } U_i(\sigma''_i, \sigma_{-i})}$$
Nash Equilibrium Existence I: Finite games

Therefore, \( U_i(\alpha \sigma'_i + (1 - \alpha) \sigma''_i, \sigma_{-i}) = \alpha U_i(\sigma'_i, \sigma_{-i}) + (1 - \alpha) U_i(\sigma''_i, \sigma_{-i}) \).

Moreover, since \( \sigma'_i, \sigma''_i \in r_i(\sigma) \), it must be that \( U_i(\sigma'_i, \sigma_{-i}) = U_i(\sigma''_i, \sigma_{-i}) \).

Hence, \( \alpha \sigma'_i + (1 - \alpha) \sigma''_i \in r_i(\sigma) \) as well, which provides the desired result since this holds for all \( i \in \mathcal{N} \).

Finally, for cond. (4), suppose by contradiction that there exists a sequence \( (\sigma^n, \hat{\sigma}^n) \rightarrow (\sigma, \hat{\sigma}) \) with \( \hat{\sigma}^n \in r(\sigma^n) \), but \( \hat{\sigma} \not\in r(\sigma) \). Thus, for some player \( i \), \( \hat{\sigma}_i \not\in r_i(\sigma) \). That means there must exist \( \epsilon > 0 \) and a \( \sigma^*_i \in \Sigma_i \) such that

\[
U_i(\sigma^*_i, \sigma_{-i}) > U_i(\hat{\sigma}_i, \sigma_{-i}) + 3 \epsilon
\]
Nash Equilibrium Existence I: Finite games

\[
\iff \quad U_i(\sigma_i^*, \sigma_{-i}) - \epsilon > U_i(\hat{\sigma}_i, \sigma_{-i}) + 2\epsilon. \tag{3}
\]

Moreover, since \( U_i \) is continuous and the sequence \((\sigma^n, \hat{\sigma}^n)\) converges to \((\sigma, \hat{\sigma})\), for \( n \) large enough

\[
U_i(\sigma_i^*, \sigma^n_{-i}) > U_i(\sigma_i^*, \sigma_{-i}) - \epsilon, \tag{4}
\]

and

\[
U_i(\hat{\sigma}_i, \sigma_{-i}) + 2\epsilon > U_i(\hat{\sigma}_i^n, \sigma^n_{-i}) + \epsilon. \tag{5}
\]

Combining (3)-(5), we have

\[
U_i(\sigma_i^*, \sigma^n_{-i}) > U_i(\hat{\sigma}_i^n, \sigma^n_{-i}) + \epsilon,
\]

implying that \( \hat{\sigma}_i^n \not\in r_i(\sigma^n) \), a contradiction.
Nash Equilibrium Existence II: Infinite games

Many economic models have an uncountable number of actions (e.g. Cournot, Bertrand, Hotelling, etc.).

These games with continuum action/strategy spaces are called infinite games.

Their study is important not only because they are used to represent many economic situations, but also because when the infinite game does not have a Nash equilibrium, the equilibria corresponding to the fine, discrete grids (whose existence is guaranteed by Theorem 1), could be very sensitive to which finite grid is actually chosen.

If there were equilibria of the finite game insensible to the grid chosen, one could take a sequence of finer and finer grids converging to the continuum, and the limit of a convergent subsequence of the discrete-action-space equilibria would be a Nash equilibrium of the continuum game.
Nash Equilibrium Existence II: Infinite games

Consider a strategic game \( G = (S_i, u_i)_{i \in \mathcal{N}} \) whose strategy spaces \( S_i \) are nonempty, compact, and convex subsets of \( \mathbb{R}^m \) (\( m \) finite).

**Theorem 2 (Debreu 1952-Glicksberg 1952- Fan 1952)**

*If for all \( i \in \mathcal{N} \), the payoff function \( u_i(\cdot) \) is continuous in \( s \) and quasi-concave in \( s_i \), then \( G \) has a pure strategy equilibrium.*

The proof is similar to that of Nash’s theorem.

- Continuous payoffs imply nonempty and closed-graph reaction correspondences;
- Quasi-concavity in players’ own actions implies convex valued reaction correspondences.

When payoffs are continuous but they are not quasi-concave, mixed strategies can be used to ensure convex valued reaction correspondences.
Nash Equilibrium Existence II: Infinite games

Consider a strategic game $G = (S_i, u_i)_{i \in \mathcal{N}}$ whose strategy spaces $S_i$ are nonempty and compact subsets of a metric space.

**Theorem 3 (Glicksberg 1952)**

*If for all $i \in \mathcal{N}$, the payoff functions $u_i(\cdot)$ is continuous, then $G$ has a mixed strategy equilibrium.*

Here the mixed strategies are (Borel) probability measures over the pure strategies, which we endow with the topology of weak convergence.

N.B. A sequence of probability measures $\mu^n$ on a metric space $A$ converges weakly to a limit $\mu$ if $\int f d\mu^n \to \int f d\mu$ for every real valued and continuous function $f$ on $A$.

N.B. The set of probability measures with the topology of weak convergence is a compact set.
Nash Equilibrium Existence II: Infinite games

The proof of Theorem 3 applies again a fixed point theorem to the reaction correspondences.

The introduction of mixed strategies makes:

- The strategy sets convex;
- The payoffs linear in own strategy and continuous in all strategies (due to weak convergence and continuity of $u_i$);
- The reaction correspondences convex valued.

With infinite many pure strategies, the space of mixed strategies is infinite dimensional, so a more powerful FPT than Kakutani’s is required.

Alternatively, one can approximate the strategy spaces by a sequence of finite grids, each of which has a mixed strategy equilibrium by Theorem 1.
Nash Equilibrium Existence II: Infinite games

Since the space of probability measures is weakly compact, the sequence of the discrete MSE has an accumulation point.

Finally, the limit point is a MSE of the original game due to the continuity of the expected payoffs in the space of probability measures.

When the payoff functions $u_i$ are discontinuous pure as well as mixed strategy equilibria can easily fail to exist.

In particular, Glicksberg’s theorem can’t be used because the discontinuity of the pure-strategy payoffs implies that the expected payoffs are also discontinuous in the space of probability measures.

Thus, best responses may fail to exist for some of the opponents’ strategies.
Application: Election with Discontinuous Payoffs

A number of games in the economic literature have discontinuous and/or non-quasi-concave payoff functions. Here is an example.

Consider two political candidates/parties, \( i = L, R \), which compete in a winner-take-all election by simultaneously announcing a policy platform \( x_i \in X = [0, 1] \).

The electorate is made of a continuum of voters.

Each voter has a preferred policy \( \theta \in X \) and a utility function \( u_\theta(x) = -|x - \theta| \), where \(|\cdot|\) denotes the absolute value on \( \mathbb{R} \).

Voters vote sincerely for the platform they like the most.

Candidate \( i \) wins the election if its platform \( x_i \) gets more than half of the votes, with ties broken by a fair coin toss.

Apart from the uncertainty due to the possibility of a tie, candidates have uncertainty about voters’ preferences.
Application: Election with Discontinuous Payoffs

We assume that the median voter’s ideal point, denoted by $\theta_m$, is uniformly distributed over $[1/2 - \beta, 1/2 + \beta]$, with $\beta > 0$.

Thus, the probability that candidate $L$ attaches to winning the election is:

- $p(x_L, x_R) = \text{Prob} \left( \theta_m \in \left[ 0, \frac{x_L + x_R}{2} \right] \right)$ if $x_L \leq x_R$, and
- $p(x_L, x_R) = \text{Prob} \left( \theta_m \in \left[ \frac{x_L + x_R}{2}, 1 \right] \right)$ if $x_L > x_R$.

Candidate $R$’s probability of winning is $1 - p(x_L, x_R)$. 

\[
\begin{align*}
\text{if } x_L \leq x_R, & \quad \text{then } p(x_L, x_R) = \text{Prob} \left( \theta_m \in \left[ 0, \frac{x_L + x_R}{2} \right] \right) \\
\text{if } x_L > x_R, & \quad \text{then } p(x_L, x_R) = \text{Prob} \left( \theta_m \in \left[ \frac{x_L + x_R}{2}, 1 \right] \right)
\end{align*}
\]
Application: Election with Discontinuous Payoffs

Candidates possess mixed or hybrid motives for running for office:

- They value winning the election; but
- They care about policy too.

Formally, the payoffs for any profile \((x_L, x_R) \in X^2\) are:

\[
\Pi_L(x_L, x_R) = p(x_L, x_R) \cdot (u_{\theta_L}(x_L) + \chi_L) + [1 - p(x_L, x_R)] \cdot u_{\theta_L}(x_R),
\]

and

\[
\Pi_R(x_L, x_R) = [1 - p(x_L, x_R)] \cdot (u_{\theta_R}(x_R) + \chi_R) + p(x_L, x_R) \cdot u_{\theta_R}(x_L),
\]

where \(\theta_i\) stands for candidate \(i\)'s ideological (preferred) position on \(X\), and \(\chi_i > 0\) denotes its payoff from being in power (office rents).

We assume that \(\theta_L < 1/2 < \theta_R\), and \(\beta < \min\left\{ \frac{1}{2} - \theta_L + \frac{\chi_L}{2}, \theta_R - \frac{1}{2} + \frac{\chi_R}{2} \right\}\).
Application: Election with Discontinuous Payoffs

Unfortunately, these payoff functions are not quasi-concave.

For instance, when $\chi_R = 0.2$, $\chi_L = 0.6$, $\beta = 0.25$, $\theta_L = 0.2$, and $\theta_R = 0.9$, the conditional payoff functions look like this:

Figure 1: Party L’s conditional payoff function given $x_R^* = 0.65$.

Figure 2: Party R’s conditional payoff function given $x_L^* = 0.55$.

For these values of the parameters a PSE doesn’t exist.
Application: Election with Discontinuous Payoffs

Nonetheless, the election game has an equilibrium in mixed strategies.

Let $\Delta$ be the space of probability measures on the Borel subsets of $X$. A mixed strategy for $i$ is a probability measure $\mu_i \in \Delta$, with support $\text{supp}(\mu_i) \equiv \{x \in X : \forall \epsilon > 0, \mu_i((x - \epsilon, x + \epsilon) \cap X) > 0\}$.

We extend each $\Pi_i$ to $\Delta^2$ by

$$U_i(\mu_L, \mu_R) = \int_{X^2} \Pi_i(x_L, x_R) d(\mu_L(x_L) \times \mu_R(x_R)).$$

Note that $U_i$ is well defined because the set of discontinuities of $\Pi_i$, namely $\{(x_L, x_R) \in X^2 : x_L = x_R \neq 1/2\}$, has measure zero.

The support of the mixed strategy equilibrium when the right-wing candidate is relatively more ideological is characterized in the next proposition.
Application: Election with Discontinuous Payoffs

If $\chi_R/2 < \beta < \beta^C_1$, the election game $G = (X, \Pi_i)_{i=L,R}$ has a mixed strategy equilibrium $(\mu^*_L, \mu^*_R) \in \Delta^2$ with the property that,

(a) If $\beta \leq \frac{\chi_L + \chi_R}{4}$, then $\text{supp}(\mu^*_i) = [x, \bar{x}]$ for all $i = L, R$, with $x = \tilde{x}_L(\beta, \chi_R)$ and $\bar{x} = \frac{1}{2} + \beta - \frac{\chi_R}{2} = x^*_R$; and

(b) If $\beta > \frac{\chi_L + \chi_R}{4}$, then $\text{supp}(\mu^*_L) = [x, \bar{x}]$ and $\text{supp}(\mu^*_R) = [x, \bar{x}] \cup \{x^*_R\}$, with $x = \tilde{x}_L(\beta, \chi_R)$ and $\bar{x} = \frac{1}{2} - \beta + \frac{\chi_L}{2} = x^*_L$.

$\triangleright \quad \beta^C_1 \equiv \frac{\chi_L - \chi_R}{4} + \frac{\sqrt{\chi_L \chi_R}}{2}$;

$\triangleright \quad \tilde{x}_L(\beta, \chi_R)$ is the solution to

$$\Pi_R(x'_L, x^*_R) - \limsup x_R \to -x'_L \Pi_R(x'_L, x_R) = 0.$$
Nash Equilibrium Existence III: Discontinuous Payoffs

Suppose each strategy set $S_i$ is a nonempty, compact and convex subset of some finite dimensional Euclidean space.

With discontinuous payoffs, a compact strategy space no longer ensures that a player’s optimal response to his opponents’ strategies exists.

To guarantee existence, we assume that the payoffs are upper semi-continuous (u.s.c.). In words, a function is u.s.c. if it doesn’t jump down.

**Definition 2 (u.s.c.)**

A payoff function $u_i(\cdot)$ on $S = \prod_{i=1}^{N} S_i$ is upper semi-continuous at $s$ if for any sequence $s^n$ converging to $s$, $\limsup_{n \to \infty} u_i(s^n) \leq u_i(s)$.

N.B. The limit superior (limit sup) of a sequence $x^n$ in $\mathbb{R}$ is the smallest $x$ such that for all $\epsilon > 0$ there exists an integer $N$ such that $x^n \leq x + \epsilon$ for all $n > N$.

Note that the conditional payoffs of Figs. 1 and 2 fail to be u.s.c. at the point of discontinuity.
Nash Equilibrium Existence III: Discontinuous Payoffs

Define the set of player $i$’s optimal pure reactions to pure strategies $s_{-i}$ as

$$r_i^*(s_{-i}) = \left\{ s_i \in S_i : u_i(s_i, s_{-i}) \in \max_{\bar{s}_i \in S_i} u_i(\bar{s}_i, s_{-i}) \right\}.$$  \hfill (6)

Note that $r_i^*$ differs from the reaction correspondence $r_i$ in that for a given $s_{-i}$, $r_i$ is the convex hull of the points in $r_i^*$ (i.e., includes mixed best responses).

If $S_i$ is compact and $u_i(\cdot, s_{-i})$ is u.s.c., then $r_i^*(s_{-i})$ is nonempty.

Indeed, take a sequence $\{s^n_i\} \subset S_i$, such that $\lim_{n \to \infty} u_i(s^n_i, s_{-i}) = \sup_{\bar{s}_i \in S_i} u_i(\bar{s}_i, s_{-i})$ (which exists due to u.s.c.). Since $S_i$ is compact, $\{s^n_i\}$ has a convergent subsequence, with limit $\hat{s}_i \in S_i$ say. By u.s.c.,

$$u_i(\hat{s}_i, s_{-i}) \geq \limsup_{n \to \infty} u_i(s^n_i, s_{-i}) \geq \lim_{n \to \infty} u_i(s^n_i, s_{-i}) = \sup_{\bar{s}_i \in S_i} u_i(\bar{s}_i, s_{-i}).$$

Thus, $u_i(\hat{s}_i, s_{-i}) \geq \sup_{\bar{s}_i \in S_i} u_i(\bar{s}_i, s_{-i})$, which implies that a maximand exists.
Nash Equilibrium Existence III: Discontinuous Payoffs

Let \( r^* : S \rightrightarrows S \) be the pure strategy reaction correspondence. Formally, \( r^* \) is the Cartesian product of the \( r^*_i \), such that \( r^*(s) = \prod_{i=1}^{N} r^*_i(s) \) for all \( s \in S \).

To prove that \( r^* \) has a fixed point we use Kakutani’s theorem.

Nonemptyness of \( r^*(s) \) is ensured by u.s.c.

To guarantee that \( r^*(s) \) is convex valued for all \( s \in S \), we assume that for all \( i \in \mathcal{N} \) and \( s_{-i} \in S_{-i} \), the conditional payoff \( u_i(\cdot, s_{-i}) \) is quasi-concave in \( s_i \).

Finally, the following condition together with u.s.c. ensures that \( r^* \) has a closed graph.

**Definition 3**

A function \( u_i(\cdot) \) has a **continuous maximum** if \( u^*_i(s_{-i}) = \max_{s_i \in S_i} u_i(s_i, s_{-i}) \) is continuous in \( s_{-i} \).
Nash Equilibrium Existence III: Discontinuous Payoffs

A continuous maximum and u.s.c. imply that \( r^*_i \) has a closed graph. Suppose not. Then, there must exist a sequence \( s^n \to s \), with \( s^n_i \in r^*_i(s^n_{-i}) \) and \( s_i \not\in r^*_i(s_{-i}) \). Since \( s_i \) is not a best response to \( s_{-i} \), we have

\[
\max_{s'_i \in S_i} u_i(s'_i, s_{-i}) > u_i(s_i, s_{-i}).
\]  

(7)

By upper semi-continuity,

\[
u_i(s_i, s_{-i}) \geq \limsup_{n \to \infty} u_i(s^n_i, s_{-i}) = \limsup_{n \to \infty} u_i(s^n_i, s^n_{-i}).
\]  

(8)

Moreover, since by hypothesis \( s^n_i \in r^*_i(s^n_{-i}) \),

\[
\limsup_{n \to \infty} u_i(s^n_i, s^n_{-i}) = \limsup_{n \to \infty} \left( \max_{s'_i \in S_i} u_i(s'_i, s^n_{-i}) \right).
\]  

(9)

Thus, combining (7)-(9), we get

\[
\max_{s'_i \in S_i} u_i(s'_i, s_{-i}) > \limsup_{n \to \infty} \left( \max_{s'_i \in S_i} u_i(s'_i, s^n_{-i}) \right),
\]

contradicting the assumption of a continuous maximum.
Nash Equilibrium Existence III: Discontinuous Payoffs

Consider a strategic game \( G = (S_i, u_i)_{i \in \mathcal{N}} \) whose strategy spaces \( S_i \) are nonempty, compact, and convex subsets of \( \mathbb{R}^m \) (\( m \) finite).

**Theorem 4 (Dasgupta-Maskin 1986)**

*If for all \( i \in \mathcal{N} \), the payoff function \( u_i(\cdot) \) is u.s.c. in \( s \), is quasi-concave in \( s_i \), and has a continuous maximum, then \( G \) has a pure strategy equilibrium.*

If either u.s.c. or quasi-concavity do not hold, one way to go is to explore the existence of mixed equilibria.

Consider a sequence of finite approximations \( S^n_i \) of \( S_i \) converging to \( S_i \) for all \( i \).

By Theorem 1, each discretized game \( G^n = (S^n_i, u_i) \) has a mixed strategy equilibrium \( \sigma^n \); that is,

\[
U_i(\sigma^n_i, \sigma^n_{-i}) \geq U_i(s_i, \sigma^n_{-i}) \quad \text{for all} \quad s_i \in S^n_i, \quad \text{and all} \quad i \in \mathcal{N}. \quad (10)
\]
Nash Equilibrium Existence III: Discontinuous Payoffs

Since the space of probability measures on $S_i$ is compact under the topology of weak convergence, a subsequence of $\sigma^n$ converges to some MSE $\sigma^*$.

If either (i) $u_i(\cdot)$ is continuous, or (ii) $\sigma^n$ assigns vanishingly small probability on the discontinuity points of the payoff functions, we would have that

$$U_i(\sigma^n_i, \sigma^n_{-i}) \to U_i(\sigma^*_i, \sigma^*_{-i})$$

and

$$U_i(s_i, \sigma^n_{-i}) \to U_i(s_i, \sigma^*_{-i}).$$

Thus, (10) would imply that $U_i(\sigma^*_i, \sigma^*_{-i}) \geq U_i(s_i, \sigma^*_{-i})$ for all $s_i \in S_i$, and all $i \in \mathcal{N}$. That is, $\sigma^*$ would be a MSE of the limit game $G = (S_i, u_i)_{i \in \mathcal{N}}$.

If neither (i) nor (ii) are satisfied, the challenge is to find conditions that ensure that the discontinuities do not matter in the limit game.

The two conditions needed are (1) u.s.c. of the sum of the payoffs $\sum_i u_i$, and (2) weakly lower semi-continuity (w.l.s.c.).
Nash Equilibrium Existence III: Discontinuous Payoffs

The definition of w.l.s.c. requires some notation. Let \( S^{**}(i) \) denote the set of strategy profiles \( s \in S \) such that \( u_i \) is discontinuous at \( s \); and let \( S^{**}(s_i) = \{ s_{-i} \in S_{-i} : (s_i, s_{-i}) \in S^{**}(i) \} \).

In the election game discussed above, \( S^{**}(i), i = L, R \), would be the diagonal of the strategy space, that is, \( S^{**}(i) = \{ (x_L, x_R) \in X^2 : x_L = x_R \neq 1/2 \} \).

Suppose the discontinuities occur only on a subset of measure 0 in which a player’s strategy is “related” to another player’s.

That is, for any two players \( i, j \in N \), assume there exists a finite number of functions \( f_{ij}^d : S_i \rightarrow S_j \), where \( d \) is an index, that are one-to-one and continuous, such that for each player \( i \in N \)

\[
S^{**}(i) \subseteq S^*(i) = \{ s \in S : \exists j \neq i, \exists d \text{ such that } s_j = f_{ij}^d(s_i) \}.
\]

In the election game the function that relates the strategies of both players at the discontinuities is the identity function.
Nash Equilibrium Existence III: Discontinuous Payoffs

Definition 4 (w.l.s.c.)
The payoff function \( u_i(s_i, s_{-i}) \) is w.l.s.c. in \( s_i \) if for all \( s_i \) there exists \( \lambda \in [0, 1] \) such that, for all \( s_{-i} \in S_{-i}(s_i) \),

\[
   u_i(s_i, s_{-i}) \leq \lambda \liminf_{s_i' \rightarrow -s_i} u_i(s_i', s_{-i}) + (1 - \lambda) \liminf_{s_i' \rightarrow +s_i} u_i(s_i', s_{-i}). \tag{11}
\]

The inequality (11) says that \( u_i \) does not jumps up when \( s_i' \) tends to \( s_i \) either from the left, from the right, or both. Instead, if \( u_i(s_i, s_{-i}) \) were greater than “both lim inf”, then there would no \( \lambda \in [0, 1] \) for which (11) would hold.

Roughly, w.l.s.c. implies that player \( i \) can do as well with strategies near \( s_i \) as with \( s_i \), even if the rivals’ strategies put weight on the discontinuities of \( u_i \).

Theorem 5 (Dasgupta-Maskin 1986)
Let \( S_i = [s_i, \overline{s}_i] \subset \mathbb{R} \). Suppose \( u_i \) is continuous except on \( S^{**}(i) \subseteq S^*(i) \), is bounded, and w.l.s.c. in \( s_i \). If \( \sum u_i(s) \) is u.s.c., then the game \( G = (S_i, u_i)_{i \in \mathbb{N}} \) has a mixed strategy equilibrium.
Nash Equilibrium Existence IV: Discontinuous Payoffs

An strategic game \( G = (S_i, u_i)_{i \in \mathcal{N}} \) is said to be compact if \( S_i \) is a nonempty compact subset of a topological vector space, and \( u_i : S \rightarrow \mathbb{R} \) is bounded.

In addition, \( G \) is said to be quasi-concave if \( S_i \) is convex and for every \( s_{-i} \), \( u_i(\cdot, s_{-i}) \) is quasi-concave on \( S_i \).

**Definition 5 (Security)**

Player \( i \) can secure a payoff of \( \alpha \) at \( s \in S \) if there exists \( \tilde{s}_i \in S_i \), such that \( u_i(\tilde{s}_i, s'_{-i}) \geq \alpha \) for all \( s'_{-i} \) in some open neighborhood of \( s_{-i} \).

Thus, a payoff can be secured by \( i \) at \( s \) if \( i \) has a strategy that guarantees at least that payoff even if the other players deviate slightly from \( s \).

**Definition 6 (b.r.s.)**

A game \( G = (S_i, u_i)_{i \in \mathcal{N}} \) is better-reply secure if whenever \( (s^*, u^*) \) is in the closure of the graph of its vector payoff function and \( s^* \) is not an equilibrium, some player \( i \) can secure a payoff strictly above \( u^*_i \) at \( s^* \).
Nash Equilibrium Existence IV: Discontinuous Payoffs

In words, a game is b.r.s. if for every non-equilibrium strategy $x^*$ and every payoff vector limit $u^*$ resulting from strategies approaching $x^*$, some player $i$ has a strategy yielding a payoff strictly above $u_i^*$ even if the others deviate slightly from $x^*$.

Games with continuous payoff functions are better-reply secure, since any better reply will secure a payoff strictly above a player’s inferior non-equilibrium payoff and those generated by nearby strategies.

**Theorem 6 (Reny 1999)**

*If $G = (S_i, u_i)_{i \in \mathcal{N}}$ is compact, quasi-concave, and better-reply secure, then it possesses a pure strategy Nash equilibrium.*

The above result on the existence of pure strategy equilibria generalizes the mixed strategy equilibrium existence results of Nash 1950, Glicksberg 1952, and Dasgupta and Maskin 1986.
Nash Equilibrium Existence IV: Discontinuous Payoffs

While b.r.s. is straightforward to verify, it is sometimes even simpler to verify other conditions leading to it.

Better-reply security combines and generalizes two conditions, reciprocal upper semi-continuity (r.u.s.c.) and payoff security (p.s.).

**Definition 7 (p.s.)**
A game $G = (S_i, u_i)_{i \in \mathcal{N}}$ is payoff secure if for every $s \in S$ and every $\epsilon > 0$, each player $i$ can secure a payoff of $u_i(s) - \epsilon$ at $s$.

A game is p.s. if for every joint strategy $s$, each player has a strategy that virtually guarantees the payoff he receives at $s$, even if the others play slightly differently than at $s$.

It embeds the idea of robustness of one’s payoff to perturbations in the others’ strategies (a kind of “continuity” in the others’ actions).
Nash Equilibrium Existence IV: Discontinuous Payoffs

Definition 8 (r.u.s.c.)

A game $G = (S_i, u_i)_{i \in \mathcal{N}}$ is reciprocally upper semi-continuous if, whenever $(s^*, u^*)$ is in the closure of the graph of its vector payoff function and $u_i(s^*) \leq u_i^*$ for every player $i$, then $u_i(s^*) = u_i^*$ for every $i$.

This condition was introduced by Simon (1987) under the name of “complementary discontinuities.” It requires that some player’s payoff jumps up when some other player’s payoff jumps down.

The role played by r.u.s.c. in ensuring existence of equilibrium is similar to that played by u.s.c. of a function in guaranteeing the existence of a maximum of a real-valued function.

It generalizes the condition introduced by Dasgupta and Maskin (1986) that the sum of the players’ payoffs be u.s.c. Since the u.s.c.-sum condition is cardinal in nature, this generalization is a distinct improvement.
Nash Equilibrium Existence IV: Discontinuous Payoffs

None of the three conditions, b.r.s., p.s. and r.u.s.c., is an ordinal property. However, they are all virtually so in the following sense.

If $f_i : \mathbb{R} \rightarrow \mathbb{R}$ is continuous and strictly increasing for every $i \in \mathcal{N}$, $(S_i, u_i)_{i \in \mathcal{N}}$ is b.r.s. (resp., r.u.s.c./p.s.) if and only if $(S_i, f_i \circ u_i )_{i \in \mathcal{N}}$ is.

Proposition 1 (Reny 1999)

If $G = (S_i, u_i)_{i \in \mathcal{N}}$ is r.u.s.c. and p.s., then it is b.r.s.

When one moves to mixed strategies, securing any particular payoff becomes both easier and more difficult.

- It becomes easier because one can now employ mixed strategies to attempt to secure a payoff and this can increase the payoff one can secure.
- On the other hand, one’s payoff must be secure against perturbations from the larger set of mixed strategies of the others, which can reduce the payoff one can secure.
Nash Equilibrium Existence IV: Discontinuous Payoffs

As a result, although r.u.s.c. of the mixed extension $\overline{G} = (\Sigma_i, U_i)$ implies that of $G = (S_i, u_i)$, b.r.s. (resp., p.s.) of $\overline{G}$ neither implies nor is implied by b.r.s. (resp., p.s.) of $G$.

The following result provides a convenient means for checking r.u.s.c. of $\overline{G}$.

**Proposition 2 (Reny 1999)**

If $\sum_{i=1}^{N} u_i(s)$ is u.s.c. in $s$ on $S$, the mixed extension $\overline{G} = (\Sigma_i, U_i)$ is r.u.s.c.

The result follows from the fact that $\sum_{i=1}^{N} U_i(\sigma)$ is u.s.c. in $\sigma$ on $\Sigma$.

**Theorem 7 (Reny 1999)**

Suppose that $G = (S_i, u_i)$ is a compact (Hausdorff) game. Then $G$ possesses a mixed strategy Nash equilibrium if its mixed extension, $\overline{G} = (\Sigma_i, U_i)$, is b.r.s. Moreover, $\overline{G}$ is b.r.s. if it is both r.u.s.c. and p.s.
Nash Equilibrium Existence V: Other Approaches

An alternative approach to the “the topological approach to existence of equilibrium” is based upon lattice-theoretical concepts, and at its heart lies Tarski’s (1955) fixed point theorem.

Payoffs need not be quasi-concave and continuous. The key property is that best-replies are increasing in the opponents’ strategies.

This method typically yields the existence of pure strategy equilibria.

Now, while lattice-theoretic methods do not require payoffs to be continuous, enough continuity must be assumed in order to guarantee the existence of best replies. So, typically payoffs are required to be u.s.c. in one’s own strategy, an assumption that fails to hold in virtually all auctions, as well as in the classic games of Bertrand and Hotelling.

Consequently, most practical applications of lattice-theoretical techniques tend to be confined to continuous games.
Nash Equilibrium Existence V: Other Approaches

A third approach, although still fundamentally topological in nature, has been introduced by Simon and Zame (1990).

They modify the vector of payoffs at points of discontinuity so that they correspond to points in the convex hull of limits of nearby payoffs, which ensures a mixed strategy equilibrium of such a suitably modified game.

As an example of their approach, Simon and Zame note that the discontinuities in Hotelling’s location model arise only when two firms choose the same location, and consumers are indifferent between them.

Consequently, rather than insist that the two firms split the market evenly, Simon and Zame suggest that one ought to be content with any division of consumers among the two firms.
Nash Equilibrium Existence V: Other Approaches

Simon and Zame’s result ensures that there is a sharing rule specifying how consumers are divided among firms when indifference arises such that the resulting game possesses a mixed strategy equilibrium.

While in some settings involving discontinuities this approach is remarkably helpful, i.e., Hotelling’s location game., in others it is less so.

For example, in a mechanism design environment where discontinuities are sometimes deliberately introduced auction design, for example., the participants must be presented with a game that fully describes the strategies and payoffs. One cannot leave some of the payoffs unspecified, to somehow be endogenously determined.

In addition, this method is only useful in establishing the existence of a mixed, as opposed to pure, strategy equilibrium.
Dynamic Games

Reference: Acemoglu, D.: Political Economy Lecture Notes, mimeo, Ch. 3.

Consider the following class of (dynamic or stochastic) games:

- Finite set $\mathcal{N} = \{1, \ldots, N\}$ of players, with generic element $i \in \mathcal{N}$;
- (Infinite) set $K \subset \mathbb{R}^n$ of state vectors, with generic element $k \in K$;
- Infinite time horizon $t = 1, 2, \ldots$;
- Player $i \in \mathcal{N}$ has a strategy set $A_i(k_t) \subset \mathbb{R}^{n_i}$ at every $t$, with generic element $a_{it} \in A_i(k_t)$;
- Player $i \in \mathcal{N}$ has at time $t$ an instantaneous utility $u_i(a_t, k_t)$, where $a_t \in A(k_t) \equiv \prod_{i \in \mathcal{N}} A_i(k_t)$ is an action profile at $t$, and $u_i(\cdot)$ is assumed to be continuous, bounded, and time independent;
Dynamic Games

- The law of motion of $k_t$ is given by a (Markovian) transition process $q(k_{t+1} \mid a_t, k_t)$, which denotes the probability density of each $k_{t+1}$ given $k_t \in K$ and $a_t \in A_t(k_t)$, with $\int_{-\infty}^{+\infty} q(k \mid a_t, k_t) \, dk = 1$;

  N.B. This process $q(\cdot)$ is said to be Markovian because it depends only on the current action profile $a_t$ and the current state $k_t$.

- Players observe the realizations of all past actions (perfect observability);

- Let $H^t$ be the set of public histories at time $t$, with generic element $h^t = (a_0, k_0, \ldots, a_t, k_t)$.

- A mixed strategy for $i \in \mathcal{N}$ at $t$ is a mapping $\sigma_{it} : H^{t-1} \times K \to \Delta(A_i)$ that determines a probability distribution over $A_i$ given the entire past history $h^{t-1}$ and the current state $k_t$;

  If for every pair $(h^{t-1}, k_t) \in H^{t-1} \times K$, $\sigma_{it}(h^{t-1}, k_t)$ assigns probability one to a single action $a_{it} \in A_i(k_t)$, then $\sigma_{it}$ is said to be a pure strategy;
Dynamic Games

- Let $S_i$ be player $i$’s set of strategy profiles for the whole game, with generic element $\sigma_i = (\sigma_{i1}, \sigma_{i2}, \ldots, \sigma_{i\infty})$;
- Let $S_i[t]$ be player $i$’s set of continuation strategy profiles after time $t$ induced by each $\sigma_i \in S_i$, with generic element $\sigma_i[t] = (\sigma_{it}, \ldots, \sigma_{i\infty})$;
- Player $i$’s objective at time $t$ is to maximize the discounted payoff

$$U_{it}(\sigma_i[t], \sigma_{-i}[t]) = E_t \left( \sum_{s=0}^{\infty} \beta^s u_i(a_{t+s}, k_{t+s}) \right),$$

where $\beta \in (0, 1)$ is a common discount factor, and $E_t(\cdot)$ is the expectation operator conditional on the information available at $t$ (not indexed by $i$ because of perfect observability).
- Player $i$’s best response correspondence is $BR_i(\sigma_{-i}[t] \mid h^{t-1}, k_t) = \{\sigma_i[t] \in S_i[t] : \sigma_i[t] \in \arg \max_{S_i[t]} U_{it}(\sigma_i[t], \sigma_{-i}[t]) \text{ s.t. } \sigma_{-i}[t] \in S_{-i}[t]\}$. 
Subgame Perfect Equilibrium

Definition 9 (SPE)

A subgame perfect equilibrium (SPE) is a strategy profile \( \sigma^* = (\sigma_1^*, \ldots, \sigma_N^*) \in S \equiv \prod_{i \in N} S_i \) such that \( \sigma_i[t] \in BR_i(\sigma_{-i}[t] | h^{t-1}, k_t) \) for all \( t = 0, 1, \ldots \), all \((h^{t-1}, k_t) \in \mathcal{H}^{t-1} \times K\), and all \( i \in N \).

A SPE requires strategies to be best responses to each other, at every possible subgame following a particular history, for all possible histories of the game that lead to that particular history.

Because strategies in a SPE depend on the whole history of the game, there are typically many SPE.

This prompted game theorists to focus on subsets of SPE, such as Markov Perfect Equilibrium (MPE), hoping to improve the predictions of the models.
Markov Strategies

MPE differs from SPE in only conditioning on the payoff-relevant state rather than on the entire history of play.

Given the Markovian transition function assumed above, in our case the payoff-relevant state is simply $k_t \in K$.

Thus a (mixed) Markovian strategy is a mapping $\hat{\sigma}_i : K \rightarrow \Delta(A_i)$, and we denote by $\hat{S}_i$ the corresponding set of all those strategies.

- Given the stationarity of the per-period payoff $u_i(a_t, k_t)$, subscript $t$ has been dropped from the Markovian strategy since it isn’t payoff-relevant.
- $\hat{\sigma}_i$ and $\sigma_i$ have different dimensionality; the former assigns a probability distribution over $A_i$ to each possible state; the second does so for each subgame $(h_{t-1}, k_t)$ and all $t$.
- A Markovian strategy with the same dimension than $\sigma_i$ can be defined as $\hat{\sigma}_i' : K \times \mathcal{H}_{t-1} \rightarrow \Delta(A_i)$ such that $\hat{\sigma}_i'(k_t, h_{t-1}) = \hat{\sigma}_i(k_t)$ for all $h_{t-1} \in \mathcal{H}_{t-1}$, all $k_t \in K$, and all $t$. 

Markov Perfect Equilibrium

Definition 10 (MPE)
A Markov perfect equilibrium is a profile \( \hat{\sigma} = (\hat{\sigma}_i, \hat{\sigma}_{-i}) \in \hat{S} \equiv \prod_{i \in \mathcal{N}} \hat{S}_i \) of Markovian strategies such that the extension \( \hat{\sigma}' = (\hat{\sigma}'_i, \hat{\sigma}'_{-i}) \) of these strategies satisfy that for all \( i \in \mathcal{N}, \) all \( (h^{t-1}, k_t) \in \mathcal{H}^{t-1} \times K, \) and all \( t = 0, 1, \ldots, \)

\[
\hat{\sigma}'_i[t] \in \arg \max_{\hat{\sigma}_i[t]} U_{it}(\sigma_i[t], \hat{\sigma}'_{-i}[t]) \text{ s.t. } \hat{\sigma}'_{-i}[t] \in \hat{S}'_{-i}[t].
\]

- Player \( i \)'s deviations are not required to be Markovian.
- A Markovian strategy \( \hat{\sigma}_i \) must be a best response to \( \hat{\sigma}_{-i} \) among all strategies \( \sigma_{it} \) available at time \( t. \)
- MPE is a SPE.
Application: Common Pool Games

An example of dynamic games is common pool games, where individuals decide over time how much to exploit a common resource. This class of games is useful to illustrate how MPE can be computed in practice.

- Suppose $u_i(a_t, k_t) = \log(a_{it})$, where $a_{it}$ denotes consumption of individual $i \in N$ at time $t$.

- Assume society has a common resource $k_t$ which follows the non-stochastic law of motion $k_{t+1} = \alpha k_t - \sum_{i \in N} a_{it}$, where $\alpha \geq 1$, $k_0$ is given, and $k_t \geq 0$ must be satisfied in every period.
  - $\alpha = 1$ corresponds to a fixed resource game, where the resource is being run down.
  - $\alpha > 1$ corresponds to a case where growth in the capital stock is also allowed.

- The stage game is as follows: at every date all players simultaneously announce $a_{it} \geq 0$. If $\alpha k_t - \sum_{i \in N} a_{it} \geq 0$, then each individual consumes $a_{it}$. Otherwise, $\alpha k_t$ is equally allocated among the players.
Single Person Decision Problem

Let’s start with the benchmark case where \((a_{it})_{i \in \mathcal{N}}\) is chosen by a benevolent planner wishing to maximize the total discounted payoffs of all agents,

\[
E_t \left( \sum_{i \in \mathcal{N}} \sum_{s=0}^{\infty} \beta^s \log(a_{i, t+s}) \right).
\]

By concavity, it is clear that this planner will allocate consumption equally across all individuals at every date.

Thus each individual will consume \(c_t/N\), where \(c_t\) denotes total consumption at time \(t\), i.e., \(c_t = \sum_{i \in \mathcal{N}} a_{it}\).

This implies that the program of the planner at time \(t = 0\) is:

\[
\max_{(c_t)} E_0 \left( \sum_{t=0}^{\infty} \beta^t \left[ \log(c_t) - \log(N) \right] \right) \quad \text{s.t.} \quad k_{t+1} = \alpha k_t - c_t, \quad k_0 \text{ given.} \quad (13)
\]

The second term in the objective function can be dropped since it is constant.
Single Person Decision Problem

Since there is no uncertainty, expectation can be dropped too.

Defining $s$ as the savings left for next period, so that $c = \alpha k - s$, we can write (13) as a dynamic programming recursion

$$V(k) = \max_{s \leq \alpha k} \left[ \log(\alpha k - s) + \beta V(s) \right].$$

(14)

A solution to this problem is a pair of functions $V(k)$ and $h(k)$ such that $s \in h(k)$ is optimal.

Standard arguments of dynamic programming imply that

- $V(k)$ is uniquely defined, is continuous, concave and also differentiable whenever $s \in (0, \alpha)$;
- $h(k)$ is single valued.
Single Person Decision Problem

Given these, whenever $s$ is interior the optimal plan $h(k)$ must satisfy the following necessary and sufficient first-order condition:

$$\frac{-1}{\alpha k - s} + \beta V'(s) = 0. \quad (15)$$

Moreover, since $V(k)$ is differentiable, using the envelope condition, we have

$$V'(k) = \frac{\alpha}{\alpha k - s} + \left[ \frac{-1}{\alpha k - s} + \beta V'(s) \right] \cdot \frac{\partial s}{\partial k} = \frac{\alpha}{\alpha k - s}. \quad (16)$$

Substituting the optimal plan $s = h(k)$ and (16) into (15) we have (recall $V'(\cdot)$ is a function of the next period’s initial stock, i.e. the current saving $h(k)$)

$$\frac{1}{\alpha k - h(k)} - \frac{\beta \alpha}{\alpha h(k) - h(h(k))} = 0. \quad (17)$$

which solves for $h(\cdot)$ as the optimal policy function.
Single Person Decision Problem

Since we know that $V(\cdot)$ and $h(\cdot)$ are uniquely defined, if we can find one pair of functions that satisfy the necessary conditions, we are done.

Thus, using the “guess and verify” method, the solution is

$$h(k) = \beta \alpha k; \quad (18)$$

and the value function of this optimization problem is

$$V(k) = \frac{(1 - \beta) \log(1 - \beta) + \beta \log(\beta) + \log(\alpha)}{(1 - \beta)^2} + \frac{\log(k)}{1 - \beta}. \quad (19)$$

Going back to the original variables (remember $a_i = \sum_{j=1}^{N} a_j/N$ and $\sum_j a_j \equiv c = \alpha k - s$), it follows from (18) that

$$a_{it} = \frac{(1 - \beta) \alpha}{N} \cdot k_t \quad \text{for all } t = 0, 1, 2, \ldots \quad (20)$$
Multi-agent Common Pool Games: MPE

Suppose $N > 1$; otherwise, we are back to the benchmark case!

The (multi-agent) common pool game has some uninteresting MPEs.

For example, all individuals announcing $a_{i0} = \alpha k_0$ is an MPE, since there are no profitable deviations by any agents.

- Since $\sum_i a_{i0} = N\alpha k_0 > \alpha k_0$, in the equilibrium candidate each individual gets a share $\alpha k_0/N$, with instantaneous utility at time 0 equal to $\log(\alpha k_0) - \log(N)$, and $\log(0) = -\infty$ for all $t > 0$.
- If $N > 2$, any unilateral deviation produces no change in the payoff of the deviating player (same reasoning than above).
- If $N = 2$, the deviating player gets the same payoff if he deviates to any $a'_{i0} > 0$, and a lower payoff if $a'_{i0} = 0$ ($i$ mainly looses the instantaneous payoff at time 0, as the whole pie goes to $-i$).
Multi-agent Common Pool Games: MPE

Instead, we will look for a more interesting, continuous and symmetric MPE.

The symmetry requirement is for simplicity, and implies that all agents will use the same Markovian strategy. Let that strategy be denoted by $a(k)$.

As before the law of motion is

$$s = \alpha k - \sum_{\ell=1}^{N} a_\ell.$$ 

Thus individual consumption can be defined as $a_i = \alpha k - \sum_{j \neq i} a_j - s$, with $s$ as the capital stock left for the next period.

Given our restriction to symmetric Markovian strategies (and dropping conditioning on $i$), this gives

$$a(k) = \alpha k - (N - 1)a(k) - s$$ (21)
Multi-agent Common Pool Games: MPE

Given this, each individual’s optimization problem can again be written recursively as follows:

\[
V(k) = \max_{s \leq \alpha k - Na(k)} [\log(\alpha k - (N - 1)a(k) - s) + \beta V(s)].
\] (22)

The solution of (22) is a value function \(V(k)\), a consumption policy \(a(k)\) and a saving policy \(h(k)\), such that \(s \in h(k)\) is best response.

Assuming differentiability (not guaranteed yet because \(a(k)\) is unknown), the first-order condition of the maximization problem in (22) is

\[
-\frac{1}{\alpha k - (N - 1)a(k) - s} + \beta V'(s) = 0.
\] (23)

Moreover, using the envelope condition, we have

\[
V'(k) = \frac{1}{\alpha k - (N - 1)a(k) - s} \cdot \left(\alpha - (N - 1) \frac{\partial a(k)}{\partial k}\right).
\] (24)
Multi-agent Common Pool Games: MPE

Notice the term $\partial a(k)/\partial k$ in the numerator. This is there because individuals realize that by their own action they will affect the state variable, and by affecting the state variable, they will influence the consumption decision of others. This is where the subtlety of dynamic games come in.

Using the “guess and verify” method, the equilibrium savings rate in the MPE of the common pool problem is

$$h(k) = \frac{\beta \alpha}{N - \beta(N - 1)} k; \quad (25)$$

Notice that when $N = 1$, this is exactly equal to the optimal value obtained from the single-person decision problem (or the social planner’s problem). Moreover,

$$\frac{\partial h(k)}{\partial N} = \frac{-(\beta \alpha k)(1 - \beta)}{(N - \beta(N - 1))^2} < 0; \quad (26)$$
Multi-agent Common Pool Games: MPE

That is, the greater the number of players drawing resources from the common pool, the lower the savings rate of the economy. This captures the well-known free-rider or tragedy of the commons problem.

The inability of the players in this game to coordinate their actions leads to too much consumption and too little savings.

For example, it is quite possible that

$$\alpha \beta > 1 > \frac{\beta \alpha}{N - \beta(N - 1)},$$

so that, the social planner’s solution would involve growth, while the MPE would involve the resources shrinking over time.

Finally, it is useful to remember that there are many MPE in this game. Perhaps unsurprisingly, there also exist non-symmetric MPEs; and there may also exist discontinuous MPEs that implement the planner’s solution.
Multi-agent Common Pool Games: SPE

The common pool problem can also be used to illustrate the difference between SPE and MPE.

We already saw that there exists an equilibrium in which all individuals receive negative infinite utility (they all play \( a_{i0} = \alpha k_0 \) at time 0).

This immediately implies that for any discount factor \( \beta \), any allocation can be supported as an SPE. In particular, the social planner’s solution of saving the society’s resources at the rate \( \beta \alpha \) is an SPE.

The (grim) strategy profile that would support this SPE is one in which everybody follows the social planner’s allocation until one agent deviates from it, and as soon as there is such a deviation, all agents switch to demanding the whole capital stock of the economy.
Some Basic Results

Suppose that the instantaneous payoff function of each player is uniformly bounded, i.e., there exists $B_i < \infty$ for all $i \in \mathcal{N}$ such that

$$\sup_{k \in K, a \in A(k)} u_i(a, k) < B_i.$$ 

**Theorem 8 (One-Stage Deviation Principle–SPE)**

A strategy profile $\sigma^* \in S$ is a SPE if and only if for all $i \in \mathcal{N}$, all $(h^{t-1}, k_t) \in \mathcal{H}^{t-1} \times K$, all $t$, and all $a_{it} \in A(k_t)$,

$$U_{it}(a_{it}, \sigma^*_i[t + 1], \sigma^*_{-i}[t]) \leq U_{it}(\sigma^*_i[t], \sigma^*_{-i}[t]).$$

**Theorem 9 (One-Stage Deviation Principle–MPE)**

A strategy profile $\hat{\sigma} \in \hat{S}$ is a MPE if and only if for all $i \in \mathcal{N}$, all $(h^{t-1}, k_t) \in \mathcal{H}^{t-1} \times K$, all $t$, and all $a_{it} \in A(k_t)$,

$$U_{it}(a_{it}, \hat{\sigma}'_i[t + 1], \hat{\sigma}'_{-i}[t]) \leq U_{it}(\hat{\sigma}'_i[t], \hat{\sigma}'_{-i}[t]).$$
Some Basic Results

These theorems basically imply that in dynamic games, we can check whether a strategy is a best response to other players’ strategies by looking at one-stage deviations, keeping the rest of the strategy profile of the deviating player as given.

The basic idea of the proof is as follows. The problem of individual $i$ after fixing the strategy of the other players is equivalent to a dynamic optimization problem. Given the uniform boundedness of instantaneous payoffs and $\beta < 1$, 
$$\lim_{T \to \infty} \sum_{s=0}^{T} \beta^s u_i(a_{t+s}, k_{t+s}) = 0$$
for all $\{a_{t+s}, k_{t+s}\}_{s=0}^{T}$ and all $t$. Hence, we can apply the principle of optimality from dynamic programming, to obtain of the one-stage deviation principle.

The uniform boundedness assumption can be weakened to require “continuity at infinity,” which essentially means that discounted payoffs converge to zero along any history (and this assumption can also be relaxed further).
Some Basic Results

Theorem 10 (Existence MPE)

If $K$ and $A_i(k)$ for all $i \in \mathcal{N}$ and $k \in K$ are two finite sets, then there always exists a Markov perfect equilibrium.

The proof of 10 is as follows. Consider an extended game in which each player is an element $(i, k) \in \mathcal{N} \times K$, with a payoff function given by player $i$’s original payoff function starting in state $k$, as in (12), and a strategy set $A_i(k)$.

The set of players $\mathcal{N} \times K$ is finite, and the strategy set $A_i(k)$ is also finite. The set of mixed strategies $\Delta(A_i(k))$ for player $(i, k)$ is the simplex over $A_i(k)$. Therefore, the standard proof of existence of Nash equilibrium based on Kakutani’s fixed point theorem applies and leads to the existence of an equilibrium $(\hat{\sigma}_{(i,k)})_{(i,k) \in \mathcal{N} \times K}$ in this extended game.
Some Basic Results

Now going back to the original game, construct the strategy \( \hat{\sigma}_i \) for each player \( i \in \mathcal{N} \) such that \( \hat{\sigma}_i(k) = \hat{\sigma}_{(i,k)} \), i.e., \( \hat{\sigma}_i : K \rightarrow \Delta(A_i) \) (in words: player \( i \in \mathcal{N} \) plays the original game following the strategies of the players \( (i, k)_{k \in K} \)).

This strategy profile \( \hat{\sigma} = (\hat{\sigma}_i)_{i \in \mathcal{N}} \) is Markovian.

Consider the extension of \( \hat{\sigma} \) to \( \hat{\sigma}' \) as above, i.e., \( \hat{\sigma}'_i(k_t, h^{t-1}) = \hat{\sigma}_i(k_t) \) for all \( h^{t-1} \in \mathcal{H}^{t-1}, k_t \in K, i \in \mathcal{N} \) and \( t \).

Then, by construction, given \( \hat{\sigma}'_{-i} \), it is impossible to improve over \( \hat{\sigma}'_i \) with a deviation at any \( k \in K \); thus Theorem 9 implies that \( \hat{\sigma}'_i \) is best response to \( \hat{\sigma}'_{-i} \) for all \( i \in \mathcal{N} \), and therefore \( \hat{\sigma}' \) is a MPE.

Similar existence results can be proved for countably infinite sets \( K \) and \( A_i(k) \), and also for uncountable sets, but in this latter instance, some additional requirements are necessary.
Some Basic Results

Theorem 11 (Markov vs. Subgame Perfect Equilibria)

The set of (extended) MPE strategies is a subset of the set of SPE strategies.

This theorem follows immediately by noting that since \( \hat{\sigma} \) is a MPE strategy profile, the extended strategy profile, \( \hat{\sigma}' \), is such that \( \hat{\sigma}'_i \) is a best response to \( \hat{\sigma}'_{-i} \) for all \( h^{t-1} \in H^{t-1}, k_t \in K, i \in \mathcal{N} \) and \( t \). Thus it is subgame perfect.

Theorem 12 (Existence SPE)

If \( K \) and \( A_i(k) \) for all \( i \in \mathcal{N} \) and \( k \in K \) are two finite sets, then there always exists a subgame perfect equilibrium.

Theorem 10 shows that a MPE exists and since a MPE is a SPE (Theorem 11), the existence of a SPE follows.

When \( K \) and \( A_i(k) \) are uncountable sets, existence of pure strategy SPEs can be guaranteed by imposing compactness and convexity of \( K \) and \( A_i(k) \) and quasi-concavity of \( U_{it}(\cdot) \) in \( \sigma_i[t] \) for all \( i \in \mathcal{N} \) (in addition to continuity).
Some Basic Results

In the absence of convexity of $K$ and $A_i(k)$ or quasi-concavity of $U_{it}(\cdot)$, mixed strategy equilibria can still be guaranteed to exist under some very mild additional assumptions.

Suppose that the same stage game is played an infinite number of times, so that payoffs are given by

$$U_{it}(\sigma_i[t], \sigma_{-i}[t]) = E_t \left( \sum_{s=0}^{\infty} \beta^s u_i(a_{t+s}) \right),$$

which is only different from (12) because there is no conditioning on the state variable $k_{t+s}$. Let us refer to the game $\{u_i(a), a \in A\}$ as the stage game.

Let $m_i = \min_{a_{-i}} \max_{a_i} u_i(a)$ be agent $i$’s minimax payoff in this stage game.

Denote $V \in \mathbb{R}^N$ the set of feasible per period payoffs for the $N$ players, with $v_i$ being player $i$’s payoff (so that the discounted payoff is $v_i/(1 - \beta)$).
Some Basic Results

Theorem 13 (Folk Theorem for Repeated Games)

If $A_i$ is compact for all $i \in \mathcal{N}$, then for any $v \in V$ such that $v_i > m_i$ for all $i \in \mathcal{N}$, there exists $\bar{\beta} \in [0, 1)$ such that for all $\beta > \bar{\beta}$, $v$ can be supported as the payoff profile of a SPE.

Theorem 14 (Unique MPE in Repeated Games)

If the stage game $\{u_i(a), a \in A\}$ has a unique equilibrium $a^*$, then there exists a unique MPE in which $a^*$ is played at every date.

In repeated games, there is no state vector; so Markov strategies cannot be conditioned on anything. We can only look at the strategies that are best response in the stage game. If the latter has a unique equilibrium, that must also be the equilibrium of the repeated game.