# Notes on Consumer Theory

Alejandro Saporiti

# Consumer theory

Reference: Jehle and Reny, *Advanced Microeconomic Theory*, 3rd ed., Pearson 2011: Ch. 1.

The economic model of consumer choice has 4 ingredients:

1. The consumption set;
2. The preference relation;
3. The feasible (budget) set;
4. Behavioral assumptions (e.g., rationality).

This basic structure gives rise to a general theory of choice which is used in several social sciences (e.g., economics & political science).

For concreteness, we focus on explaining the behavior (choices) of a representative consumer, a central actor in much of economic theory.

# Consumption set

The consumption or choice set represents the set of all alternatives available to the (unrestricted) consumer.

In economics, these alternatives are called consumption plans.

A consumption plan represents a bundle of goods, and is written as a vector $x$ consisting of $n$ different consumption goods, $x = (x_1, \ldots, x_n)$.

Typical assumptions on $X$ are:

1. $\emptyset \neq X \subseteq \mathbb{R}^n_+$ (i.e., nonempty & each good measured in infinitely divisible and nonnegative units);
2. $X$ is closed;
3. $X$ is convex;
4. $0 \in X$.

# Consumer preferences

Consumer's preferences represent his attitudes toward the objects of choice.

The consumer is born with these attitudes, i.e. preferences are a 'primitive' in classical consumer theory.

To represent them formally, we use the at least as good as binary relation $\succsim$ on $X$; and for any two bundles $x^1$ and $x^2$, we say that,

1. The consumer is indifferent between $x^1$ and $x^2$, denoted by $x^1 \sim x^2$, if and only if (iff) $x^1 \succsim x^2$ and $x^2 \succsim x^1$;
2. The consumer strictly prefers $x^1$ over $x^2$, indicated by $x^1 \succ x^2$, iff $x^1 \succsim x^2$ and $\neg[x^2 \succsim x^1]$.

# Consumer preferences

We require $\succsim$ to satisfy the following axioms:

1. Completeness: For all $x^1$ and $x^2$ in $X$, either $x^1 \succsim x^2$ or $x^2 \succsim x^1$;
2. Transitivity: For any three bundles $x^1$, $x^2$ and $x^3$ in $X$, if $x^1 \succsim x^2$ and $x^2 \succsim x^3$, then $x^1 \succsim x^3$.

When $\succsim$ satisfies these axioms it is said to be a rational preference relation.

Under completeness and transitivity, for any two bundles $x^1, x^2 \in X$, exactly one of the following three possibilities holds: either

- $x^1 \succ x^2$,
- $x^2 \succ x^1$, or
- $x^1 \sim x^2$.

Thus, the rational preference relation $\succsim$ offers a weak or partial order of $X$ (complete and transitive), ranking any finite number of alternatives in $X$ from best to worst, possibly with some ties.

# Consumer preferences

Apart from Axioms 1 & 2, we demand three additional properties on $\succsim$:

3. Continuity: For all $x^0 \in X$, the sets $\{x \in X : x \succsim x^0\}$ and $\{x \in X : x^0 \succsim x\}$ are closed in $X \subset \mathbb{R}^n_+$;

- Continuity rules out open areas in the indifference set; that is, the set $\{x \in X : x \sim x^0\} = \{x \in X : x \succsim x^0\} \cap \{x \in X : x^0 \succsim x\}$ is closed.
- It rules out 'sudden preference reversals' such as the one happening in Fig 1, where $(b, 0) \succ_\ell (x, 1)$ for all $x < b$, but $(b, 1) \succ_\ell (b, 0)$.

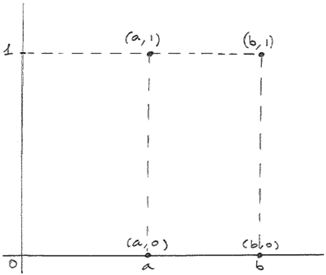



Figure 1: Lexicographic preferences $\succsim_\ell$ on $\mathbb{R}^2_+$; $(a_1, a_2) \succsim_\ell (b_1, b_2)$ iff either $a_1 > b_1$, or $a_1 = b_1$ and $a_2 \geq b_2$.

# Consumer preferences

4. Strict monotonicity: For all $x^1, x^2 \in X$, (a) if $x^1$ contains at least as much of every commodity as $x^2$, then $x^1 \succsim x^2$; (b) if $x^1$ contains strictly more of every commodity, then $x^1 \succ x^2$.

- This axiom rules out the possibility of having 'indifference zones'; it also eliminates indifference sets that bend upward and contain positively sloped segments.
- Strict monotonicity implies that the better (resp. worse) than set is above (resp. below) the indifference set.

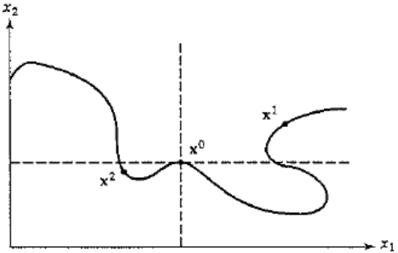



Figure 2: Violation of strict monotonicity.

# Consumer preferences

5. Strict convexity: For all $x^1, x^2 \in X$, if $x^1 \neq x^2$ and $x^1 \succsim x^2$, then for all $\alpha \in (0,1)$, $\alpha x^1 + (1-\alpha)x^2 \succ x^2$.

   - It rules out concave to the origin segments in the indifference sets.
   - It prevents the consumer from preferring extremes in consumption.

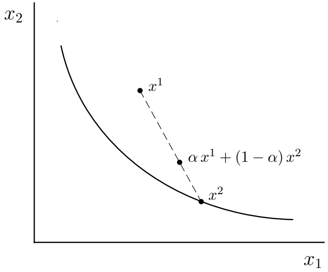



Figure 3: Convex

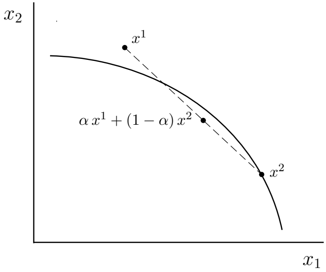



Figure 4: Nonconvex

# Consumer preferences

Axioms 3-5 exploit the structure of the space $X$:

- Continuity uses the ability to talk about closeness.
- Monotonicity uses the orderings on the axis (the ability to compare bundles by the amount of any particular commodity).
- It gives commodities the meaning of 'goods': More is better.
- Convexity uses the algebraic structure (the ability to speak of the sum of two bundles and the multiplication of a bundle by a scalar).
- That is, it assumes the existence a "geography" of the set of alternatives, so that we can talk about one alternative being between two others.
- It's appropriate when the argument "if a move is an improvement so is any move part of the way" is legitimate, while the argument "if a move is harmful then so is a move part of the way" is not.

# Utility representation

When preferences are defined over large sets of alternatives, it is usually convenient to employ calculus methods to work out the best options.

To do that, we'd like to represent the information conveyed by the preference relation $\succsim$ through a function.

A real-valued function $u : \mathbb{R}^n_+ \to \mathbb{R}$ is said to be a utility representation of the preference relation $\succsim$ if for all $x^1, x^2 \in X \subseteq \mathbb{R}^n_+$

$$x^1 \succsim x^2 \Leftrightarrow u(x^1) \geq u(x^2). \tag{1}$$

## Theorem 1 (Debreu, 1954)

*If the preference relation $\succsim$ is complete, transitive and continuous, then it possesses a continuous utility representation $u : \mathbb{R}^n_+ \to \mathbb{R}$.*

Consistent pair-wise comparability over $X$ and some topological regularity are enough for a numerical representation of $\succsim$.

# Invariance of the utility function

The ordinal nature of the utility representation embedded in (1) implies that a utility function $u : \mathbb{R}^n_+ \to \mathbb{R}$ is unique up to any strictly increasing transformation.

More formally, suppose $u : \mathbb{R}^n_+ \to \mathbb{R}$ represents the preference relation $\succsim$.

Then, for any strictly increasing transformation $f : \mathbb{R} \to \mathbb{R}$, the function

$$v(x) \equiv f(u(x)) \ \forall x \in X, \tag{2}$$

also represents $\succsim$.

To see this, recall that $u(\cdot)$ represents $\succsim$ if for all $x, y \in X$,

$$x \succ y \quad \Leftrightarrow \quad u(x) > u(y), \tag{3}$$

$$x \sim y \quad \Leftrightarrow \quad u(x) = u(y). \tag{4}$$

# Invariance of the utility function

Moreover, note that for any two real numbers $a, b \in \mathbb{R}$,

(i) $f(a) > f(b)$ if and only if $a > b$; and

(ii) $f(a) = f(b)$ if and only if $a = b$.

Thus, using the definition of $v$ given in (2) and (i)-(ii), for all $x, y \in X$

$$\begin{aligned} v(x) > v(y) \;&\Leftrightarrow\; f(u(x)) > f(u(y)) \;\;\Leftrightarrow\;\; u(x) > u(y), \quad (5) \\ v(x) = v(y) \;&\Leftrightarrow\; f(u(x)) = f(u(y)) \;\;\Leftrightarrow\;\; u(x) = u(y). \quad (6) \end{aligned}$$

Combining (5) with (3), we have

$$v(x) > v(y) \;\Leftrightarrow\; x \succ y.$$

Similarly, combining (6) with (4), we have

$$v(x) = v(y) \;\Leftrightarrow\; x \sim y.$$

Hence, $v(\cdot)$ represents $\succsim$.

# Quasiconcavity

A function $u(\cdot)$ is quasiconcave on a convex set $X \subseteq \mathbb{R}^n$ if and only if for all $x, y \in X$, and for all $\lambda \in (0, 1)$,

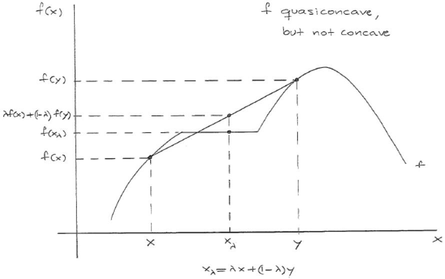
$$u(\lambda x + (1 - \lambda) y) \geq \min\{u(x), u(y)\}. \qquad (*)$$

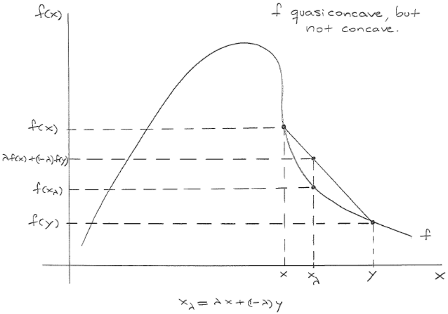
Strict quasiconcavity is defined analogously by replacing the weak inequality in $(*)$ with the strict inequality.

A function $g(\cdot)$ is quasiconvex on a convex set $X \subseteq \mathbb{R}^n$ if and only if for all $x, y \in X$, and for all $\lambda \in (0, 1)$,

$$g(\lambda x + (1 - \lambda) y) \leq \max\{g(x), g(y)\}.$$

# Quasiconcavity




Alternatively, we could say that a function $u(\cdot)$ is quasiconcave on a convex set $X \subseteq \mathbb{R}^n$ if for all $c \in \mathbb{R}$ the upper contour set $\{x \in X : u(x) \geq c\}$ is convex.

# Quasiconcavity

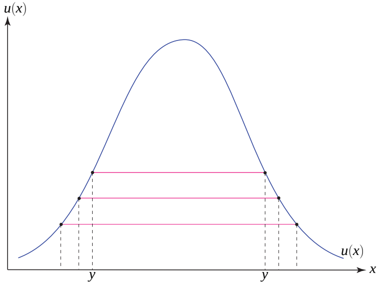



Figure 5: Quasiconcave

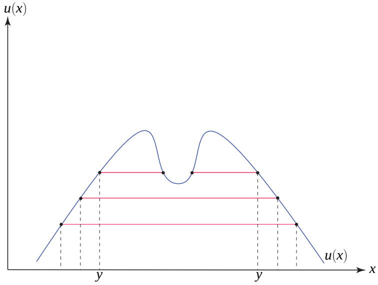



Figure 6: Not quasiconcave

In the the picture above, every horizontal cut through the function must be convex for the function to be quasi-concave. Thus, $u(\cdot)$ is quasiconcave in Fig. 5, whereas it isn't in Fig. 6.

# Quasiconcavity

An important property of quasiconcavity is that it's preserved under increasing transformations.

### Proposition 1

*If $u : X \subseteq \mathbb{R}^n \to \mathbb{R}$ is quasiconcave on $X$ and $\phi : \mathbb{R} \to \mathbb{R}$ is a monotone increasing transformation, then $\phi(u(\cdot))$ is quasiconcave.*

The proof of Proposition 1 is as follows. We wish to show that for all $x, y \in X$, and for all $\lambda \in (0, 1)$,

$$\phi(u(\lambda x + (1-\lambda)y)) \geq \min\{\phi(u(x)), \phi(u(y))\}.$$

# Quasiconcavity

Since $u$ is quasiconcave, for all $x, y \in X$, and for all $\lambda \in (0,1)$,

$$u(\lambda x + (1-\lambda) y) \geq \min\{u(x), u(y)\}. \tag{7}$$

Applying $\phi$ to both sides of (7),

$$\phi(u(\lambda x + (1-\lambda) y)) \geq \phi(\min\{u(x), u(y)\}). \tag{8}$$

But,

$$\phi(\min\{u(x), u(y)\}) = \min\{\phi(u(x)), \phi(u(y))\}. \tag{9}$$

Hence, combining (8) with (9), we get the desired result; that is, for all $x, y \in X$, and for all $\lambda \in (0,1)$,

$$\phi(u(\lambda x + (1-\lambda) y)) \geq \min\{\phi(u(x)), \phi(u(y))\}.$$

# Indifference curve & marginal utility

Given a utility function $u(\cdot)$, the indifference curve (level contour set) that passes through the bundle $\bar{x} \in X$ is defined as

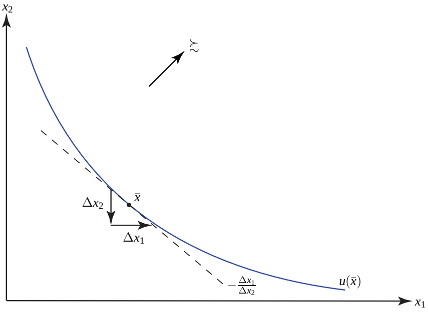
$$\{x \in X : u(x) = u(\bar{x})\}.$$

If $u(\cdot)$ is quasiconcave, then the indifference curves are convex (recall Fig. 5: the upper contour sets of a quasi-concave function are convex).

If $u$ is differentiable, then for all $i = 1, \ldots, n$, the marginal utility of $x_i$ at $x = (x_1, \ldots, x_n)$ is

$$MU_i(x) = \frac{\partial u(x)}{\partial x_i}.$$

At $\bar{x}$, consumer is willing to substitute $x_1$ against $x_2$ at the rate of $\approx -\Delta x_1/\Delta x_2$.



# Marginal rate of substitution

If we'd like to vary $x_i$ by a small amount $dx_i$, while keeping utility $u(\bar{x})$ constant, how much do we need to change $x_j \neq x_i$?

Formally, the total differential of $\bar{u} = u(\bar{x})$ is

$$0 = \frac{\partial u(\bar{x})}{\partial x_1} dx_1 + \ldots + \frac{\partial u(\bar{x})}{\partial x_n} dx_n.$$

Since we only care about changes in $x_j$ caused by changes in $x_i$, we set $dx_h = 0$ for all $h = 1, \ldots, n$, with $h \neq i, j$.

Thus, the total differential simplifies to

$$\frac{\partial u(\bar{x})}{\partial x_i} dx_i + \frac{\partial u(\bar{x})}{\partial x_j} dx_j = 0.$$

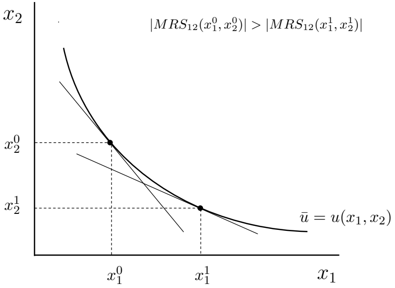
# Marginal rate of substitution

Rearranging the terms, we get that the marginal rate of substitution (MRS) between good $i$ and good $j$ is given by

$$MRS_{ij}(\bar{x}) = \left.\frac{dx_j}{dx_i}\right|_{du=0} = -\frac{\partial u(\bar{x})/\partial x_i}{\partial u(\bar{x})/\partial x_j} = -\frac{MU_i(\bar{x})}{MU_j(\bar{x})}.$$

$MRS_{ij}(\bar{x})$ is the rate at which good $j$ can be exchanged per unit of good $i$ without changing consumer's utility.

- The absolute value of the MRS is equal to the ratio of marginal utilities of $i$ and $j$ at $\bar{x}$;
- The MRS equals the slope of the indifference curve.
- When preferences are convex, the MRS between two goods is decreasing.



# Budget set

Obviously, the consumer must be able to afford his consumption bundle. This generally restricts his choice set. We assume

- Each good has a strictly positive price $p_i > 0$, for all $i = 1, \ldots, n$;
- The consumer is endowed with income $y > 0$.

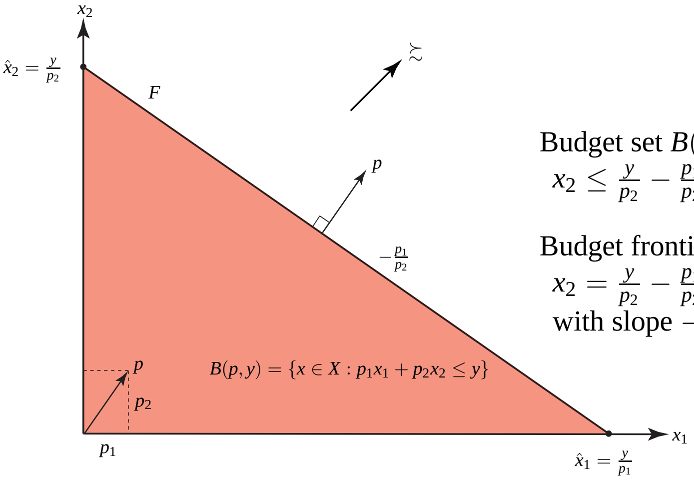
Thus consumer's purchases are restricted by the budget constraint

$$\sum_{i=1}^{n} p_i x_i \leq y.$$

The budget set is the set of bundles that satisfy this constraint:

$$B(p, y) = \{x \in X : p_1 x_1 + \ldots + p_n x_n \leq y\}.$$

# Budget set (2-goods)




Budget set $B(p, y)$:
$x_2 \leq \frac{y}{p_2} - \frac{p_1}{p_2} x_1.$

Budget frontier $F$:
$x_2 = \frac{y}{p_2} - \frac{p_1}{p_2} x_1$
with slope $-p_1/p_2$.

# Behavioral assumption: Rationality

So far, we have a model capable of representing consumer's feasible choices and his preferences over them.

Now we restrict consumer's behavior assuming that he is a rational agent, in the sense that he chooses the best alternative in the budget set $B(p, y)$ according with his preference relation $\succsim$.

Thus rationality in microeconomics has two different meanings:

1. Consumer orders consistently (transitively) *all* possible alternatives;
2. Consumer chooses the best alternative among those in the feasible set.

We are now ready to study consumer's optimal choices!

# Utility maximization

Formally, consumer's utility maximization problem (UMP) is

$$\begin{array}{ll} \max\limits_{x \in X} & u(x_1, \ldots, x_n) \\ \text{s.t.} & \sum_{i=1}^{n} p_i x_i \leq y. \end{array} \tag{10}$$

The utility function $u$ is a real valued and continuous function.

The budget set $B$ is a nonempty ($0 \in X$), closed, bounded ($p_i > 0 \ \forall i$) and thus a compact subset of $\mathbb{R}^n$.

Therefore, by the Weierstrass theorem, a maximum of $u$ over $B$ exists.

Let's assume the solution of (10), denoted $x^*$, is interior, i.e. $\forall i$, $x_i^* > 0$; then UMP can be solved using the Kuhn-Tucker method.

Set up the Lagrange function

$$\mathcal{L}(x_1, \ldots, x_n, \lambda) = u(x_1, \ldots, x_n) + \lambda \left[ y - \sum_{i=1}^{n} p_i x_i \right]. \tag{11}$$

# Interior solution ($x_i^* > 0$)

Differentiating $\mathcal{L}(\cdot)$ w.r.t. each argument, we get the Kuhn-Tucker first order conditions (FOC's) at the critical bundle $x^*$:

$$\frac{\partial \mathcal{L}(x^*, \lambda^*)}{\partial x_i} = \frac{\partial u(x^*)}{\partial x_i} - \lambda^* p_i = 0, \ \forall i = 1, \ldots, n, \tag{12}$$

$$\frac{\partial \mathcal{L}(x^*, \lambda^*)}{\partial \lambda} = y - \sum_{i=1}^{n} p_i x_i^* \geq 0, \tag{13}$$

$$\lambda^* \geq 0, \text{ and } \lambda^* \cdot \left( y - \sum_{i=1}^{n} p_i x_i^* \right) = 0. \tag{14}$$

Imposing strict monotonicity, (13) must be satisfied with equality, and therefore (14) becomes redundant.

# Interior solution ($x_i^* > 0$)

Assuming $MU_i(x^*) > 0$ for some $i$, it becomes clear from (12) that

$$\frac{\partial u(x^*)}{\partial x_1}\frac{1}{p_1} = \frac{\partial u(x^*)}{\partial x_2}\frac{1}{p_2} = \cdots = \frac{\partial u(x^*)}{\partial x_n}\frac{1}{p_n} = \lambda^* > 0.$$

Hence, the (absolute value of the) marginal rate of substitution at $x^*$ between any two goods equals the price ratio of those goods:

$$|MRS_{ij}(x^*)| = \frac{MU_i(x^*)}{MU_j(x^*)} = \frac{p_i}{p_j}. \tag{15}$$

Otherwise, the consumer can improve by substituting a good with a lower MU by a good with a higher MU.

The Lagrange multiplier $\lambda^*$ is called the shadow price of money, and it gives the utility of consuming one extra currency unit.

# Interior solution (2 goods)

Graphically, in the case of two goods (15) is equivalent to the tangency between the highest indifference curve and the budget constraint.

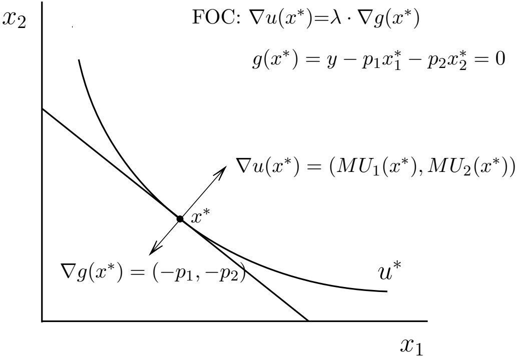



Figure 7: Interior solution $x_i^* > 0$.

# Interior & corner solution (2 goods)

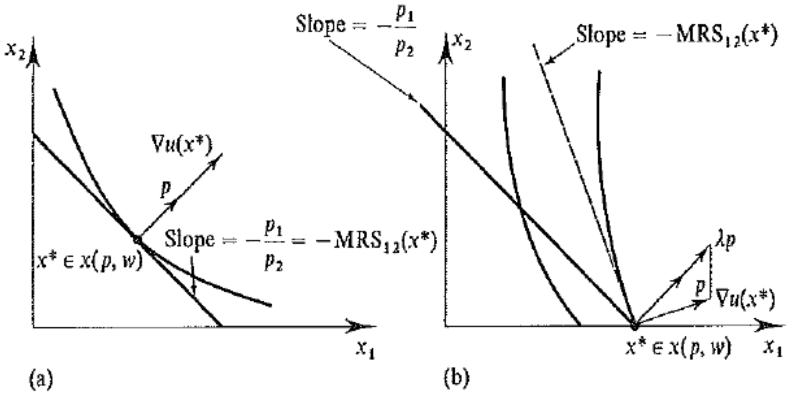



Figure 8: (a) Interior solution $x_i^* > 0$; (b) Corner solution $x_i^* \geq 0$.

# Second order conditions

Strictly speaking, for any critical point $x^*$ that satisfies the FOCs, we must check that the second order conditions (SOCs) for maxima are satisfied at $x^*$.

However, if $u(\cdot)$ is quasiconcave on $\mathbb{R}^n_+$ and $(x^*, \lambda^*) \gg 0$ solves the FOCs of the Lagrange maximization problem, then $x^*$ solves (10).

The consumer's optimal choices $x_i^*(p, y)$, as a function of all prices $p_1, \ldots, p_n$ and income $y$, are called the Walrasian demands.

N.B. They are also called sometimes Marshallian demands.

From now on, we assume $x_i^*(p, y)$ is differentiable.

# Example

Let's find the Walrasian demands for the case in which

$$u(x_1, x_2) = (x_1^\rho + x_2^\rho)^{1/\rho}, \quad \rho \in (0, 1).$$

Consumer's utility maximization problem is

$$\begin{array}{ll} \max\limits_{(x_1,x_2)\in\mathbb{R}^2_+} & (x_1^\rho + x_2^\rho)^{1/\rho} \\ \text{s.t.} & p_1 x_1 + p_2 x_2 \leq y. \end{array}$$

The associated Lagrange function is

$$\mathcal{L}(x_1, x_2, \lambda) = (x_1^\rho + x_2^\rho)^{1/\rho} + \lambda(y - p_1 x_1 - p_2 x_2).$$

Because preferences are strictly monotonic, the consumer will spend his whole budget in $x_1$ and $x_2$.

# Example

Thus an interior solution exists, and the FOCs are:

$$\begin{aligned}
\frac{\partial \mathcal{L}(x,\lambda)}{\partial x_1} &= (x_1^\rho + x_2^\rho)^{1/\rho - 1} x_1^{\rho-1} - \lambda p_1 = 0, \\
\frac{\partial \mathcal{L}(x,\lambda)}{\partial x_2} &= (x_1^\rho + x_2^\rho)^{1/\rho - 1} x_2^{\rho-1} - \lambda p_2 = 0, \\
\frac{\partial \mathcal{L}(x,\lambda)}{\partial \lambda} &= y - p_1 x_1 - p_1 x_1 = 0.
\end{aligned}$$

Equalising the first two FOCs and rearranging terms, one gets

$$\begin{aligned}
x_1 &= x_2 \left( \frac{p_1}{p_2} \right)^{\frac{1}{\rho-1}} \\
y &= p_1 x_1 + p_2 x_2.
\end{aligned}$$

# Example

Plugging the first expression into the second, we have

$$y = p_1 x_2 \left(\frac{p_1}{p_2}\right)^{\frac{1}{\rho-1}} + p_2 x_2 = x_2 (p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}) p_2^{-\frac{1}{\rho-1}}.$$

Finally, solving for $x_2$ and then for $x_1$, we find that

$$x_2^*(p, y) = \frac{y p_2^{\frac{1}{\rho-1}}}{p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}} \text{ and}$$

$$x_1^*(p, y) = \frac{y p_1^{\frac{1}{\rho-1}}}{p_1^{\frac{\rho}{\rho-1}} + p_2^{\frac{\rho}{\rho-1}}}.$$

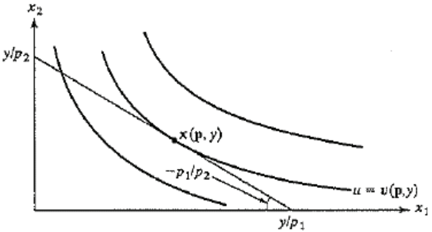
# Indirect utility

The function mapping out the maximum attainable utility for different prices and income is called the indirect utility function.

It is defined as

$$\begin{aligned} V(p,y) &= \max\left\{u(x_1,\dots,x_n) : \sum_{i=1}^{n} p_i x_i \leq y\right\}, \\ &= u(x_1^*(p,y),\dots,x_n^*(p,y)). \end{aligned}$$

Geometrically, $V(p,y)$ is equal to the utility level associated with the highest indifference curve the consumer can achieve with income $y$ and at prices $p$.



# Roy's identity

An important result in consumer theory, known as Roy's identity, shows that the Walrasian demands can be recovered from the indirect utility.

To be precise, if $V$ is differentiable at $(p, y)$ and $\partial V(p,y)/\partial y \neq 0$, then

$$x_j^*(p,y) = -\frac{\frac{\partial V(p,y)}{\partial p_j}}{\frac{\partial V(p,y)}{\partial y}} \text{ for all } j = 1, \ldots, n. \tag{16}$$

The proof of (16) rests on the envelope theorem.

The envelope theorem states that the effect of changing a parameter $\alpha_k$ over the optimized value of the objective function $v^*$ is given by the first-order partial derivative of the Lagrange function with respect to $\alpha_k$, evaluated at the optimal (interior) point $(x^*, \lambda^*)$.

$$\frac{\partial v^*(\alpha)}{\partial \alpha_k} = \frac{\partial L(x^*, \lambda^*)}{\partial \alpha_k}, \; k = 1, \ldots, l. \tag{17}$$

# Roy's identity

In the utility maximization problem $\max_x\ u(x)$ s.t. $g(x,p,y) = y - p \cdot x = 0$, the Lagrange function is $L(x,\lambda) = u(x) + \lambda(y - p \cdot x)$. If $x^* \gg 0$,

- $\frac{\partial V(p,y)}{\partial p_j} \left( = \frac{\partial L(x^*,\lambda^*)}{\partial p_j} \right) = -\lambda^* \cdot x_j^*(p,y)$; and
- $\frac{\partial V(p,y)}{\partial y} \left( = \frac{\partial L(x^*,\lambda^*)}{\partial y} \right) = \lambda^*$ ($> 0$ cos $p \gg 0$ & $MU_i(x^*) > 0$ for some $i$).

Therefore,

$$\frac{\frac{\partial V(p,y)}{\partial p_j}}{\frac{\partial V(p,y)}{\partial y}} = \frac{-\lambda^* \cdot x_j^*(p,y)}{\lambda^*} = -x_j^*(p,y),$$

which is precisely Roy's identity.

# Marginal utility of income

As noted above, by the Envelope theorem,

$$\lambda^* = \frac{\partial V(p,y)}{\partial y}.$$

That is, in the utility maximization problem the Lagrange multiplier is said to be the marginal utility of income.

Alternatively, $\lambda^*$ is also called the shadow price of (the resource) $y$.

In words, in the UMP the Lagrange multiplier measures the change of the optimal value of the utility function as we relax in one unit the budget constraint.

# Indirect utility's properties

Assuming $x^* \gg 0$ and $MU_i(x^*) > 0$ for some $i$, the indirect utility function satisfies the following properties:

For all $(p, y) \in \mathbb{R}^{n+1}_{++}$, $V(p, y)$ is (i) decreasing in $p_j$, $j = 1, \ldots, n$, and (ii) increasing in $y$; i.e., for all $(p, y) \in \mathbb{R}^{n+1}_{++}$,

- $\frac{\partial V(p,y)}{\partial p_j} = -\lambda^* \cdot x_j^*(p, y) < 0, j = 1, \ldots, n$; and
- $\frac{\partial V(p,y)}{\partial y} = \lambda^* > 0$.

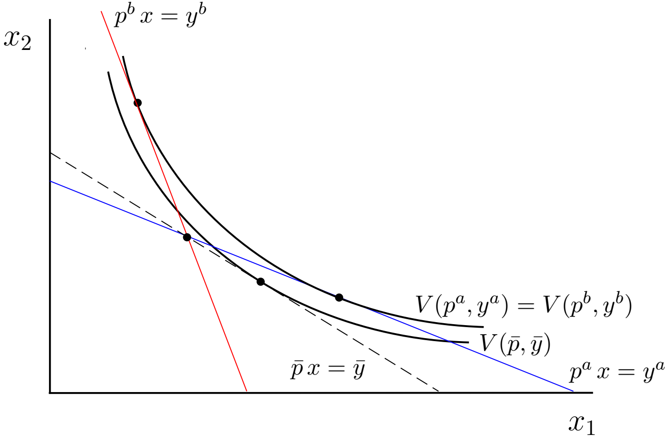
For all $(p, y) \in \mathbb{R}^{n+1}_{++}$, $V(p, y)$ is quasi-convex in prices and income; i.e., for all $(p^a, y^a)$, $(p^b, y^b)$, and $\beta \in (0, 1)$,

$$V(\overline{p}, \overline{y}) \leq \max\{V(p^a, y^a), V(p^b, y^b)\},$$

where $\overline{p} = \beta p^a + (1 - \beta)p^b$ and $\overline{y} = \beta y^a + (1 - \beta)y^b$.

# Indirect utility's properties

The intuition as to why $V(p, y)$ is quasi-convex in prices and income is given in the following graph.



# Indirect utility's properties

For all $(p, y) \in \mathbb{R}^{n+1}_{++}$, $V(p, y)$ is homogeneous of degree zero in prices and income; i.e., for all $(p, y) \in \mathbb{R}^{n+1}_{++}$ and $\alpha > 0$,

$$V(\alpha p, \alpha y) = V(p, y).$$

To see this, fix any $(p, y) \in \mathbb{R}^{n+1}_{++}$ and $\alpha > 0$. By definition,

$$\begin{aligned} V(\alpha p, \alpha y) &= \max\{u(x) : (\alpha p) \cdot x \leq \alpha y\} \\ &= \max\{u(x) : p \cdot x \leq y\} \\ &= V(p, y). \end{aligned}$$

N.B. Bear in mind that $x(\alpha p, \alpha y) = x(p, y)$; i.e. Walrasian demands are homogeneous of degree zero in prices and income (no monetary illusion).

# Expenditure minimization

The primary utility maximization problem studied before

$$\begin{array}{ll} \max\limits_{x \in X} & u(x_1, \ldots, x_n) \\ \text{s.t.} & p_1 x_1 + \cdots + p_n x_n \leq y, \end{array}$$

has the following dual expenditure minimization problem (EMP)

$$\begin{array}{ll} \min\limits_{x \in X} & p_1 x_1 + \cdots + p_n x_n \\ \text{s.t.} & u(x_1, \ldots, x_n) \geq \bar{u}, \end{array}$$

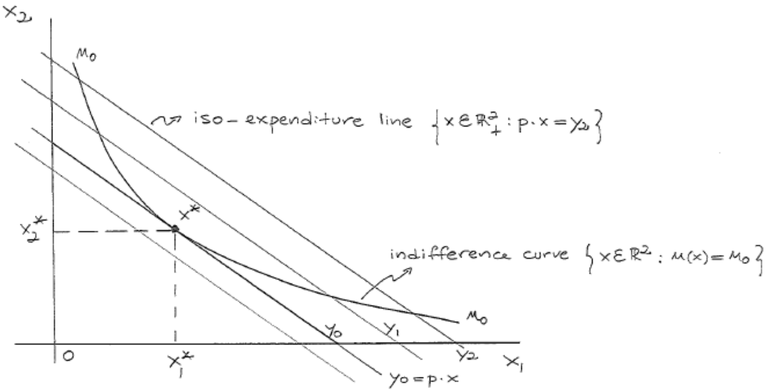
where the utility level $\bar{u}$ is maximal at $(p, y)$, i.e., $\bar{u} = V(p, y)$.

The solution of the EMP gives the lowest possible expenditure to achieve utility $\bar{u}$ at prices $p$.

# Expenditure minimization

Graphically, the problem for the utility-maximizing consumer is to move along the budget line $y_0$ until he achieves the highest IC $u_0$.

The problem for the expenditure-minimizing consumer is to move along the $u_0$-IC until he reaches the lowest iso-expenditure line $y_0$.



# Expenditure minimization

More generally, the minimum expenditure required to attain utility $w$ given prices $p \in \mathbb{R}^n_{++}$ is found by solving

$$\min_{x \in \mathbb{R}^n_+} \; p \cdot x \quad \text{s.t.} \quad u(x) \geq w. \tag{18}$$

Note that, for $p \gg 0$ and $x \in \mathbb{R}^n_+$, the set of expenditures $p \cdot x$ that satisfies the restriction $u(x) \geq w$ is closed and bounded below by zero. Therefore, a minimum always exists.

The Lagrange function corresponding to (18) is:

$$L(x, \lambda) = -p \cdot x + \lambda(u(x) - w); \tag{19}$$

and the Kuhn-Tucker conditions in an interior point $x^* \gg 0$ are as follows:

# Expenditure minimization

1. $\frac{\partial L(x^*,\lambda^*)}{\partial x_i} = -p_i + \lambda^* \cdot \frac{\partial u(x^*)}{\partial x_i} = 0,\, i = 1, \ldots, n;$
2. $u(x^*) \geq w;$
3. $\lambda^* \geq 0$ and $\lambda^* \cdot (u(x^*) - w) = 0.$

If $p \neq 0$, then the constraint must be binding at $x^*$; i.e., $\lambda^* > 0$ and $u(x^*) = w$.

Thus, in the interior solution $x^* \gg 0$ we have that:

$$MRS_{ij}(x^*) = -\frac{u_i(x^*)}{u_j(x^*)} = -\frac{p_i}{p_j}, \forall i \neq j \text{ and}$$

$$u(x^*) = w.$$

# Hicksian demands

The solution to the EMP, denoted by $x^h(p, w) = (x_1^h(p, w), \ldots, x_n^h(p, w))$, provides what is known as the Hicksian or compensated demands.

$x_j^h(\cdot)$ depends on prices $p \in \mathbb{R}^n_{++}$ and welfare $w \in \mathbb{R}$, as opposite to the Walrasian demand $x_j^*(\cdot)$, that depends on prices $p \in \mathbb{R}^n_{++}$ and income $y > 0$.

This is because the Hicksian demands must satisfy the utility constraint, whereas the Walrasian demands must satisfy the budget constraint.

Walrasian demands explain consumer's observable market demand behavior.

The Hicksian demands instead are not observable (depend on utility!); however, their analytic importance will become evident when we explain the effect of a price change over the quantities demanded of each good.

# Expenditure function

The value function of the expenditure minimization problem is called the expenditure function, and is defined as follows

$$\begin{aligned} E(p,u) &= \min\left\{\sum_{i=1}^{n} p_i x_i : u(x_1,\ldots,x_n) \geq u\right\} \\ &= p_1\, x_1^h(p,u) + \ldots + p_n\, x_n^h(p,u). \end{aligned}$$

The assumptions 1-5 on consumers' preferences imply that the expenditure function $E(p,u)$ verifies the following properties:

For every utility level $u \in \mathbb{R}$, $E(p,u)$ is concave in $p$; i.e., for all $p'$, $p''$, and $\alpha \in (0,1)$,

$$E(p_\alpha,u) \geq \alpha \cdot E(p',u) + (1-\alpha) \cdot E(p'',u),$$

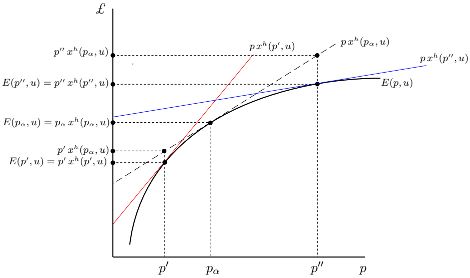
where $p_\alpha = \alpha \cdot p' + (1-\alpha) \cdot p''$.

# Expenditure function's properties

To see this formally, fix $u \in \mathbb{R}$, $p'$, $p'' \in \mathbb{R}^n_{++}$, and $\alpha \in (0, 1)$.

By definition,

$$\begin{aligned} E(p', u) &= p' \cdot x^h(p', u), \\ E(p'', u) &= p'' \cdot x^h(p'', u), \\ E(p_\alpha, u) &= p_\alpha \cdot x^h(p_\alpha, u). \end{aligned}$$




Since $x^h(p', u)$, $x^h(p'', u)$, and $x^h(p_\alpha, u)$ are solutions to their respective EMPs

$$p' \cdot x^h(p', u) \leq p' \cdot x^h(p_\alpha, u), \tag{20}$$

and

$$p'' \cdot x^h(p'', u) \leq p'' \cdot x^h(p_\alpha, u). \tag{21}$$

# Expenditure function's properties

If we multiply (20) by $\alpha$ and (21) by $(1-\alpha)$, and we add the inequalities, then

$$\begin{aligned} \alpha p' \cdot x^h(p', u) + (1-\alpha) p'' \cdot x^h(p'', u) &\leq [\alpha p' + (1-\alpha) p''] \cdot x^h(p_\alpha, u), \\ &\leq p_\alpha \cdot x^h(p_\alpha, u). \end{aligned}$$

Therefore,

$$\alpha \cdot E(p', u) + (1-\alpha) \cdot E(p'', u) \leq E(p_\alpha, u).$$

(Shephard's Lemma) The Hicksian demands are equal to the partial derivatives of the expenditure function with respect to the prices; i.e., for all $j = 1, \ldots, n$, and all $(p, u) \in \mathbb{R}^n_{++} \times \mathbb{R}$,

$$x_j^h(p, u) = \frac{\partial E(p, u)}{\partial p_j}.$$

# Expenditure function's properties

The proof rests on the Envelope theorem.

The Lagrange function associated to the EMP $\min_x p \cdot x$ subject to $u(x) - u \geq 0$ is $L(x, \lambda) = p \cdot x + \lambda[u - u(x)]$, (with $\lambda > 0$ cos $p \gg 0$).

Hence, using (17), we get the Shephard's Lemma:

$$\frac{\partial E(p,u)}{\partial p_i} = \frac{\partial L(x^*, \lambda^*)}{\partial p_i} = x_i^h(p,u). \tag{22}$$

N.B. The pair $(x^*, \lambda^*)$ in (22) denotes the interior solution of the EMP.

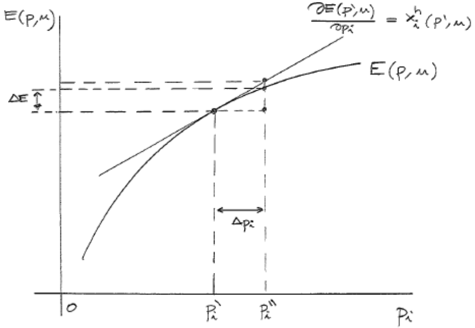
Intuitively, Shephard's Lemma says the following. Suppose Peter buys 12 units of $x_i$ at \$1 each. Assume that $p_i$ increases to \$1.1.

# Expenditure function's properties

Shephard's lemma says that, to maintain utility $u$ constant, the expenditure must increase by $\Delta E = x_i \cdot \Delta p_i = 12 \cdot 0.1 = \$1.2$ (if $x_i$ doesn't change!).

Since $E(p,u)$ is concave in $p$, $x_i \cdot \Delta p_i$ overstates the required increase (cos $x_i$ actually changes when $p_i$ changes).

But, for $\Delta p_i$ small enough, the difference can be ignored.



# Expenditure function's properties

Given $u \in \mathbb{R}$, for all $i = 1, \ldots, n$, $E(p, u)$ is increasing in $p_i$; i.e., $\frac{\partial E(p,u)}{\partial p_i} > 0$, with strict inequality because $x_i^h(p, u) > 0$ (interior solution).

This property follows from (22); it simply means that higher prices $\Rightarrow$ a greater expenditure is needed to attain $u$.

If there exists an interior solution for the expenditure minimization problem, then by the Envelope theorem,

$$\frac{\partial E(p, u)}{\partial u} = \lambda^* > 0.$$

Hence, given $p \in \mathbb{R}^n_{++}$, $E(p, u)$ is increasing in $u$.

In the EMP the Lagrange multiplier represents the rate of change of the minimized expenditure w.r.t. the utility target, i.e.,the utility's marginal cost.

# Expenditure function's properties

Given $u \in \mathbb{R}$, $E(p,u)$ is homogeneous of degree one in $p$.

To see this, note that Hicksian demands are homogeneous of degree zero in prices; i.e., for all $(p,u)$ and all $k > 0$, $x^h(kp,u) = x^h(p,u)$.

The reason is any equi-proportional change in all prices does not alter the slope of the iso-expenditure curves!

Therefore, given $u \in \mathbb{R}$, for all $p \in \mathbb{R}^n_{++}$ and $k > 0$,

$$E(kp,u) = (kp) \cdot x^h(kp,u) = k(p \cdot x^h(p,u)) = k \cdot E(p,u).$$

# Duality

Though the indirect utility and the expenditure function are conceptually different, there is a close relationship between them.

Indeed, under assumptions 1-5, for all $p \in \mathbb{R}^n_{++}$, $y > 0$, and $u \in \mathbb{R}$, we have

$$V(p, E(p,u)) = u; \text{ and} \tag{23}$$

$$E(p, V(p,y)) = y. \tag{24}$$

The intuition for (23) is as follows. (A similar reasoning applies to (24) too.) Given a budget $E(p,u)$, the maximum attainable utility at prices $p$ must be equal to $u$. Instead,

- If $V(p, E(p,u)) > u$, then it would be possible to take some money away from $E(p,u)$ and still get $u$, which would contradict that $E(p,u)$ is the minimum expenditure necessary to attain utility $u$;
- If $V(p, E(p,u)) < u$, then it would be necessary to spend more than $E(p,u)$ to get $u$, which would contradict again that $E(p,u)$ is the minimum expenditure to attain utility $u$.

# Duality

The relationships stated in (23) and (24) indicate that we don't need to solve both UMP and EMP to find the indirect utility and the expenditure function.

From (23), holding prices $p$ constant, we can invert $V(p,\cdot)$ to get

$$E(p,u) = V^{-1}(p,u). \tag{25}$$

Notice that $V^{-1}(p,\cdot)$ exists because $V(p,\cdot)$ is increasing in income.

Similarly, from (24), holding $p$ fixed, we can invert $E(p,\cdot)$ to get

$$V(p,y) = E^{-1}(p,y). \tag{26}$$

Notice that $E^{-1}(p,\cdot)$ exists because $E(p,\cdot)$ is increasing in utility.

Formally, (25) and (26) reflect that the indirect utility and the expenditure function are simply the appropriately chosen inverses of each other.

# Duality

In view of this, it is natural to expect a close relationship between the Hicksian and the Walrasian demands too.

In effect, for all $p \in \mathbb{R}^n_{++}$, $y > 0$, $u \in \mathbb{R}$, and $i = 1, \ldots, n$, we have

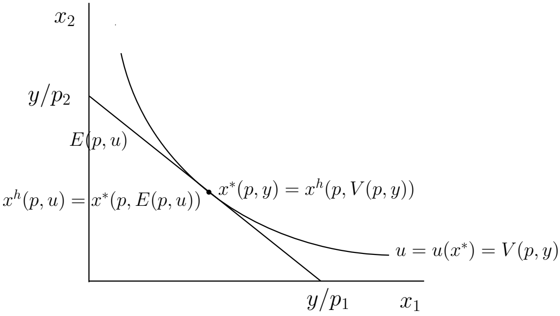
$$\begin{aligned} x_i^*(p,y) &= x_i^h(p, V(p,y)); \text{ and} && (27) \\ x_i^h(p,u) &= x_i^*(p, E(p,u)). && (28) \end{aligned}$$

- (27) says that the Walrasian demand at prices $p$ and income $y$ is equal to the Hicksian demand at prices $p$ and the maximum utility at $(p, y)$;
- (28) says that the Hicksian demand at prices $p$ and utility $u$ is equal to the Walrasian demand at prices $p$ and the minimum expenditure at $(p, u)$.

# Duality

Another way to read (27) and (28) is the following.

- If $x^*$ solves $\max u(x)$ s.t. $p \cdot x \leq y$, then $x^*$ solves $\min(p \cdot x)$ s.t. $u(x) \geq u^* \equiv u(x^*)$;
- Conversely, if $x^*$ solves $\min(p \cdot x)$ s.t. $u(x) \geq u$, then $x^*$ solves $\max u(x)$ s.t. $p \cdot x \leq y^* \equiv p \cdot x^*$.




For that reason, we say $x^*$ has a dual nature.

# Slutsky equation

Fixing an income level $y = E(p,u)$, the expression in (28) implies that

$$x_i^h(p,u) = x_i^*(p,y). \tag{29}$$

Differentiating (29) w.r.t. $p_j$, we have

$$\frac{\partial x_i^h(p,u)}{\partial p_j} = \frac{\partial x_i^*(p,y)}{\partial p_j} + \frac{\partial x_i^*(p,y)}{\partial y} \cdot \underbrace{\frac{\partial E(p,u)}{\partial p_j}}_{=x_j^h(p,u)}.$$

Moving the last term of the RHS to the LHS and using the fact that $x_j^h(p,u) = x_j^*(p,y)$ when $y = E(p,u)$, we get the Slutsky equation:

$$\frac{\partial x_i^*(p,y)}{\partial p_j} = \frac{\partial x_i^h(p,u)}{\partial p_j} - x_j^*(p,y) \cdot \frac{\partial x_i^*(p,y)}{\partial y}. \tag{30}$$

# Slutsky equation

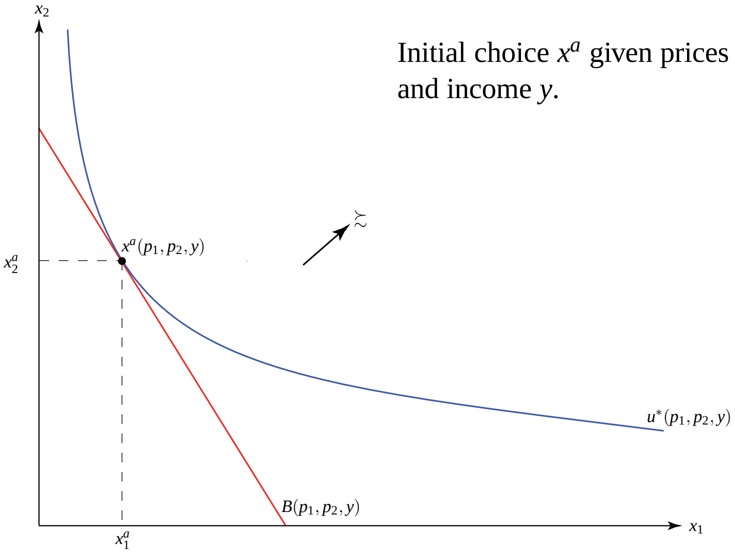
The Slutsky equation is sometimes called the fundamental equation of demand theory.

If we differentiate (29) w.r.t. the own-price $p_i$, then (30) tells us that the slope of the Walrasian demand is the sum of two effects:

- An unobservable substitution effect, $\partial x_i^h(p,u)/\partial p_i$, given by the slope of the Hicksian demand; and
- An observable income effect, $-x_i^*(p,y) \cdot [\partial x_i^*(p,y)/\partial y]$;

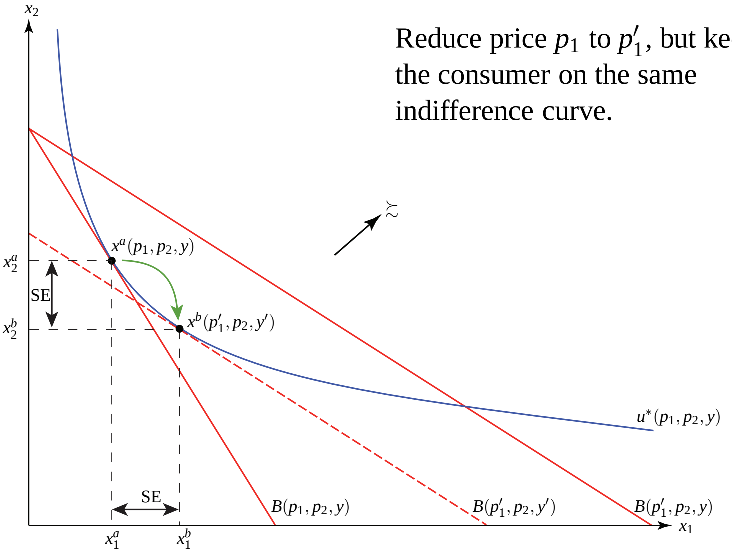
$$\frac{\partial x_i^*(p,y)}{\partial p_i} = \underbrace{\frac{\partial x_i^h(p,u)}{\partial p_i}}_{\text{SE}} - \underbrace{x_i^*(p,y) \cdot \frac{\partial x_i^*(p,y)}{\partial y}}_{\text{IE}} .$$

# Slutsky decomposition



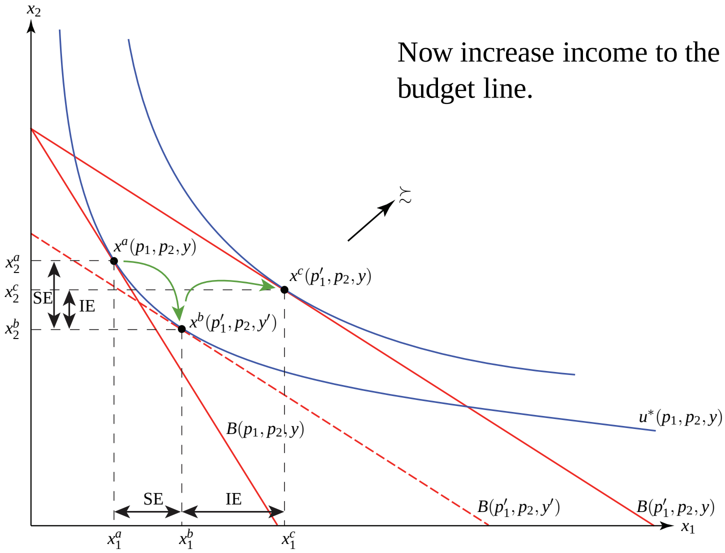

Initial choice $x^a$ given prices $p$ and income $y$.

# Slutsky decomposition: Substitution effect




Reduce price $p_1$ to $p_1'$, but keep the consumer on the same indifference curve.

# Slutsky decomposition: Income effect




Now increase income to the new budget line.

# Upshot from the Slutsky decomposition

A price change has two effects

- substitution effect: always negative, and
- income effect: cannot be signed—depends on preferences.

Depending on the sum of these two effects, (Walrasian) demand may change either way following a price reduction.

However, we know that

- If $\partial x_i^* / \partial y > 0$ (normal good), SE and IE goes in the same direction and the Walrasian demand has a negative slope;
- If $\partial x_i^* / \partial y < 0$ (inferior good), the sign of the slope of the Walrasian demand depends on the size of $|\partial x_i^h / \partial p_i|$ in relation to $|x_i^* \cdot \partial x_i^* / \partial y|$.

# Slutsky matrix

Consider the $n \times n$-matrix of first-order partial derivatives of the Hicksian demands:

$$\sigma(p,u) = \begin{pmatrix} \frac{\partial x_1^h(p,u)}{\partial p_1} & \cdots & \frac{\partial x_1^h(p,u)}{\partial p_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_n^h(p,u)}{\partial p_1} & \cdots & \frac{\partial x_n^h(p,u)}{\partial p_n} \end{pmatrix}.$$

By Shephard's Lemma, $\sigma(p,u)$ is the matrix of second-order partial derivatives of the expenditure function, which is concave in prices. Thus, $\sigma(p,u)$ is negative semi-definite.

Moreover, by definition of negative semi-definiteness, the elements of the diagonal are non-positive; i.e.

$$\frac{\partial x_i^h(p,u)}{\partial p_i} = \frac{\partial^2 E(p,u)}{\partial p_i^2} \leq 0. \tag{31}$$

# Demands' slopes

That means the Hicksian demands cannot have a positive slope!

Finally if $E(p,u)$ is twice continuously differentiable, $\sigma(p,u)$ is symmetric because Young's theorem implies

$$\frac{\partial x_i^h(p,u)}{\partial p_j} = \frac{\partial x_j^h(p,u)}{\partial p_i}.$$

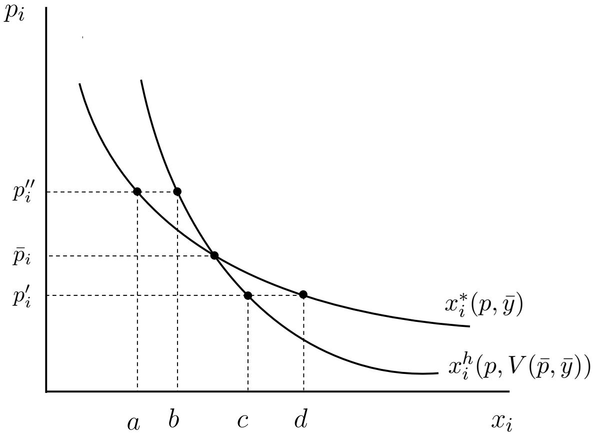
Assuming that the Walrasian demand for $x_i$ is also downward sloping, the relationship between the slopes is as follows.

For a normal good:

- When $p_i \downarrow$, Walrasian $\uparrow$ more than Hicksian because IE reinforces SE;
- Equally, when $p_i \uparrow$, Walrasian $\downarrow$ more than Hicksian because IE reinforces SE.

# Demands' slopes: normal good

As a result, Walrasian demand is flatter than Hicksian demand when $x_i$ is a normal good.



# Demands' slopes: inferior good

For an inferior good:

- When $p_i \downarrow$, Walrasian $\uparrow$ less than Hicksian because IE offsets part of the SE;
- Equally, when $p_i \uparrow$, Walrasian $\downarrow$ less than Hicksian because IE offsets part of the SE.

As a result, assuming both have negative slopes, Hicksian demand is flatter than Walrasian demand when $x_i$ is an inferior good.