The Invertible Matrix Theorem - Proofs

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1 The Invertible Matrix Theorem

In this collection of proofs, I will go through each relevant characterisation of an Invertible (non-singular) matrix and give its proof.

Some statements that follow too closely to others are ommitted.

Let $A$ be an $n \times n$ matrix. The following statements are logically equivalent and, therefore, form the Invertible Matrix Theorem.

1. $A$ is invertible.

Proof. Trivial, as for $A$ to be invertible, $\exists$ a matrix $B$ such that $AB = BA = I_n$.

2. $A$ is row equivalent to the identity matrix $I_n$.

Proof. If $A$ is row equivalent to the identity matrix, $\exists E_p, E_{p-1}, \ldots, E_1$ such that

$$(E_pE_{p-1}\ldots E_1)A = I_n$$

So, as the $E_i$ are invertible, we write

$$(E_pE_{p-1}\ldots E_1)^{-1}(E_pE_{p-1}\ldots E_1)A = A = (E_pE_{p-1}\ldots E_1)^{-1}I_n$$

So

$$A^{-1} = ((E_pE_{p-1}\ldots E_1)^{-1})^{-1}I_n$$

So

$$A^{-1} = (E_pE_{p-1}\ldots E_1)I_n$$

So, if $A$ is row equivalent to $I_n$, its inverse $A^{-1}$ exists.

This proof is why we can use the *Double Matrix* method to find the inverse of an invertible matrix. That method relies on us reducing $A$ to the identity matrix by applying the inverse of the row transforms that transform $I_n$ into $A$, hence transforming $I_n$ into $A^{-1}$.
3. \( A \) has \( n \) pivot positions.

Proof. Trivial - as the inverse of a matrix is unique (see the small proof below), then \( A\vec{x} = \vec{b} \iff \vec{x} = A^{-1}\vec{b} \), so \( \forall \vec{b} \in \mathbb{R}^n, \exists \) a unique solution \( \vec{x} \) such that \( A\vec{x} = \vec{b} \). As \( n \) pivot positions means there must be a unique solution (there can be no free variables), we have \( \vec{x} = A^{-1}\vec{b} \), where \( A^{-1} \) is unique.

The Inverse of a Matrix is Unique.

Proof. Let \( B \) and \( C \) be inverses of \( A \). Then

\[ I_n = BA = CA \]

and

\[ I_n = AB = AC \]

Now, \( AB = AC \iff BAB = BAC \Rightarrow IB = IC \iff B = C \).
So \( B = C = A^{-1} \), so the inverse of a matrix is unique.

4. \( A\vec{x} = \vec{0} \) has only the trivial solution.

Proof. This is a direct and immediate consequence of the previous point. The invertibility of a matrix means that its associated matrix equation \( A\vec{x} = \vec{b} \) has a unique solution \( \vec{x} \) for every \( \vec{b} \in \mathbb{R}^n \), and so its associated homogeneous matrix equation has a unique solution. As every homogeneous equation of this form must have the trivial solution, this unique solution must be the trivial solution and so, if a matrix is invertible, its homogeneous matrix equation has only the trivial solution.

5. The columns of \( A \) form a linearly independent set.

Proof. If a set is linearly indepedent, then the only weights for which its vectors can be made to equal zero collectively are all zero. If the columns of \( A \) are linearly independent, and \( A\vec{x} \) gives a linear combination of the columns of \( A \) with the weights given by \( \vec{x} \), then \( \vec{x} = \vec{0} \) and so the matrix equation associated with the matrix \( A \) has only the trivial solution, hence \( A\vec{x} = \vec{0} \iff \vec{x} = \vec{0} \).
By the previous characterisation of an invertible matrix, linear independence among columns guarantees invertibility, that is the existence of \( A^{-1} \).

6. The linear transformation \( \vec{x} \to A\vec{x} \) is injective/one-to-one.
Proof. Let \( \vec{x} \in \mathbb{R}^n \) and let \( T : \mathbb{R}^n \rightarrow \mathbb{R}^n \), defined by

\[
T(\vec{x}) = A\vec{x}
\]

where \( A \) is the standard matrix of the linear transformation \( T \). Saying \( T \) is injective is equivalent to saying that, \( \forall \vec{b} \in \mathbb{R}^n, \exists \) some unique \( \vec{x} \) such that \( T(\vec{x}) = A\vec{x} = \vec{b} \), hence is equivalent to saying that the matrix equation \( A\vec{x} = \vec{b} \) has a unique solution for each \( \vec{b} \in \mathbb{R}^n \). If there is a unique solution to \( A\vec{x} = \vec{b} \) for each \( \vec{b} \), then we have already established that the matrix \( A \) is invertible.

More interestingly, the injectivity of \( T \) means that \( T \) is also invertible as a linear transformation. Specifically, the standard matrix of \( T^{-1} \) is \( A^{-1} \), hence defined \( T^{-1} : \mathbb{R}^n \rightarrow \mathbb{R}^n \) by

\[
T^{-1}(\vec{x}) = A^{-1}\vec{x}
\]

And,

\[
T(T^{-1}(\vec{x})) = A(A^{-1}\vec{x}) = I_n \vec{x} = \vec{x}
\]

Hence, the identity transformation, written here as \( T(T^{-1}(\vec{x})) \), has the standard matrix \( I_n \).

7. The matrix \( A^T \) is invertible.

Proof. We assume that \( A \) is invertible, that is

\[
AA^{-1} = I_n
\]

So, as the identity matrix is its own transpose, that is \( I_n = I_n^T \) (this is true of any symmetric/diagonal matrix), we write

\[
I_n^T = (AA^{-1})^T = (A^{-1})^T A^T = I_n
\]

Hence, when \( A \) is invertible, \( A^T \) is also invertible.

A key point from this proof is that, for any matrix’s transpose to be invertible, the transpose of the transpose (the original matrix) must be invertible.

8. The columns of \( A \) form a basis for \( \mathbb{R}^n \).

Proof. A basis, by definition, is a set that generates a subspace - hence the set must be linearly independent and must span the subspace it is a basis of.

A square matrix has \( n \) columns, therefore its columns may be seen as a set of \( n \) vectors. If the matrix is invertible, its columns are linearly independent (by previous proofs) and, as there are \( n \) vectors taken from \( n \) columns, we automatically have a basis.

9. The column space \( \text{Col}\{A\} \) of \( A \) spans \( \mathbb{R}^n \).

Proof. A direct consequence of the previous characterisation - \( \text{Col}\{A\} \) is the set of all possible linear combinations of the columns of the matrix as vectors. If the columns form a basis, then \( \text{Col}\{A\} = \text{Span}\{\vec{a}_i, \ldots, \vec{a}_n\} \), where the \( \vec{a}_i \) are the columns of \( A \).
10. The dimension of the column space of $A$, $\dim \text{Col}(A)$, is equal to $n$.

Proof. The dimension of a subspace is the number of vectors required to form a basis of that subspace. As we have previously proven that $n$ vectors are enough to form a basis, then the dimension of the column space of $A$ is $n$.

Additionally, if the null space $\text{Nul}(A)$ is the set of all solutions to the homogeneous equation $A\vec{v} = \vec{0}$, then $\text{Nul}(A) = \{\vec{0}\}$ - as we have previous proven that the only solution to the homogeneous equation of an invertible matrix is the trivial one, hence we do not need any vectors to form a basis of the null space of an invertible matrix. Also, the dimension of an invertible matrix’s null space is zero, $\dim \text{Nul}(A) = 0$.

\[\square\]

11. Zero is not an eigenvalue of $A$.

Proof. An eigenvalue of $A$ is some $\lambda \in \mathbb{R}$ such that $A\vec{v} = \lambda \vec{v}$ for some non-zero $\vec{v} \in \mathbb{R}^n$. If any of the $\lambda_i$ (eigenvalues of the matrix) is equal to zero, say $\lambda_k$, then we have $A\vec{v} = \lambda_k \vec{v} = 0$, so $A\vec{v} = 0$, where $\vec{v} \neq \vec{0}$. This means that we cannot consider the trivial solution of the homogeneous equation $A\vec{v} = 0$, and so there must exist some non-trivial solution to the homogeneous equation, hence the matrix $A$ is not invertible.

Therefore, $A$ is invertible if and only if $\nexists \lambda_i$ with $\lambda_i = 0$.

\[\square\]

12. $\det(A) \neq 0$

Proof. Suppose $\det(A) = 0$ and $A^{-1}$ exists such that $AA^{-1} = I_n$.

Then, $\det(I_n) = \det(AA^{-1}) = \det(A) \det(A^{-1}) = \det(A) \det(A)^{-1} = \frac{\det(A)}{\det(A)} = 0$

$0$ is indeterminate, and certainly $\frac{0}{0} \neq 1 = \det(I_n)$, so we have a contradiction and $\det(A) \neq 0$ if $A$ is invertible.

\[\square\]

13. The orthogonal complement of the column space of $A$ is equal to the zero subspace, $\{\vec{0}\}$.

Proof. If the matrix $A$ is invertible, its column space spans and is therefore equal to $\mathbb{R}^n$, so the set of all vectors that are orthogonal to every vector in this subspace (the orthogonal complement), is the set $\{\vec{0}\}$ as $\forall \vec{v} \in \mathbb{R}^n, \vec{0} \cdot \vec{v} = 0$, so the zero vector $\vec{0}$ is orthogonal to every vector (including itself).

A direct consequence of this is that the orthogonal complement of the null space $\text{Nul}(A)^\perp = \mathbb{R}^n$, because $\text{Nul}(A) = \{\vec{0}\}$, and every vector $\vec{v} \in \mathbb{R}^n$ has $\vec{v} \cdot \vec{0} = 0$.

\[\square\]

14. The row space of $A$ spans $\mathbb{R}^n$.

Proof. The column space of a matrix $A$, $\text{Col}(A)$, spans $\mathbb{R}^n$ if $A$ is invertible, as the columns of $A$ for a basis for $\mathbb{R}^n$. As $A^T$ is also invertible (as we proved earlier), $\text{Col}(A^T) = \text{Row}(A)$, so as $\text{Col}(A^T) = \mathbb{R}^n$, $\text{Row}(A) = \mathbb{R}^n$.

\[\square\]