AERO III/IV Mathematics Tutorial Sheet 1

(1) Solve the Euler-Lagrange equation which makes the integral

$$I[y] = \int_0^{\pi/2} (2xyy' + (y')^2)dx$$

stationary, given $y = 0$ when $x = 0$ and $y = 1$ when $x = \pi/2$.
(2) Find the minimizing curves of the integral

$$I[\theta] = \int r^2 \left[1 + r^2 \left(\frac{d\theta}{dr}\right)^2\right]^{1/2} dr.$$

(3) Show that the extremal curves of the integral

$$I = \int \left[1 + \left(\frac{d\phi}{d\theta}\right)^2 \sin^2\theta\right]^{1/2} d\theta$$

can be written in the form $\sin(\alpha - \phi) = \beta \cot\theta$ where α and β are constants.
(4) Find the stationary value of the integral $I = \int_0^1 (y')^2 dx$ subject to the constraint $J = \int_0^1 y = 1$ and the end conditions $y(0) = 0$ and $y(1) = 0$.
(5) Prove that the extremal curve $y = y(x)$ integral

$$I = \int_1^2 \left[x^2 \left(\frac{dy}{dx}\right)^2 + 2y^2\right] dx$$

which passes through the points $(1, 0)$ and $(2, 1)$ in the x-y plane is given by

$$y = \frac{4}{7}\left(x - \frac{1}{x^2}\right).$$

If the constraint $\int_1^2 y/x dx = 1/4$ is added to the problem, find the new exremal curve of I.
(6) (a) Prove that the extremal curve $y = y(x)$ of the integral

$$I = \int_0^{2\pi} \left[m^2 y^2 - \left(\frac{dy}{dx}\right)^2\right] dx$$

(m an integer) satisfying the conditions

$$y = 1 \text{ on } x = 0, \quad dy/dx = \pi/2 \text{ on } x = 2\pi,$$

$$\int_0^{2\pi} y(x) \cos nx dx = \pi/2 \quad (n \text{ an integer})$$

is given by

$$y = \frac{1}{2}\left(\cos mx + \frac{\pi}{m}\sin mx + \cos nx\right) \qquad \text{provided } m \neq n.$$

(b) Find the form of the extremal curve of I subject to the same conditions for the special case of $m = n$. (Note that this is not just a case of putting $m = n$ in the agove expression for $y(x)$!!)

(7) (Isoperimetric problem) Show that the plane closed curve of given length which encloses the greatest area is a circle. In polar coordinates (as in the Figure), this is equivalent to find the extremal curve $r(\theta)$ of

$$\min \int_0^\pi \frac{1}{2} r^2 d\theta,$$

subject to the total length constraint

$$\int_0^\pi (r^2 + r'^2)^{1/2} d\theta = \ell.$$

Here the boundary condition, suggested by the fact that the horizontal axis is tangent to the curve, is given by the overdetermined boundary conditions

$$r(0) = r(\pi) = 0, r'(0) = r'(\pi) = 0.$$

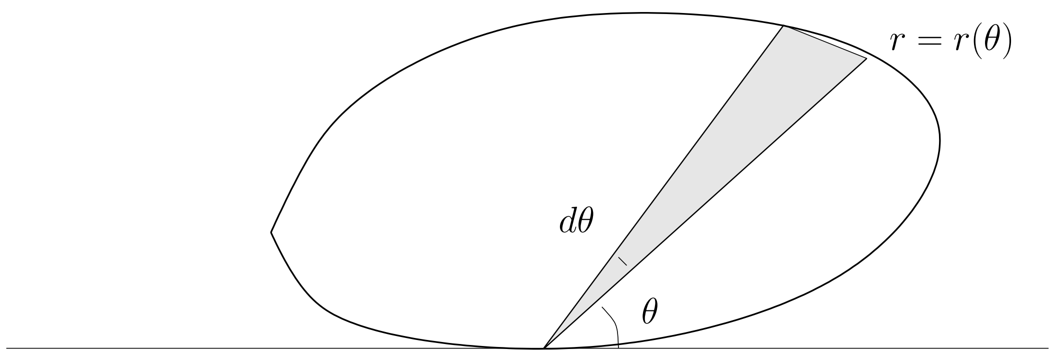

Figure 1: The isoperimetric problem in polar coordinates.

Answers: **(1)** $y = \sin x$; **(2)** $r^3 = c_1 \sec(3\theta - c_2)$; **(4)** 12;**(5)** $y = 2/x^2 + x - 3/x$; **(6)** $y = \cos mx + \frac{\pi}{2m}(1 - 4m^2)\sin mx + mx \sin mx$; **(7)** $r = \frac{\ell}{\pi}\sin\theta$