AERO III/IV Mathematics Tutorial Sheet 4: Laplace Transform

(1) Using the **convolution theorem** (of course partial fractions works too) show that

$$\mathcal{L}^{-1}\left[\frac{1}{p(p+a)}\right] = \frac{1-e^{-at}}{a}.$$

Hence obtain the solution of the ODE

$$x''(t) + ax'(t) = f(t)$$

with the initial value $x(0) = x_0, x'(0) = x_1$ and $f(t)$ is an arbitrary function. Show that the **steady-state** value of x is given by

$$x(\infty) = \lim_{t\to\infty} x(t) = \frac{1}{a}\int_0^\infty f(u)du + x_0 + \frac{x_1}{a},$$

when $a > 0$.

(2) Using the convolution theorem, show that if

$$y(x) = f(x) + \int_0^x g(x-u)y(u)du$$

then $Y(p) = F(p)/(1-G(p))$, where $Y(p), F(p), G(p)$ are the Laplace transforms of $y(x), f(x), g(x)$. Hence solve the integral equation

$$y(x) = \sin 3x + \int_0^x \sin(x-u)y(u)du.$$

(3) (i) Show that if $f(t) = \int_t^\infty \frac{g(u)}{u}du$, then $F(p) = \frac{1}{p}\int_0^p G(q)dq$. Here $F(p)$ and $G(p)$ are the Laplace transform of $f(t)$ and $g(t)$, respectively.

(ii) Show that the Laplace transform of

$$f(t) = \int_t^\infty \frac{e^{-u}}{u}du$$

is $F(p) = \frac{1}{p}\ln(p+1)$.

(4) Solve the PDE

$$\frac{\partial u}{\partial t} + x\frac{\partial u}{\partial x} = x, \qquad u(x,0) = 0,\ u(0,t) = 0;$$

for $u(x,t)$ (with $x > 0, t > 0$) by taking the Laplace transform with respect to t.

(5)

(a) By using the residue theorem and the complex inversion formula for the Laplace transform, prove that

$$\mathcal{L}^{-1}\left[\frac{1}{p(p^2+2p+3)}\right] = \frac{1}{3}\left(1 - e^{-t}\cos\sqrt{2}t - \frac{e^{-t}}{\sqrt{2}}\sin\sqrt{2}t\right)$$

(b) By taking a Laplace transform and using the result of (a) solve

$$\frac{\partial^2 u}{\partial t^2} + 2\frac{\partial u}{\partial t} + x\frac{\partial u}{\partial x} = x^3; \quad u(x,0) = \frac{\partial u}{\partial t}(x,0) = u(0,t) = 0.$$

Answer:

(1) $x(t) = \frac{1}{a}\int_0^t f(u)(1 - e^{-a(t-u)}du + x_0 + \frac{x_1}{a}(1 - e^{-at}),$

(2) $y(t) = \frac{x}{3} + \frac{8}{9}\sin 3x;$

(4) $x(1 - e^{-t})$

(5) (b) $u(x,t) = \frac{x^3}{3}\left(1 - e^{-t}\cos\sqrt{t} - \frac{e^{-t}}{\sqrt{2}}\sin\sqrt{2}t\right)$