AERO III/IV Mathematics Tutorial Sheet 5: Conformal Mapping

(1) Sketch the image of the quarter circle $|z| < 1, 0 < \arg z < \pi/2$ under each of the following transformations:

$$\text{(i) } w = z^2 \qquad \text{(ii) } w = iz \qquad \text{(iii) } w = 1/z \qquad \text{(iv)} w = \ln z.$$

(2) Sketch the region

$$|xy| < \pi/4, \qquad |y| < x$$

in the x-y plane, and its image under the transformation $w = u + iv = (x + iy)^2 = z^2$.

The function $\phi(x, y)$ is bounded and harmonic throughout this region. Find ϕ in terms of x and y with boundary condition:

$$\phi = 0 \ \text{ on } xy = \pm\pi/4, x > |y|; \qquad \phi = \cos 2xy \ \text{ on } x = \pm y, |xy| < \pi/4.$$

(3) Show that the function

$$w = \frac{z - i}{z + i}$$

maps the upper half-plane $y > 0$ onto the interior of the circle $|w| < 1$. Solve

$$\phi_{xx} + \phi_{yy} = 0 \quad (y > 0)$$

subject tot the boundary conditions

$$\phi(x, 0) = (x^2 - 1)/(x^2 + 1); \quad \phi \text{bounded at infinity,}$$

by transforming to the u-v plane, where $w = u + iv$.

(4) Solve the Laplace equation

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = 0$$

on the shaded region in Figure 1 by the conformal map $w = \frac{z-1}{z+1}$.

(a) Determine the region $\tilde{D}$ on the w-plane where the shaded region is transformed to and the specify the boundary condition for $H(u, v)$ on this region.

(b) Solve the Laplace equation for $H(u, v)$ on $\tilde{D}$

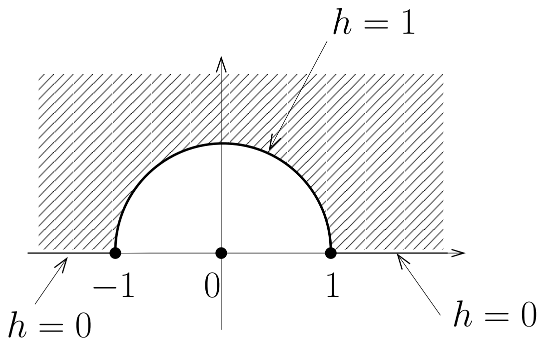

Figure 1: Solving the Laplace equation for h on shaded region, with $h(x,0)=0$ for $|x|>1$ and $h(x,y)=1$ on the upper unit circle.

(c) Write u, v as functions of x, y.

(d) Get the solution on the original shaded region by $h(x,y) = H(u(x,y), v(x,y))$.