AERO III/IV Mathematics Tutorial Sheet 3: Laurent series and contour integrals

(1) Expand the function

$$f(z) = \frac{z}{(z-1)(2-z)}$$

in a Laurent series valid for

$$\text{(i) } 0 < |z-1| < 1, \quad \text{(ii) } |z-1| > 1, \quad \text{(iii) } 0 < |z-2| < 1, \quad \text{(iv) } |z-2| > 1,$$

giving in each case the first three non-zero terms in the expansion. Hence otherwise determine the residues of $f(z)$ at each of its poles.

(2) Expand the following functions in Laurent series valid in the regions given

$$\text{(i)} \frac{e^z}{z(z^2+1)} \quad \text{for } 0 < |z| < 1;$$

$$\text{(ii)} \frac{1}{(z^2+1)(z^2+2)} \quad \text{for (a) } |z| > \sqrt{2}, \quad \text{(b) } 1 < |z| < \sqrt{2}, \quad \text{(c) } |z| < 1.$$

(3) Expand the following functions in a Laurent series about $z = 0$, naming the type of singularity in each case:

$$\text{(i)} \frac{\sin z}{z}, \quad \text{(ii)} \frac{\cosh 1/z}{z}, \quad \text{(iii)} \frac{e^{z^2}}{z^4}.$$

(4) Find the residues of the following functions at the points indicated:

$$\text{(i) } \frac{1}{(e^z-1)^2} \text{ at } z = 2n\pi i; \qquad \text{(ii) } \frac{z^3}{z^2+1} \text{ at } z = \pm i; \qquad \text{(iii) } e^z \tan z \text{ at } z = 3\pi/2.$$

(5) By integrating the function

$$\frac{e^{iz}}{(z^2+a^2)(z^2+b^2)} \qquad (a > b > 0)$$

around a semicircular contour, show that

$$\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+a^2)(x^2+b^2)} dx = \frac{\pi}{a^2-b^2} \frac{ae^{-b} - be^{-a}}{ab}.$$

(6) Evaluate by contour integration:

$$\text{(i)} \int_0^{2\pi} \frac{\cos\theta}{5-4\cos\theta} d\theta; \quad \text{(ii)} \int_0^{\infty} \frac{dx}{1+x^n} (n \text{ is a positive integer greater than two});$$

$$\text{(iii)} \int_{-\infty}^{\infty} \frac{\sin x}{x(x^2+1)} dx;$$

(7) Using the indicated contour in Figure 1(a), find the following integral:

$$\int_{-\infty}^{\infty} \frac{\cos x}{e^x + e^{-x}} dx = \frac{\pi}{e^{\pi/2} + e^{-\pi/2}}.$$

(8) Using the indicated contour in Figure 1(b), find the following integral:

$$\text{(a)} \quad \int_0^{\infty} \frac{(\log x)^2}{1+x^2} dx = \frac{\pi^3}{8}; \qquad \text{(b)} \quad \int_0^{\infty} \frac{\log x}{(1+x^2)^2} dx = -\frac{\pi}{4}.$$

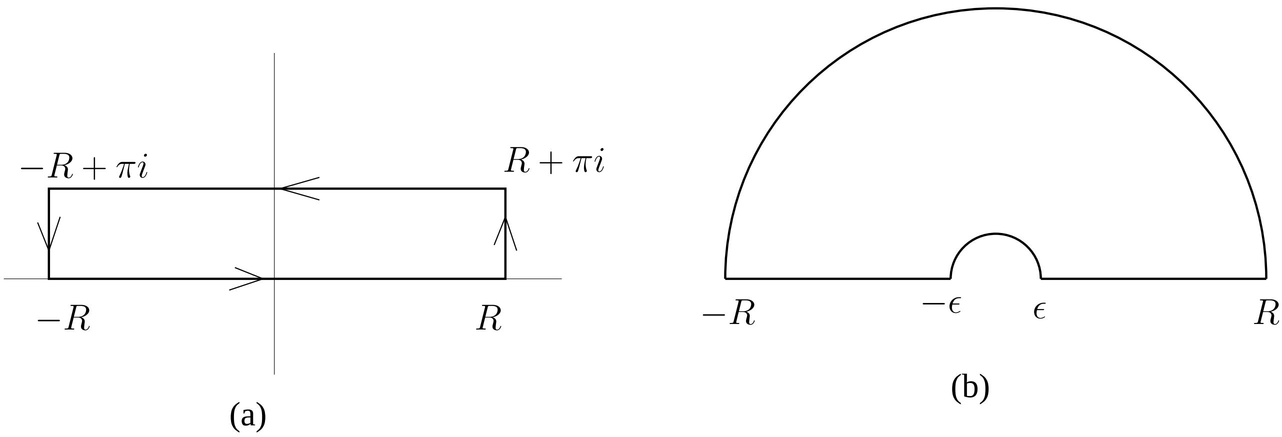

Figure 1: The contours for problem (7) and (8).

Answers:

(1) (i) $1/(z-1) + 2 + 2(z-1) + \cdot$; (ii) $-1/(z-1) - 2/(z-1)^2 - 2/(z-1)^3 + \cdots$;
(iii) $-2/(z-2) + 1 - (z-2) + \cdots$; (iv) $-1/(z-2) - 1/(z-2)^2 + 1/(z-2)^3 + \cdots$.
$\mathrm{Res}(f, z=1) = 1$, $\mathrm{Res}(f, z=2) = -2$

(2) (i) $1/z + 1 - z/2 - 5z^2/6 + \cdots$; (ii) (a) $1/z^4 - 3/z^6 + \cdots$;
(b) $\cdots + 1/z^6 - 1/z^4 + 1/z^2 - 1/2 + z^2/4 + \cdots$; (c) $1/2 - 3z^2/4 + 7z^4/8 - \cdots$.

(3) (i) $1 - z^2/3! + z^4/5! + \cdots$; removable singularity
(ii) $1/z + 1/(2z^3) + \cdots$; essential singularity;
(iii) $1/z^4 + 1/z^2 + 1/2 + \cdots$; pole of order 4 at $z = 0$ (with residue zeros).

(4) (i) -1; (ii) $-1/2$; (iii) $-e^{3\pi/2}$;

(6) (i) $\pi/3$; (ii) $\dfrac{\pi}{n}\dfrac{1}{\sin\pi/n}$; (iii) $\pi(e-1)/e$