(1) By writing $f(z) = u(x, y) + iv(x, y)$ where $z = x + iy$, find the real function $u(x, y)$, $v(x, y)$ in each of the following cases:

(i) $z^2 + 2z$ (ii) $1/z \quad (z \neq 0)$

(iii) $\sin z$ (iv)$z/(e^z - 1) \quad (z \neq 0)$

(2) By using the Cauchy-Riemann equations show that the function

$$f(z) = \left(1 + \frac{i}{4\pi}\right) \log z$$

is analytic in any region that does not encircle the origin. Show that the equation of the contours on which $u(x, y)$ is constant may be written in the form

$$r = Ae^{\theta/4\pi}$$

where A is a constant and (r, θ) are polar coordinates. Sketch these contours and also those which $v(x, y)$ is constant. Which physical situation does these contours describe?

(3) Determine all the possible values of

(i) $\log(1 - i)$; (ii) $\arctan(2i)$; (iii) $(1 + i)^{2i}$; (v) $(1 + i)^{1/2}$; (iv) 2^{3+2i}.

(4) Evaluate the following using Cauchy's integral formula:

(i)

$$\int_{\mathcal{C}} \frac{e^z}{z + i\pi/2} dz$$

where $\mathcal{C}$ is the boundary of the square defined by the lines $x = \pm 2, y = \pm 2$;

(ii)

$$\int_{\mathcal{C}} \frac{e^z}{z(z + 1)} dz$$

where $\mathcal{C}$ is the boundary of the circle $|z - 1| = 3$.

(5) Evaluate the following integrals, where $\mathcal{C}$ is the boundary of the circle $|z| = 2$:

(i)

$$\int_{\mathcal{C}} \frac{z^3 + 5}{z - i} dz$$

(ii)

$$\int_{\mathcal{C}} \frac{\cos z}{z^n} dz, \qquad n = 1, 2, \cdots$$

(iii)

$$\int_{\mathcal{C}} z^n (1 - z)^m dz, \quad m = 0, 1, \cdots; \ n = 0, \pm 1, \pm 2, \cdots.$$

(6) By completing the contour on the complex plane, find the following integrals:

(i)

$$\int_{-\infty}^{\infty} e^{-x^2-2ix}\,dx,$$

where you are given that $\displaystyle\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$.

(ii)

$$\int_{-\infty}^{\infty} e^{ix^2}\,dx.$$

Answers:

(1) (i) $u = x^2 - y^2 + 2x, v = 2y(1+x)$; (ii) $u = x/(x^2+y^2), v = -y/(x^2+y^2)$; (iii) $u = \sin x \cosh y, v = \cos x \sinh y$; (iv) $u = (xe^x \cos y - x + ye^x \sin y)/(e^{2x} + 1 - 2e^x \cos y), v = (ye^x \cos y - xe^x \sin y - y)/(e^{2x} + 1 - 2e^x \cos y)$;

(2) contours on which v is constant are $r = Be^{-4\pi\theta}$, B arbitrary constant;

(3) (i) $\frac{1}{2}\ln 2 - i(\frac{\pi}{4} + 2n\pi)$; (ii) $\frac{\pi}{2} + n\pi + \frac{i}{2}\ln 3$; (iii) $e^{-\frac{\pi}{2}-4n\pi}(\cos\ln 2 + i\sin\ln 2)$; (iv) $2^{\frac{1}{4}}e^{i\pi/8}, 2^{\frac{1}{4}}e^{i9\pi/8}$; (v) $8(\cos 2\ln 2 + i\sin 2\ln 2)$;

(4) (i) 2π; (ii) $2\pi i(1 - e^{-1})$

(5) (i) $2\pi(1+5i)$; (ii) 0 when n even; $(-1)^{\frac{n-1}{2}}\frac{2\pi i}{(n-1)!}$ when n odd; (iii)

$$\begin{cases} (-1)^{-n-1}\frac{2\pi i m!}{(m+n-1)!(-n-1)!}, & n \geq -m-1, n < 0; \\ 0, & \text{otherwise.} \end{cases}$$

(6) (i) $\sqrt{\pi}/e$; (ii) $\sqrt{\pi}e^{i\pi/4}$