Aero III/IV Laplace Transform

1 Revision on Laplace Transform and its basic properties

Definition 1.1. The **Laplace transform** of a function f on $[0, \infty)$ is defined as

$$F(s) = \mathcal{L}[f(t)](s) = \int_0^\infty f(t)e^{-st}dt.$$

Remark. Notice the notation, since t is a dummy variable in $\mathcal{L}[f(t)](s)$, we just write it as $F(s) = \mathcal{L}[f](s)$.

Example 1.1. Show that the Laplace transform of e^{-at}.

$$\mathcal{L}[e^{-at}](s) = \int_0^\infty e^{-at-st}dt = -\frac{e^{-(a+s)t}}{a+s}\bigg|_{t=0}^{\infty} = \frac{1}{a+s}.$$

We have the same formula even if a is complex, and it is valid in fact for for $s > -\mathrm{Re}a$.

In fact, we can find the transforms for other simple functions and have the "building blocks" for many other functions.

$f(t)$	$\mathcal{L}[f](s)$	valid for
1	$1/s$	Re $s > 0$
$t^n, n = 1, 2, \cdots$	$n!/s^{n+1}$	Re $s > 0$
e^{-at}	$1/(s+a)$	Re $s > 0$
$\cos \omega t$	$s/(s^2 + \omega^2)$	Re $s > \lvert\mathrm{Im}\omega\rvert$
$\sin \omega t$	$\omega/(s^2 + \omega^2)$	Re $s > \lvert\mathrm{Im}\omega\rvert$
$H(t-T) = \begin{cases} 0, & t < T \\ 1, & t > T \end{cases}$	e^{-sT}/s	Re $s > 0$

Table 1: Laplace Transforms for elementary functions

Here $H(t)$ is the **Heaviside function**

$$H(t) = \begin{cases} 0, & t < 0, \\ 1, & t \geq 0. \end{cases}$$

Basic properties of Laplace transforms

a) Linearity. $\mathcal{L}[af+bg]=a\mathcal{L}[f]+b\mathcal{L}[g]$

b) Scaling. $\mathcal{L}[f(t/a)]=a\mathcal{L}[f](as)$

c) Translation. $\mathcal{L}[e^{-at}f(t)]=\mathcal{L}[f](s+a)$

d) $\mathcal{L}(f(t-a)H(t-a)](s)=e^{-as}\mathcal{L}[f](s)$

e) Transformation of derivatives:

$$\mathcal{L}[f^{(n)}(t)](s)=s^n\mathcal{L}[f](s)-s^{n-1}f(0)-s^{n-2}f'(0)-\cdots-f^{(n-1)}(0). \tag{1}$$

f) Transformation with the factor t^n:

$$\mathcal{L}[t^nf(t)](s)=(-1)^n\frac{d^n}{ds^n}\mathcal{L}[f](s). \tag{2}$$

We can show (e) using integration by parts,

$$\begin{aligned}\mathcal{L}[f^{(n)}](s)&=\int_0^\infty f^{(n)}(t)e^{-st}dt\\&=\int_0^\infty e^{-st}df^{(n-1)}(t)\\&=e^{-st}f^{(n-1)}(t)\big|_{t=0}^{\infty}-\int_0^\infty f^{(n-1)}(t)de^{-st}\\&=-f^{(n-1)}(0)+s\int_0^\infty f^{(n-1)}(t)e^{-st}dt.\end{aligned} \tag{3}$$

Continuing this process, then we can get the previous formula (1).

For (f), taking derivative w.r.t s n-times in the definition of Laplace transform, we get

$$\begin{aligned}(-1)^n\frac{d^n}{ds^n}\mathcal{L}[f](s)&=(-1)^n\frac{d^n}{ds^n}\int_0^\infty f(t)e^{-st}dt\\&=(-1)^n\int_0^\infty f(t)\frac{d^n}{ds^n}e^{-st}dt=\int_0^\infty t^nf(t)e^{-st}dt=\mathcal{L}[t^nf(t)](s).\end{aligned} \tag{4}$$

One final useful formula is the convolution theorem, which convert convolution into multiplication.

Theorem 1.1. *If the Laplace transforms of f and g are F and G, respectively, then the Laplace transform of the convolution integral $\int_0^t f(t-\tau)g(\tau)d\tau$ is $F(s)G(s)$.*

We can show this by definition

$$\begin{aligned}
\int_0^\infty e^{-st}\left(\int_0^t f(t-\tau)g(\tau)d\tau\right)dt &= \int_0^\infty \int_\tau^\infty f(t-\tau)g(\tau)e^{-st}dtd\tau \\
&= \int_0^\infty \left(\int_\tau^\infty f(t-\tau)e^{-(t-\tau)s}dt\right) g(\tau)e^{-\tau s}d\tau \\
&= \int_0^\infty \left(\int_0^\infty f(t)e^{-ts}dt\right) g(\tau)e^{-\tau s}d\tau \\
&= \left(\int_0^\infty f(t)e^{-ts}dt\right)\left(\int_0^\infty g(\tau)e^{-\tau s}d\tau\right) = F(s)G(s).
\end{aligned}$$

Exercise Find the Laplace transform of $tf'(t)$ and $tf''(t)$.

Remark. You can see the following corresponence between a function and its Laplace transform: derivation to $f(t)$ becomes multiplication of s to $\mathcal{L}[f](s)$, and multiplication (of t) to $f(t)$ becomes derivation (with respect to s) of $\mathcal{L}[f](s)$. But you have to take care of the signs and "boundary terms".

Example 1.2. Find the inverse Laplace transform of (1) $F(s) = \frac{5}{s}+\frac{12}{s^2}+\frac{8}{s+3}$ (2) $F(s) = \frac{8s+4}{s^2+6s+13}$.

(1) Using the linearity of the Laplace transform and the table,

$$\begin{aligned}
\mathcal{L}^{-1}\left[\frac{5}{s}+\frac{12}{s^2}+\frac{8}{s+3}\right](t) &= \mathcal{L}^{-1}\left[\frac{5}{s}\right](t) + \mathcal{L}^{-1}\left[\frac{12}{s^2}\right](t) + \mathcal{L}^{-1}\left[\frac{8}{s+3}\right](t) \\
&= 5 + 12t + 8e^{-3t}.
\end{aligned}$$

(2) First the denominator can be written as (completing the square)

$$s^2+6s+13 = (s+3)^2+2^2,$$

therefore,

$$F(s) = \frac{8s+4}{s^2+6s+13} = \frac{8(s+3)-20}{(s+3)^2+2^2},$$

and the inverse Laplace transform is

$$f(t) = \mathcal{L}^{-1}[F(s)](t) = \mathcal{L}^{-1}\left[\frac{8(s+3)}{(s+3)^2+2^2}\right](t) - \mathcal{L}^{-1}\left[\frac{20}{(s+3)^2+2^2}\right](t) = e^{-3t}\Big(8\cos 2t - 10\sin 2t\Big).$$

2 Inverse Laplace Transform: Reduction to Residue Calculus

If $F(s)$ is the Laplace transform of $f(t)$, then there is a formal expression for the inverse transform.

Theorem 2.1 (Inverse Laplace Transform)**.** *If $F(s)$ is the Laplace transform of $f(t)$, then we have the inverse transform for $t > 0$,*

$$f(t) = \mathcal{L}^{-1}[F](t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} F(s)e^{st}ds,$$

where the γ is greater than the real part of any singularity of F.

Before giving a brief derivation of the inversion formula, we point out two technical points:

(1) The parameter s is allowed to be complex.

(2) If $f(t)$ is reasonably behaved (bounded or with at most exponential growth like e^{at}), then the transform $F(s)$ exists for $s = \gamma + i\omega$, as long as γ is large enough. Therefore we don't expect any singularity for $F(s)$ for large γ.

Proof. (The inversion formula) The basic idea is to write $F(s)$ as a contour integration, with with a simple dependence on s, such that the s-dependent part can be easily inverted.

If all singular points of $F(s)$ have real part less than γ, then we can construct a semicircle $|z - \gamma| = R$ as in Figure 1.

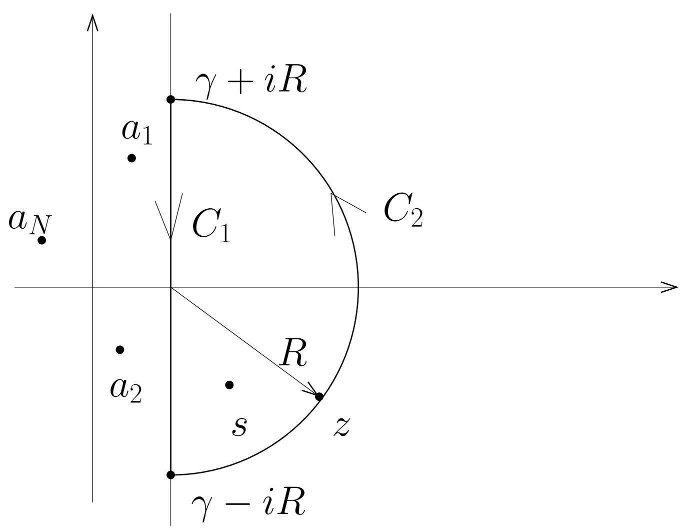

Figure 1: The contour in the proof of the inversion formula.

Then by Cauchy integral formula,

$$F(s) = \frac{1}{2\pi i}\int_{C_1+C_2} \frac{F(z)}{z-s}dz = \frac{1}{2\pi i}\int_{C_1} \frac{F(z)}{z-s}dz + \frac{1}{2\pi i}\int_{C_2} \frac{F(z)}{z-s}dz.$$

In the limit R goes to infinity, the integral on C_2 vanishes, and we have (we can justify this rigorous by assuming that $|F(s)|$ is bounded when $\mathrm{Re} \geq \gamma$)

$$F(s) = \lim_{R\to\infty}\left(\frac{1}{2\pi i}\int_{C_1} \frac{F(z)}{z-s}dz + \frac{1}{2\pi i}\int_{C_2} \frac{F(z)}{z-s}dz\right) = \frac{1}{2\pi i}\int_{\gamma+i\infty}^{\gamma-i\infty} \frac{F(z)}{z-s}dz = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{F(z)}{s-z}dz.$$

Using the inversion formula for $\frac{1}{s-z}$, we get

$$\begin{aligned} f(t) = \mathcal{L}^{-1}[F(s)](t) &= \mathcal{L}^{-1}\left[\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{F(z)}{s-z}dz\right](t) \\ &= \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} F(z)\mathcal{L}^{-1}\left[\frac{1}{s-z}\right](t)dz \\ &= \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} F(z)e^{tz}dz. \end{aligned}$$

□

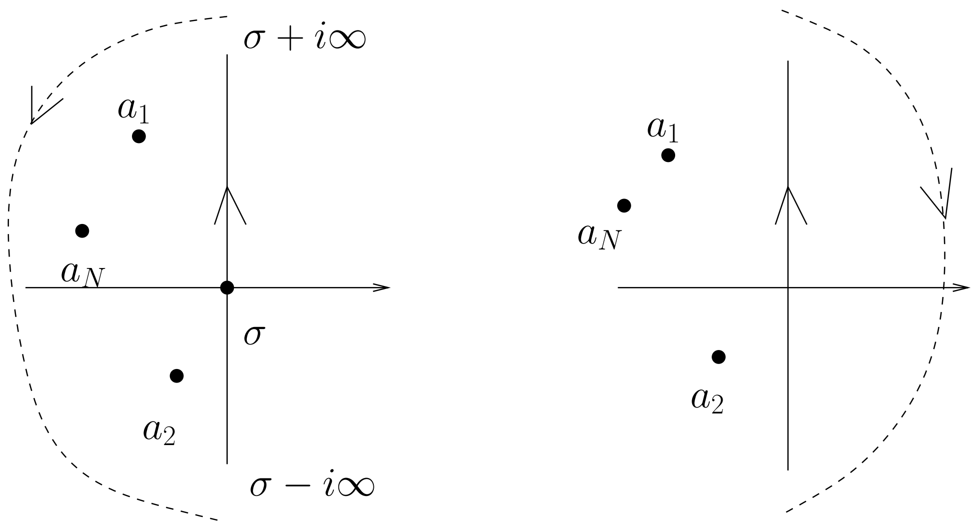

Figure 2: The complete contour depending on $t > 0$ or $t < 0$.

If $F(s)$ has only poles at $a_1, a_2, \cdots, a_N$, then we can use residue calculus to find the inverse Laplace transform:

$$f(t) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} F(s)e^{st}ds = \sum_{n=1}^{N} \mathrm{Res}\Big(F(s)e^{st}; s = a_n\Big)$$

Remark. Here the condition $t > 0$ is necessary. Otherwise, if $t < 0$, we can complete the contour, by drawing a semicircle to the right of the straight line $\sigma + is$ (right contour in Figure 2). The contribution of the integral on the semicircle vanishes (the fact $t < 0$ is decisive), and there is no singularity inside the closed contour. Therefore we have

$$\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} F(s)e^{st}ds = 0,$$

if $t < 0$.

Exercise. Verify the inverse transform of $n!/s^{n+1}, 1/(s+a), s/(s^2+\omega^2), \omega/(s^2+\omega^2), e^{-sT}/s$ in the table of transforms.

3 Efficient calculation of the inverse Laplace transform

Basic methods of inversion:

a) Using partial fractions and table of transforms

b) The inversion formula with residue theorem

Other (less common) methods of inversion and tricks:

a) Convolution theorem when $F(s) = G(s)H(s)$ is a product, then

$$\mathcal{L}^{-1}[F](t) = \int_0^t g(\tau)h(t-\tau)d\tau.$$

b) The derivative of a Laplace transform: If $F(s) = \frac{d^n}{ds^n}G(s)$ for some simple function $G(s)$, then

$$\mathcal{L}^{-1}[F](t) = \mathcal{L}^{-1}\left[\frac{d^n}{ds^n}G(s)\right](t) = (-1)^n t^n g(t).$$

c) $F(s) = G(s+a)$ is a shift of common functions $G(s)$, for example $F(s) = \frac{1}{(s+a)^2+b^2}$:

$$\mathcal{L}^{-1}[F(s)](t) = \mathcal{L}^{-1}[G(s+a)](t) = e^{-at}g(t).$$

d) $F(s) = e^{-as}G(s)$:

$$\mathcal{L}^{-1}[F(s)](t) = \mathcal{L}^{-1}[e^{-as}G(s)](t) = H(t-a)g(t-a).$$

Remark. In the last case (d), the inverse Laplace transform of $e^{-as}G(s)$ is just a shift of $g(t)$, then take the part on $t > 0$ (see Figure 3). Notice the differences between $a > 0$ and $a < 0$.

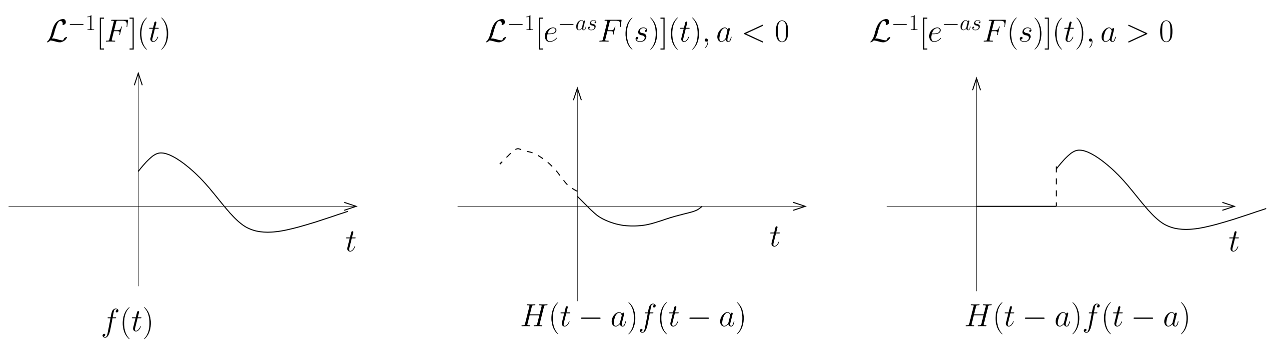

Figure 3: The inverse Laplace transform with the factor e^{-as}.

Example 3.1. Find the inverse Laplace transform of $F(s) = \frac{s+1}{s^2(s-1)}$ using partial fractions (and tables of integrals) and using the Residue calculus.
Solution: (a) Using Partial fraction. We have

$$F(s) = \frac{s+1}{s^2(s-1)} = \frac{2}{s-1} - \frac{1}{s^2} - \frac{2}{s}.$$

Then using the table, we have

$$\mathcal{L}^{-1}\left[\frac{2}{s-1}\right](t) = 2e^t, \quad \mathcal{L}^{-1}\left[\frac{1}{s^2}\right](t) = t, \quad \mathcal{L}^{-1}\left[\frac{2}{s}\right](t) = 2.$$

Therefore,

$$f(t) = \mathcal{L}^{-1}[F(s)](t) = 2e^t - t - 2.$$

(b) Using Residue Calculus. The only singularity, which are poles are $a_1 = 0$ and $a_1 = 1$. Therefore

$$\begin{aligned} f(t) &= \operatorname{Res}(F(s)e^{st}, 0) + \operatorname{Res}(F(s)e^{st}, 1) \\ &= \lim_{s\to 0} \frac{d}{ds}\left(s^2 F(s)e^{st}\right) + \lim_{s\to 1} (s-1)F(s)e^{st} \\ &= \lim_{s\to 0} \frac{d}{ds}\left(\frac{s+1}{s-1}e^{st}\right) + \lim_{s\to 1} \frac{s+1}{s^2}e^{st} \\ &= 2e^t - t - 2. \end{aligned}$$

Example 3.2. Show that the inverse Laplace transform of $F(s) = \frac{s}{(1+s^2)^2}$ is $\frac{t}{2}\sin t$ using all four methods above.
Solution:
(a) **Partial fraction**. Since

$$F(s) = \frac{i}{4}\left[\frac{1}{(s+i)^2} - \frac{1}{(s-i)^2}\right],$$

and the inverse Laplace transform of $(s \pm i)^{-2}$ is $te^{\mp it}$,

$$f(t) = \mathcal{L}^{-1}[F](t) = \frac{i}{4}\left(te^{-it} - te^{it}\right) = \frac{it}{4}(-2i)\sin t = \frac{t}{2}\sin t.$$

(b) **Residue Calculus**. Since $s = i$ and $s = -i$ are double poles for $F(s)$,

$$\begin{aligned} f(t) &= \operatorname{Res}\big(F(s)e^{st}, s = i\big) + \operatorname{Res}\big(F(s)e^{st}, s = -i\big) \\ &= \lim_{s\to i} \frac{d}{ds}\left[(s-i)^2 F(s)e^{st}\right] + \lim_{s\to -i} \frac{d}{ds}\left[(s+i)^2 F(s)e^{st}\right] \\ &= \lim_{s\to i} \frac{d}{ds}\left[\frac{s}{(s+i)^2}e^{st}\right] + \lim_{s\to -i} \frac{d}{ds}\left[\frac{s}{(s-i)^2}e^{st}\right] \\ &= \frac{t}{2}\sin t. \end{aligned}$$

(c) **The convolution theorem.** Since $F(s) = \frac{s}{1+s^2} \cdot \frac{1}{1+s^2}$,

$$\begin{aligned} f(t) &= \int_0^t \cos\tau \sin(t-\tau)d\tau \\ &= \int_0^t \frac{\sin t + \sin(t-2\tau)}{2}d\tau \\ &= \frac{1}{2}\int_0^t \sin t d\tau + \frac{1}{2}\int_0^t \sin(t-2\tau)d\tau \\ &= \frac{t}{2}\sin t. \end{aligned} \tag{5}$$

(d) **The derivation theorem.** Since

$$F(s) = \frac{s}{(1+s^2)^2} = -\frac{1}{2}\frac{d}{ds}\left(\frac{1}{1+s^2}\right) = -\frac{1}{2}\frac{d}{ds}G(s),$$

with $G(s) = 1/(1+s^2)$. Then

$$\mathcal{L}^{-1}[F](t) = -\frac{1}{2}(-1)t\mathcal{L}^{-1}[G](t) = \frac{t}{2}\sin t.$$

Example 3.3. Using the fact that the inverse Laplace transform of $F'(p)$ is $-tf(t)$, to find the inverse Laplace transform of the following functions.

$$\text{(i)}\ \frac{1}{2}\ln\frac{p+a}{p-a}, \quad a>0; \qquad\qquad \text{(ii)}\ \ln\left(\frac{p^2+b^2}{p^2+a^2}\right), \qquad a>0,\ b>0.$$

Solution: (i) For $F(p) = \frac{1}{2}\ln\frac{p+a}{p-a}$, we can get

$$F'(p) = \frac{1}{2}\left(\frac{1}{p+a} - \frac{1}{p-a}\right)$$

whose inverse Laplace transform is $(e^{-at} - e^{at})/2 = -\sinh at$. Therefore, the inverse Laplace transform of $F(p)$ is $f(t) = \frac{\sinh at}{t}$.

(ii) For $F(p) = \ln\left(\frac{p^2+b^2}{p^2+a^2}\right)$,

$$F'(p) = \frac{2p}{p^2+b^2} - \frac{2p}{p^2+a^2}$$

whose inverse Laplace transform is $2\cos bt - 2\cos at$. Therefore, the inverse Laplace transform of $F(p)$ is

$$f(t) = \frac{2\cos bt - 2\cos at}{-t} = \frac{2(\cos at - \cos bt)}{t}.$$

4 Applications of Laplace transforms

The Laplace transforms can be used to solve ordinary differential equations, partial differential equations, integral differential equation, difference and delay differential equations.

The basic idea is to transform the original equation using Laplace transform, and the resulting equations is usually algebraic (easier to solve). The equation is transformed back using inverse Laplace transform (using table of inverse Laplace transform or Residue calculus or other means).

Example 4.1 (Ordinary differential equations)**.** Find the solution of the following ordinary differential equation using Laplace transform:

$$\frac{dy}{dt} + 2y = 12, \quad y(0) = 10.$$

Solution: Taking the Laplace transform of both sides of the equation,

$$sY(s) - y(0) + 2Y(s) = \frac{12}{s},$$

or

$$Y(s) = \frac{10s+12}{s(s+2)} = \frac{6}{s} + \frac{4}{s+2}.$$

Therefore,

$$y(t) = \mathcal{L}^{-1}[Y](t) = 6 + 4e^{-2t}.$$

Example 4.2 (Ordinary differential equation with force)**.** Find the solution of the following equation

$$f''(t) + f(t) = \begin{cases} \cos t, & 0 \le t \le \pi, \\ 0, & t > \pi, \end{cases}$$

with the initial value $f(0) = f'(0) = 0$.
Solution: Taking the Laplace transform of both sides of the equation,

$$s^2F(s) + F(s) = \int_0^\pi e^{-st}\cos t dt = \mathrm{Re}\int_0^\pi e^{(i-s)t}dt = \left.\frac{e^{(i-s)t}}{i-s}\right|_{t=0}^{\pi} = \frac{s(1+e^{-\pi s})}{1+s^2}.$$

(You can also use integrating by parts to find the integral). Therefore,

$$F(s) = \frac{s(1+e^{-\pi s})}{(1+s^2)^2} = \frac{s(1+e^{-\pi s})}{(s+i)^2(s-i)^2}.$$

The inverse Laplace transform, can be found either by residue calculus or derivation formula is

$$\begin{aligned} f(t) &= \mathrm{Res}\big(F(s)e^{st}, s=i\big) + \mathrm{Res}\big(F(s)e^{st}, s=-i\big) \\ &= \lim_{s\to i}\frac{d}{ds}\big[(s-i)^2F(s)e^{st}\big] + \lim_{s\to -i}\frac{d}{ds}\big[(s+i)^2F(s)e^{st}\big] \\ &= \lim_{s\to i}\frac{d}{ds}\frac{s(1+e^{-\pi s})e^{st}}{(s+i)^2} + \lim_{s\to -i}\frac{d}{ds}\frac{s(1+e^{-\pi s})e^{st}}{(s-i)^2} \\ &= \begin{cases} \frac{1}{2}t\sin t, & 0 \le t \le \pi, \\ \frac{1}{2}\pi\sin t, & t > \pi. \end{cases} \end{aligned}$$

Alternatively, we can write

$$\frac{s}{(1+s^2)^2} = \frac{1}{1+s^2}\cdot\frac{s}{1+s^2}$$

and use the convolution theorem to get

$$\mathcal{L}^{-1}\left[\frac{s}{(1+s^2)^2}\right](t) = \int_0^t \cos\tau\sin(t-\tau)d\tau = \frac{t}{2}\sin t.$$

Using the shift theorem,

$$f(t) = \mathcal{L}^{-1}[F](t) = \frac{t}{2}\sin t + H(t-\pi)\frac{t-\pi}{2}\sin(t-\pi),$$

which is exactly the same result as above.

Example 4.3 (Variable coefficient ODEs)**.** Find the solution of the ODE

$$y'' + ty' - 2y = 4, \qquad y(0) = -1, y'(0) = 0.$$

Solution: Taking the Laplace transform of both sides,

$$s^2 Y(s) - sy(0) - y'(0) - (sY'(s) + Y(s)) - 2Y(s) = \frac{4}{s},$$

or

$$Y'(s) + \left(\frac{3}{s} - s\right) Y(s) = 1 - \frac{4}{s^2}. \tag{6}$$

Here the Laplace transform of $ty'(t)$ is $-sY'(s) - Y(s)$. First if we take the derivative of both sides of the definition $Y(s) = \int_0^\infty y(t)e^{-st}dt$, we get

$$Y'(s) = -\int_0^\infty ty(t)e^{-st}dt.$$

Therefore,

$$\int_0^\infty ty'(t)e^{-st}dt = \int_0^\infty te^{-st}dy(t) = te^{-st}y(t)\big|_0^\infty - \int_0^\infty y(t)d(te^{-st}) = -\int_0^\infty y(t)(1 - ts)e^{-st}dt.$$

The last term can be separated as

$$-\int_0^\infty y(t)e^{-st}dt + s\int_0^\infty y(t)te^{-st}dt = -Y(s) - sY'(s).$$

To find the solution of the governing equation (6) for $Y(s)$, we need to use the observation that (6) is **linear**, which can be solved using an *integrating factor*. First first order ODEs like (6), the integrating factor $I(s) = 1/Y_0(x)$, the *inverse* of the solution to the homogeneous equation $Y_0'(s) + \left(\frac{3}{s} - s\right) Y_0(s) = 0$. Since $Y_0(s) = \frac{1}{s^3}e^{s^2/2}$ can be solved from separation of variable (by dividing both sides by $Y_0(s)$ and then integrating w.r.t s), the integrating factor $I(s) = s^3 e^{-s^2/2}$ and

$$\frac{d}{ds}\left[Y(s)s^3e^{-s^2/2}\right] = \left[Y'(s) + \left(\frac{3}{s} - s\right) Y(s)\right] s^3e^{-s^2/2} = \left(1 - \frac{4}{s^2}\right) s^3e^{-s^2/2} = (s^3 - 4s)e^{-s^2/2}.$$

The general solution is obtained by integrating both sides, and is given by

$$Y(s) = \frac{2}{s^3} - \frac{1}{s} + \frac{c}{s^3}e^{s^2/2},$$

with some unknown constant c. Since when s goes to infinity, $Y(s)$ is a well-behaved function, we must have $c = 0$. Therefore, the solution, given by the inverse Laplace transform is

$$y(t) = t^2 - 1.$$

Remark. Laplace transform does not go well with variable coefficients. For example, it is impossible to write the Laplace transform of $cos(t)y'$ in a simple form. Even for terms like ty' or t^2y', these algebraic terms is related to (undesired) derivatives in the transformed equation. Remember that the power of Laplace transform is to reduce differentiation into simple powers, not vice verse.

Remark. Be careful how we eliminate the coefficient c in the above example using the behaviour of the function.

Example 4.4 (Integral equation with convolution)**.** Find the solution $m(t)$ to the integral equation

$$m(t) = 1 - e^{-\lambda t} + \lambda \int_0^t m(t-\tau)e^{-\lambda\tau}d\tau, \tag{7}$$

where λ is a given constant.
Solution: Taking the Laplace transform of both sides of the equation, we get

$$M(s) = \frac{1}{s} - \frac{1}{\lambda + s} + \frac{\lambda M(s)}{\lambda + s},$$

where $M(s) = \mathcal{L}[m](s)$ is the Laplace transform of $m(t)$. The above algebraic equation for $M(s)$ can be obtained in explicit form, i.e., $M(s) = \lambda/s^2$. Taking the inverse Laplace transform,

$$m(t) = \mathcal{L}^{-1}\left[\frac{\lambda}{s^2}\right](t) = \lambda t.$$

We can verify that $m(t) = \lambda t$ is a solution to the integral equation.

Exercise. Show that the solution of the integral equation

$$g(x) = 1 - \int_0^x (x-y)g(y)dy$$

is $g(x) = \cos x$ using Laplace transform.

Example 4.5 (Wave equation on half space)**.** Find the solution of the wave equation $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$ on $x > 0, t > 0$ with the initial and boundary condition

$$u(x,0) = 0, \quad u_t(x,0) = 0, \quad u(0,t) = f(t)$$

and $\lim_{x\to\infty} u(x,t) = 0$.
Solution: Define the Laplace of $u(x,t)$ w.r.t t,

$$U(x,s) = \int_0^\infty e^{-st}u(x,t)dt.$$

The the original wave equation is transformed into $s^2U(x,s) = \frac{\partial^2}{\partial x^2}U(x,s)$. The general solution is

$$U(x,s) = c_1(s)e^{sx} + c_2(s)e^{-sx}.$$

The initial and boundary conditions imply that $U(x,s) \to 0$ as $c_1(s) \equiv 0$ and $c_2(s) = F(s)$, where $F(s)$ is the Laplace transform of $f(t)$. Therefore, $U(x,s) = F(s)e^{-xs}$ and the inverse Laplace transform is

$$u(x,t) = \begin{cases} f(t-x), & 0 < x < t, \\ 0, & x > t. \end{cases}$$

Example 4.6 (First order partial differential equation)**.** Find the solution of

$$x\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} + u = x^2$$

on $x > 0, t > 0$ with the initial and boundary condition $u(0,t) = 0, u(x,0) = 0$.
Solution: Define the Laplace of $u(x,t)$ w.r.t t,

$$U(x,s) = \int_0^\infty e^{-st}u(x,t)dt.$$

Taking the Laplace transform of the partial differential equation, we have

$$x\frac{\partial U(x,s)}{\partial x} + sU(x,s) - u(x,0) + U(x,s) = \frac{x^2}{s},$$

or

$$\frac{\partial U}{\partial x} + \frac{s+1}{x}U = \frac{x}{s}.$$

Since the solution $U_0(x,s)$ to the homogeneous equation $\frac{\partial U_0}{\partial x} + \frac{s+1}{x}U_0 = 0$ is $U_0(x,s) = x^{-s-1}$, the integrating factor is $I_0(x,s) = \frac{1}{U_0(x,s)} = x^{s+1}$. Therefore,

$$\frac{d}{dx}\left(x^{s+1}U(x,s)\right) = x^{s+1}\left[\frac{\partial U}{\partial x} + \frac{s+1}{x}U\right] = \frac{x^{s+2}}{s}.$$

Therefore,

$$x^{s+1}U(x,s) = \tilde{c} + \int^x \frac{\tau^{s+2}}{s}d\tau = c + \frac{x^{s+3}}{s(s+3)}.$$

Therefore, the general solution is

$$U(x,s) = \frac{x^2}{s(s+3)} + cx^{-s-1}.$$

From the initial condition $u(0,t) = 0$, we must have $c = 0$, therefore $U(x,s) = \frac{x^2}{s(s+3)}$ or

$$u(x,t) = \frac{x^2}{3}(1 - e^{-3t}).$$