Aero III/IV Conformal Mapping

1 View complex function as a mapping

Unlike a real function, a complex function $w = f(z)$ cannot be represented by a curve. Instead it is useful to view it as a mapping. Write $w = f(z)$ as $u + iv = f(x + iy)$, which maps any point (x, y) in the z-plane to a corresponding point (u, v) in the w-plane. In this case, we have $u(x, y), v(x, y)$ where u, v are functions of x, y.

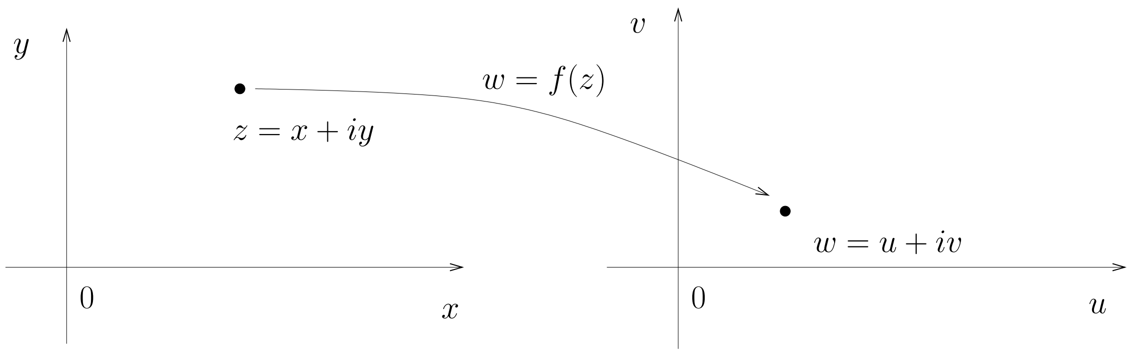


Figure 1: Complex function $w = f(z)$ as a mapping.

In general, we prefer the mapping is a *one-to-one* mapping: each point (u, v) in the w-plane, there corresponds one and only one point (x, y) in the z-plane. One useful criteria to make sure the mapping $(x, y) \to (u, v)$ one-to-one is that the Jacobian does not vanish, i.e.,

$$\det \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} \neq 0.$$

2 The basic concepts conformal mapping

A conformal mapping is a mapping that preserves angle. More precisely, if $w = f(z)$ is a conformal mapping, $\gamma_1(t)$ and $\gamma_2(t)$ are two curves on the z-plane intersect at t_0, then the angle (measured in terms of the tangent direction) between $f(\gamma_1(t))$ and $f(\gamma_2(t))$ is the same as that between $\gamma_1(t)$ and $\gamma_2(t)$.

Example 2.1 (A conformal mapping from the first quadrant to the upper-half plane)**.** Let $w = f(z) = z^2$, then f is a conformal mapping. We show this fact by by two curves

$$\gamma_1(t) = 1 + t + i, \quad \gamma_2(t) = 1 + (1 + t)i,$$

which are two straight lines on the z-plane, and intersect with each other with right angle.

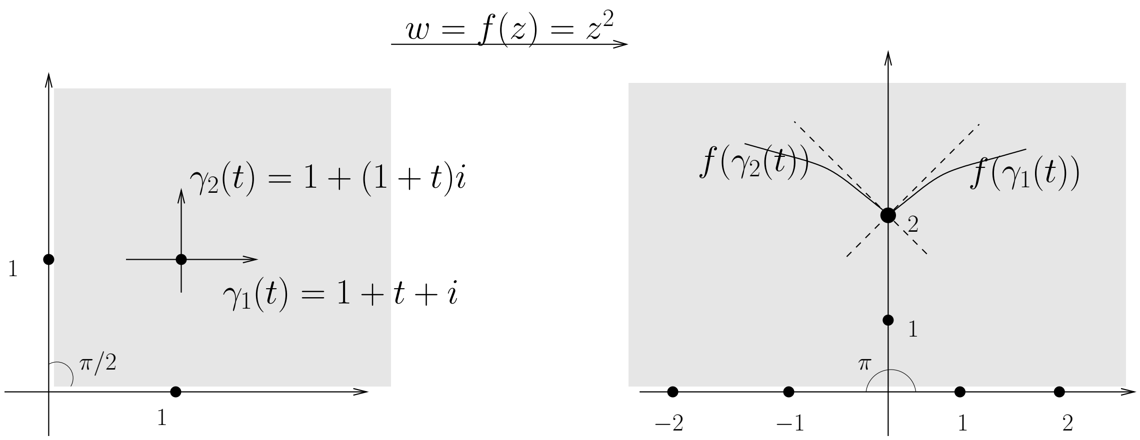


Figure 2: Two curves under the mapping $w = f(z) = z^2$. It is conformal except at the origin, where the angle is from $\pi/2$ to π.

These two curves are transformed into

$$f(\gamma_1(t)) = t^2 + 2t + 2(1+t)i, \quad f(\gamma_2(t)) = -t^2 - 2t + 2(1+t)i,$$

and intersect at $2i$. The tangents of both curves are

$$\frac{d}{dt} f(\gamma_1(t))\Big|_{t=0} = 2 + 2i, \quad \frac{d}{dt} f(\gamma_2(t))\Big|_{t=0} = -2 + 2i.$$

The angle between the two tangents are the argument of their ratio, i.e. $\pi/2$, the same as the angle between those of the original two curves.

In general, if f is analytic and γ_1, γ_2 are two curves on z-plane intersect at $z_0 = \gamma_1(t_0) = \gamma_2(t_0)$, then $f(\gamma_1(t))$ and $f(\gamma_2(t))$ intersect at $w_0 = f(z_0) = f(\gamma_1(t_0)) = f(\gamma_2(t_0))$.

We can calculate the tangents on the w-plane by chain rule,

$$\frac{d}{dt} f(\gamma_1(t))\Big|_{t=t_0} = f'(z_0)\gamma_1'(t_0), \quad \frac{d}{dt} f(\gamma_2(t))\Big|_{t=t_0} = f'(z_0)\gamma_2'(t_0).$$

Therefore, when $f'(z_0) \neq 0$,

$$\text{Arg}\frac{f'(z_0)\gamma_1'(t_0)}{f'(z_0)\gamma_2'(t_0)} = \text{Arg}\frac{\gamma_1'(t_0)}{\gamma_2'(t_0)},$$

and the angle between the curves are preserved. In fact, we have

Theorem 2.1. *If $w = f(z)$ is analytic and $f'(z_0) \neq 0$, then f is a conformal mapping.*

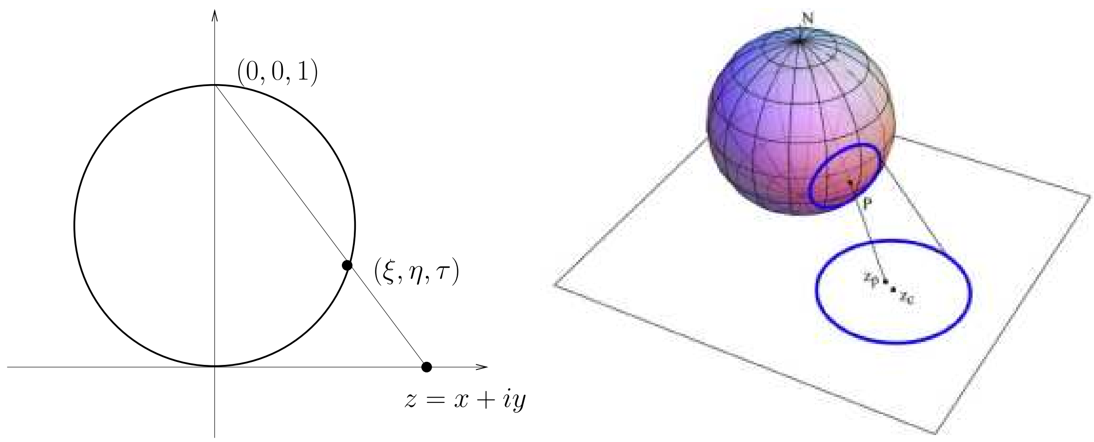


Figure 3: The stereographic projection on the complex plane to the sphere $\xi^2 + \eta^2 + (\tau - 1/2)^2 = 1$.

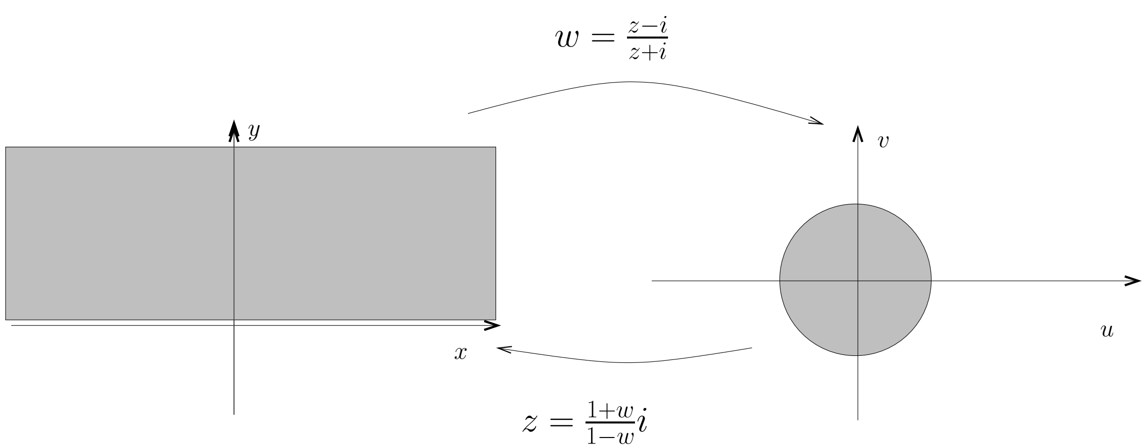


Figure 4: The transformation between the upper half plane and the unit disk by $w = (z - i)/(z + i)$ and $z = (1 + w)i/(1 - w)$.

The condition $f'(z_0) \neq 0$ is needed. For example, when $f(z) = z^2$, this condition $f'(z_0) \neq 0$ is not satisfied at the origin. In fact, the angle is doubled at this point . Two common conformal transformation are given in Figure 3 and Figure 4.

If f is a conformal mapping near z_0, then

$$|f'(z_0)| = \left| \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \right| = \lim_{z \to z_0} \frac{|f(z) - f(z_0)|}{|z - z_0|},$$

or $|f'(z_0)|$ is the *scale factor* (of the length). The scale factor of the area of the transformation $(x, y) \to (u, v)$ is given by the Jacobian matrix

$$\frac{\partial(u, v)}{\partial(x, y)} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial v}{\partial x}\frac{\partial u}{\partial y} = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = |f'(z_0)|^2.$$

3 Invariance of the Laplace equation (or harmonic functions) under conformal transformation

First, we can show that the transformation by a conformal mapping actually transform harmonic functions (whose Laplacian is zero) to harmonic functions.

Definition 3.1. A function $h(x, y)$ is said to be **harmonic** if its Laplacian is zero, i.e., $\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = 0$.

Theorem 3.1. *If the harmonic function $h(x, y)$ is transformed to $H(u, v)$ by the conformal mapping $w = f(z)$ where $z = x + iy$ and $w = u + iv$ and $f'(z) \neq 0$, then $H(u, v)$ is harmonic too.*

Proof. Using the change rule,

$$\frac{\partial h}{\partial x} = \frac{\partial}{\partial x} H(u(x, y), v(x, y)) = \frac{\partial H}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial H}{\partial v}\frac{\partial v}{\partial x},$$

$$\begin{aligned} \frac{\partial^2 h}{\partial x^2} = &\left(\frac{\partial^2 H}{\partial u^2}\frac{\partial u}{\partial x} + \frac{\partial^2 H}{\partial u \partial v}\frac{\partial v}{\partial x}\right)\frac{\partial u}{\partial x} + \frac{\partial H}{\partial u}\frac{\partial^2 u}{\partial x^2} \\ &+ \left(\frac{\partial^2 H}{\partial u \partial v}\frac{\partial u}{\partial x} + \frac{\partial^2 H}{\partial v^2}\frac{\partial v}{\partial x}\right)\frac{\partial v}{\partial x} + \frac{\partial H}{\partial v}\frac{\partial^2 v}{\partial x^2}. \end{aligned}$$

Similarly,

$$\begin{aligned} \frac{\partial^2 h}{\partial y^2} = &\left(\frac{\partial^2 H}{\partial u^2}\frac{\partial u}{\partial y} + \frac{\partial^2 H}{\partial u \partial v}\frac{\partial v}{\partial y}\right)\frac{\partial u}{\partial y} + \frac{\partial H}{\partial u}\frac{\partial^2 u}{\partial y^2} \\ &+ \left(\frac{\partial^2 H}{\partial u \partial v}\frac{\partial u}{\partial y} + \frac{\partial^2 H}{\partial v^2}\frac{\partial v}{\partial y}\right)\frac{\partial v}{\partial y} + \frac{\partial H}{\partial v}\frac{\partial^2 v}{\partial y^2}. \end{aligned}$$

Therefore,

$$0 = \frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = \frac{\partial^2 H}{\partial u^2}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right] + \frac{\partial^2 H}{\partial v^2}\left[\left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right]$$
$$+ 2\frac{\partial^2 H}{\partial u \partial v}\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right) + \frac{\partial H}{\partial u}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + \frac{\partial H}{\partial v}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right).$$

Since u and v are the real and imaginary parts of the analytic function f, we have the Cauchy-Riemann condition $u_x = v_y, u_y = -v_x$ and

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 = |f'(z)|^2, \; \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \; \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0,$$

Therefore $0 = \frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = (\frac{\partial^2 H}{\partial u^2} + \frac{\partial^2 H}{\partial v^2})|f'(z)|^2$. Since $f'(z) \neq 0$, H is harmonic too. □

4 Solution of the Laplace equation on some simple domains

Solution of the Laplace equation on the unit disk: To use the conformal mapping to find the solutions of Laplace equation, we need the solution on a simple geometry, which is either the unit disk or upper half plane. The solution of the Laplace equation on the unit disk can be obtained in different ways. One way is to use the expansion,

$$h(re^{i\phi}) = \sum_{n=-\infty}^{\infty} c_n r^{|n|} e^{in\phi},$$

where the coefficient c_n is obtained by the boundary condition on the unit circle $r = 1$, i.e.,

$$U(e^{i\theta}) = \sum_{n=-\infty}^{\infty} c_n e^{in\theta},$$

or

$$c_n = \frac{1}{2\pi}\int_0^{2\pi} U(e^{i\theta}) e^{-in\theta} d\theta.$$

then

$$h(re^{i\phi}) = \sum_{n=-\infty}^{\infty} c_n r^{|n|} e^{in\phi} = \frac{1}{2\pi}\int_0^{2\pi}\left(\sum_{n=-\infty}^{\infty} r^{|n|} e^{in(\phi-\theta)}\right) U(e^{i\theta}) d\theta$$

The infinite series inside the brackets can be evaluated explicitly, when decomposed into two geometric series,

$$\frac{1}{2\pi}\sum_{n=-\infty}^{\infty} r^{|n|} e^{in(\phi-\theta)} = \frac{1}{2\pi}\left(1 + \sum_{n=1}^{\infty}\left(r^n e^{in(\phi-\theta)} + r^n e^{-in(\phi-\theta)}\right)\right) = \frac{1}{2\pi}\frac{1-r^2}{1-2r\cos(\theta-\phi)+r^2}.$$

This gives the celebrated **Poisson integral formula**

$$h(re^{i\phi}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-r^2}{1-2r\cos(\theta-\phi)+r^2} U(e^{i\theta})d\theta. \tag{1}$$

Alternatively, we can derive the same formula using Cauchy integral formula, by assuming the solution is analytic on the unit disk. For any z inside the unit disk, we have

$$h(z) = \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{h(\xi)}{\xi - z} d\xi, \qquad 0 = \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{h(\xi)}{\xi - 1/z} d\xi,$$

where $\mathcal{C}$ is the unit disk and the second expression is zero because $1/z$ is outside the unit disk ($h(\xi)/(\xi - 1/z)$ is analytic on the unit disk). Therefore, we put these two expression, and using the parametrization $\xi = e^{i\theta}$ on the unit circle,

$$h(z) = \frac{1}{2\pi i}\int_{\mathcal{C}} h(\xi)\left(\frac{1}{\xi - z} + \frac{1}{\xi - 1/z}\right) d\xi = \frac{1}{2\pi}\int_0^{2\pi} U(e^{i\theta})\left(\frac{1}{e^{i\theta} - z} + \frac{1}{e^{i\theta} - 1/z}\right) e^{i\theta} d\theta.$$

Let $z = re^{i\phi}$ then we get the same Poisson integral formula (1).

Solve the Laplace equation on a wedge: The domain is defined to be the wedge $\{z : \theta_1 < \arg z < \theta_2\}$, on which $\Delta\phi = 0$. The boundary condition is $\phi = 0$ on $\arg z = \theta_0$ and $\phi = 1$ on $\arg z = \theta_1$.

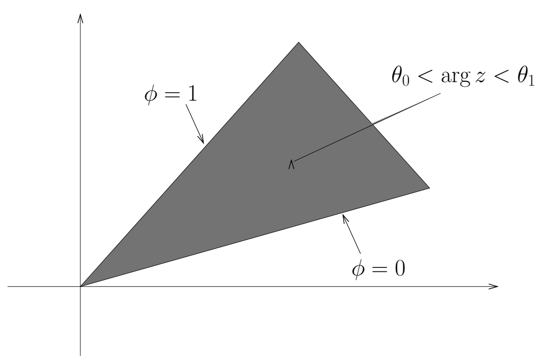


Figure 5: The Laplace equation on the wedge $\{z : \theta_1 < \arg z < \theta_2\}$, with $\phi = 0$ on $\arg z = \theta_0$ and $\phi = 1$ on $\arg z = \theta_1$.

By the symmetry of the domain and the boundary condition, ϕ is a constant on the line $\arg z = \theta$ for $\theta_1 \leq \theta \leq \theta_2$, or the solution is given by $\phi(x, y) = h(\arg z) = h(\arctan \frac{y}{x})$. Therefore,

$$\nabla\phi(x, y) = h'\big(\arctan \frac{y}{x}\big) \begin{pmatrix} -y/(x^2+y^2) \\ x/(x^2+y^2) \end{pmatrix}$$

and

$$\begin{aligned}
\Delta\phi(x,y) &= -\frac{\partial}{\partial x}\left[h'\big(\arctan\frac{y}{x}\big)\frac{y}{x^2+y^2}\right] + \frac{\partial}{\partial y}\left[h'\big(\arctan\frac{y}{x}\big)\frac{x}{x^2+y^2}\right] \\
&= h''\big(\arctan\frac{y}{x}\big)\left[\frac{y^2}{(x^2+y^2)^2} + \frac{x^2}{(x^2+y^2)^2}\right] + h'\big(\arctan\frac{y}{x}\big)\left[\frac{2xy}{(x^2+y^2)^2} - \frac{2xy}{(x^2+y^2)^2}\right] \\
&= h''\big(\arctan\frac{y}{x}\big)\frac{1}{x^2+y^2}.
\end{aligned}$$

This implies that $h'' \equiv 0$ or h is a linear function. Using the boundary condition $h(\theta_0) = 0$ and $h(\theta_1) = 1$, we get

$$h(\theta) = \frac{\theta - \theta_0}{\theta_1 - \theta_0}$$

and the solution of the Laplace equation is

$$\phi(x,y) = h(\arctan\frac{y}{x}) = \frac{\arctan\frac{y}{x} - \theta_0}{\theta_1 - \theta_0}.$$

5 Solving Laplace equation on complex domains by conformal mapping

One of the applications of conformal mapping is to find the solutions $H(u,v)$ of the Laplace equations on complex domains from the solutions $h(x,y)$ on simple domains, by identifying $H(u,v) = h(x,y)$ and the points (u,v) is related to (x,y) by the conformal mapping f by $u + iv = f(x+iy)$ or $w = f(z)$.

Example 5.1 (Solution of the Laplace equation on the upper half plane)**.** If $h(x,y)$ satisfies the Laplace equation on the upper half plane and $h(x,0) = H(x)$, find $h(x,y)$.

The general conformal mapping to transform the upper half plane to the unit circle is $w = (z - z_0)/(z - \overline{z_0})$. It is clear that the point z_0 is transformed into the origin, and the real axis is transformed into the unit disk. To show the latter, when $z = \tau$ is real,

$$|w| = \left|\frac{\tau - z_0}{\tau - \overline{z_0}}\right| = \frac{|\tau - z_0|}{|\tau - \overline{z_0}|} = \frac{|\tau - z_0|}{|\overline{\tau - z_0}|} = 1,$$

or w on the unit circle.

If we want to find the solution at $x+iy$, we can choose the arbitrary point $z_0 = x+iy$, such that the solution on the transformed w-plane is evaluated at the origin has the simple form $\frac{1}{2\pi}\int_0^{2\pi} U(e^{i\theta})d\theta$, the average on the unit circle. We still have to find transform this integral in terms of the boundary condition $H(\tau) = U(e^{i\theta})$, by the transformation $e^{i\theta} = (\tau - z_0)/(\tau - \overline{z_0})$. Then

$$e^{i\theta} i d\theta = \frac{z_0 - \overline{z_0}}{(\tau - \overline{z_0})^2}d\tau, \qquad \text{or} \qquad d\theta = \frac{1}{i}\frac{z_0 - \overline{z_0}}{(\tau - z_0)(\tau - \overline{z_0})}d\tau.$$

Finally,

$$h(x,y) = \frac{1}{2\pi}\int_0^{2\pi} U(e^{i\theta})d\theta = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{(z_0 - \overline{z_0})H(\tau)}{(\tau - z_0)(\tau - \overline{z_0})}d\tau = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{yH(\tau)}{(x-\tau)^2 + y^2}d\tau.$$

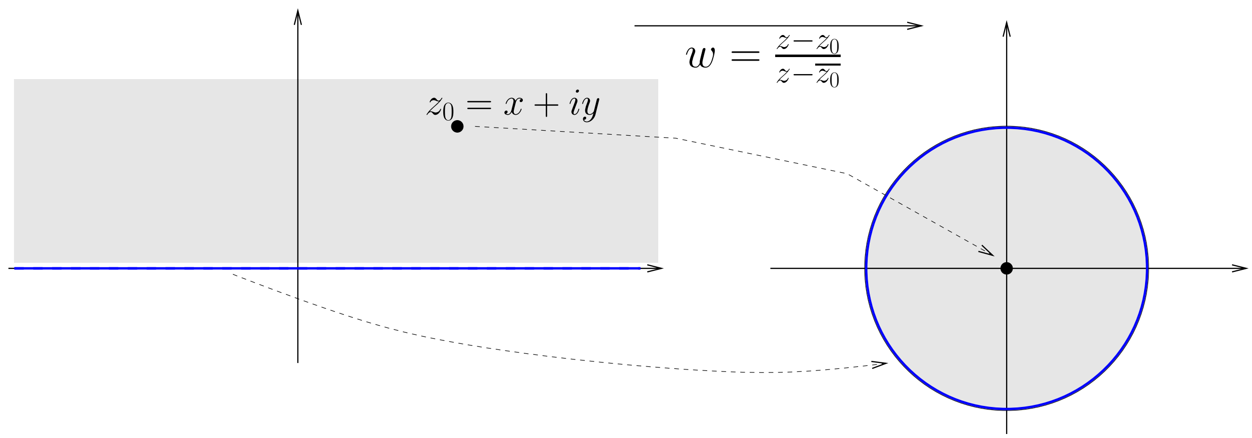
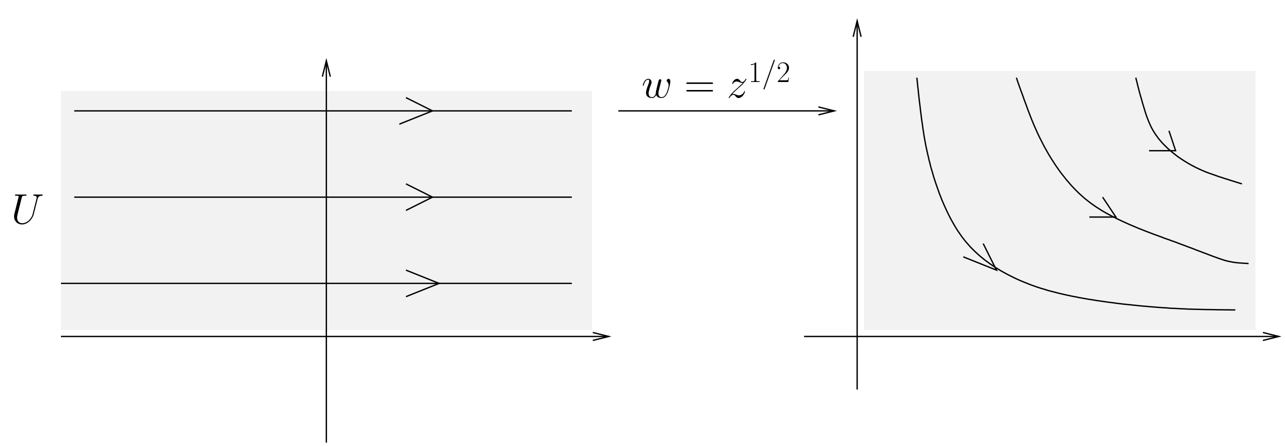


Figure 6: The upper half plane is mapped to the unit disk by $w = (z - z_0)/(z - \overline{z_0})$. Here the real axis is mapped to the unit circle and the point z_0 is mapped to the origin in the w-plane.

Example 5.2 (Transformation of the flow pattern)**.** For a uniform idea flow on the upper half plane whose velocity is given by $(u_1, u_2) = (U, 0)$. The upper half plane can be mapped to the first quadrant by $w = z^{1/2}$. To see the resulting flow on the first quadrant on the w-plane using conformal mapping, we must introduce the velocity complex function $\Omega(z) = \phi(x, y) + i\psi(x, y)$, where ϕ is the velocity potential and ψ is the stream function, i.e.,

$$u_1 = \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}, \quad u_2 = \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}.$$



For the uniform idea flow, $\Omega(z) = Uz$ and is analytic. To find the complex velocity potential $\tilde{\Omega}(w)$ on the first quadrant on the w-plane, we have

$$\tilde{\Omega}(w) = \Omega(z) = Uz = Uw^2 = U(u^2 - v^2 + 2uvi),$$

with $\tilde{\phi}(u,v) = u^2 - v^2$ and $\tilde{\psi}(u,v) = 2uv$. Therefore, the velocity on the first quadrant is

$$\tilde{u_1} = \frac{\partial \tilde{\phi}}{\partial u} = 2Uu, \qquad \tilde{u_2} = \frac{\partial \tilde{\phi}}{\partial v} = -2Uv.$$

Remark. The key point to apply the transformation is that, if (x,y) is transformed to the point (u,v), then

$$\boxed{h(x,y) = H(u,v).}$$

6 Bilinear Transformation

A **bilinear transformation** (or **Mobius transformation**) $w = f(z)$ is a mapping with

$$f(z) = \frac{az+b}{cz+d}, \qquad ad - bc \neq 0.$$

The condition $ad - bc \neq 0$ is needed here, otherwise, $f(z)$ is a constant (independent of z).

We have the following three basic Mobius Transformations:

i) **Translation**: $f(z) = z + b$

ii) **Dilation** (or **multiplication**): $f(z) = az$

iii) **Inversion**: $f(z) = 1/z$

Any general Mobius transformations can be written as a composition of the above three building blocks.

Exercise. Find the bilinear transformation that carries the points $-1, \infty, i$ on the z-plane to the following points on the w-plane:

$$(a) i,\ 1,\ 1+i; \qquad\qquad (b) \infty,\ i,\ 1.$$

The transformation between different regions are characterized by the fact that their boundaries are transformed from one to another. Therefore, we are going to focus on the transformations of the boundaries (curves on the complex plane), the corresponding domains reside on either side of the domain. In general, there are two ways to find the resulting curves, either by writing the transformation in terms of the real and imaginary parts (pure real variables) in the whole process or manipulating using the complex numbers and then transforming back to u, v, as in the following examples.

Example 6.1. Find the curve $x^2 - y^2 = 1$ under the inversion.
(*pure real variables*). We have to find the change of variable x, y in terms of u, v, using the definition of inversion.

$$x + iy = z = 1/w = \frac{1}{u+iv} = \frac{u - iv}{u^2 + v^2}$$

or $x = u/(u^2+v^2), y = -v/(u^2+v^2)$. Substituting it into the equation for the curve in z-plane,

$$1 = x^2 - y^2 = \frac{u^2 - v^2}{(u^2+v^2)^2}.$$

(*complex variables method*). The equation $x^2 - y^2 = 1$ is equivalent to

$$1 = (\text{Re}z)^2 - (\text{Im}z)^2 = \left(\frac{z+\bar{z}}{2}\right)^2 - \left(\frac{z-\bar{z}}{2i}\right)^2 = \frac{z^2+\bar{z}^2}{2}.$$

The mapping $w = 1/z$ can be written as $z = 1/w, \bar{z} = 1/\bar{w}$. Substituting them into the equation, we have

$$1 = \frac{1}{2}\left(\frac{1}{w^2} + \frac{1}{\bar{w}^2}\right).$$

Finally, put $w = u + iv, \bar{w} = u - iv$ into this equation, we get the same equation for the curve on the w-plane.

Exercise. Consider the following bilinear transformation

$$w = f(z) = \frac{2iz - 2}{2z - i}.$$

(a) Determine the invariant points of the transformation (those points such that $z = f(z)$).

(b) Find the point ξ for which the equation $f(z) = \xi$ has no solution for z in the finite complex plane.

(c) Show that the imaginary axis is mapped onto itself.

(d) Determine the image of the disc $|z| < 1$.

Example 6.2 (Laplace equation between two circles)**.** Let D be the regime between the two circles $|z| = 1$ and $|z - 2/5| = 2/5$. If f satisfies the Laplace equation on D, $f = 0$ on $|z| = 1$ and $f = 1$ on $|z - 2/5| = 2/5$, find f using the Mobius transformations $w = (z - 2)/(2z - 1)$.

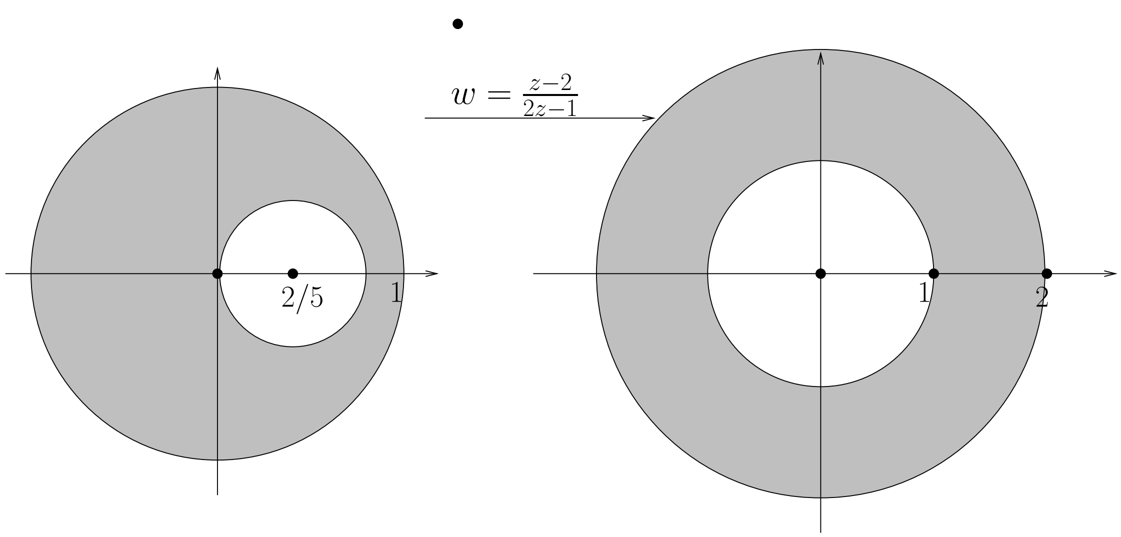


Figure 7: Laplace equation between two circles.

The transformation $w = (z-2)/(2z-1)$ can be written as $z = (w-2)/(2w-1)$. Therefore the circle $|z| = 1$ is transformed to $|(w-2)/(2w-1)| = 1$. Write the equation in terms of $w = u + iv$, $(u + iv - 2)/(2u + 2iv - 1)| = 1$ or $u^2 + v^2 = 1$. Therefore, the unit disk $|z| = 1$ is mapped to the unit disk $|w| = 1$.

Similarly, the circle $|z - 2/5| = 2/5$ is mapped to

$$\frac{2}{5} = \left|\frac{w-2}{2w-1} - \frac{2}{5}\right| = \left|\frac{w-8}{5(2w-1)}\right| = \left|\frac{u-8+iv}{5(2u-1+2iv)}\right|,$$

or $u^2 + v^2 = 4$. Therefore, the circle $|z - 2/5| = 2/5$ is mapped to $|w| = 2$. On the w-plane, let the solution of the Laplace equation with value zero on $|w| = 1$ and one on $|w| = 2$ be $F(u, v)$, then F depends only on $R = \sqrt{u^2 + v^2}$ and in this radial coordinate

$$0 = \frac{\partial^2 F}{\partial u^2} + \frac{\partial^2 F}{\partial v^2} = \frac{1}{R}\frac{d}{dR}R\frac{dF}{dR}.$$

The general solution is $F = c_1 + c_2 \ln R$. Using the boundary condition $F = 0$ when $R = 1$ and $F = 1$ when $R = 2$, we get $F = \frac{1}{\ln 2} \ln R = \frac{1}{2\ln 2} \ln(u^2 + v^2)$.

To find the solution on the z-plane, we have to transform (u, v) using (x, y). From the definition of the mapping,

$$u + iv = w = \frac{z-2}{2z-1} = \frac{x-2+iy}{2x-1+2iy} = \frac{(x-2+iy)(2x-1-2iy)}{(2x-1)^2 + 4y^2}.$$

Taking the real and imaginary parts of both sides,

$$u = \frac{2x^2 + 2y^2 - 5x + 2}{(2x-1)^2 + 4y^2}, \quad v = \frac{3y}{(2x-1)^2 + 4y^2}.$$

Finally, the solution of the Laplace equation is

$$f(x, y) = \frac{1}{2\ln 2}\ln(u^2 + v^2) = \frac{1}{2\ln 2}\ln\frac{(2x^2 + 2y^2 - 5x + 2)^2 + 9y^2}{((2x-1)^2 + 4y^2)^2}.$$

Remark. The transformation $w = (z-2)/(2z-1)$ is not unique. In fact there are infinitely many mapping the two non-concentric circles to concentric circles. Another confusing fact is that even though the circle is mapped to a circle, their centres are NOT mapped to each other.

Remark. Early we know that the equation $|z - z_1|/|z - z_2| = \lambda$ is a circle (called the Circle of Appollonius), which is equivalent to

$$\left|z - \frac{\lambda^2 z_2 - z_1}{\lambda^2 - 1}\right| = \frac{\lambda}{|\lambda^2 - 1|}|z_2 - z_1|.$$

Use this fact to verify the transformations between circles in this section.

7 Joukowski transformation

Another important transformation with applications in *airfoil design* is the **Joukowski transformation**

$$w = f(z) = z + \frac{1}{z}.$$

Sometimes, it is also written as $f(z) = \frac{1}{2}\big(z + 1/z\big)$ or $f(z) = z + c^2/z$ for some positive constant c. This transformation usually maps the circle (not centred at the origin) to some domain resembling the shape of an airfoil.

Exercise. Show that the circle $|z| = a(\neq 1)$ in the z-plane is mapped into the ellipse in the w-plane:

$$\frac{u^2}{A^2} + \frac{v^2}{B^2} = 1, \qquad A = a + 1/a, \ B = a - 1/a.$$

Show also that the circle $|z| = 1$ is mapped to the line segment from -2 to 2.

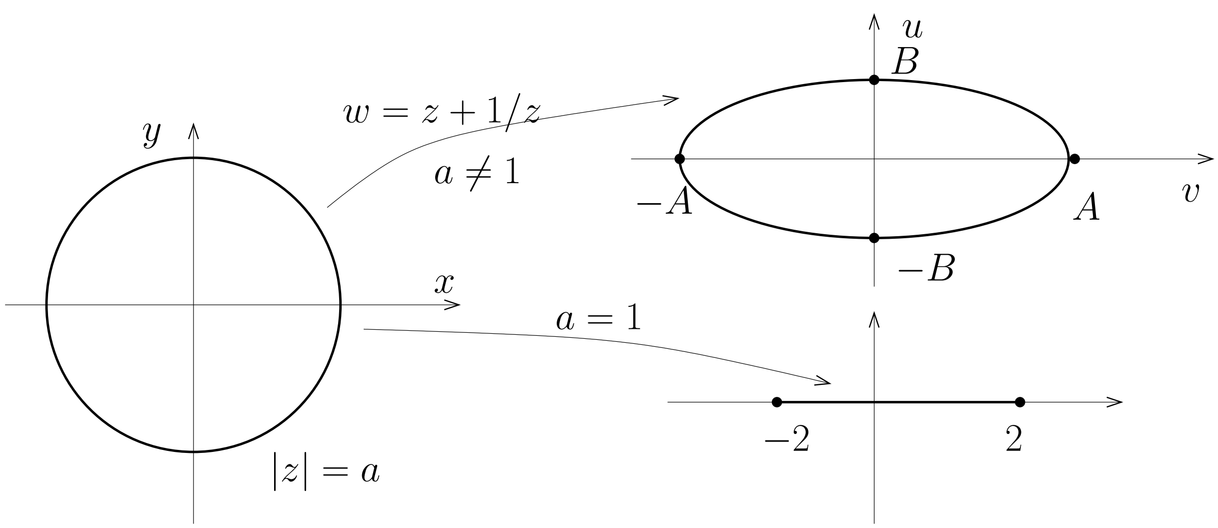


Figure 8: The circle $|z| = a$ is mapped to an ellipse or a line segment by $w = z + 1/z$.

Remark. The Joukowski transformation is a conformal mapping when $z \neq 0$ and $z \neq \pm 1$.

The goal is to choose the right center z_0 and the radius r_0, such the circle $|z - z_0| = r_0$ is mapped to an aerofoil (see Figure 9). The extra conditions are:

1. The point $z = 1$ (where $w = z + 1/z$ is NOT conformal) is always on the circle $|z - z_0| = r_0$, and is mapped to $w = 2$. This makes sure that the flow from the upper and lower edge (the trailing edge) near $w = 2$ can join smoothly.

2. The other two point $z = 0$ and $z = -1$ (where $w = z + 1/z$ is NOT conformal too) is always INSIDE the circle $|z - z_0| = r_0$.

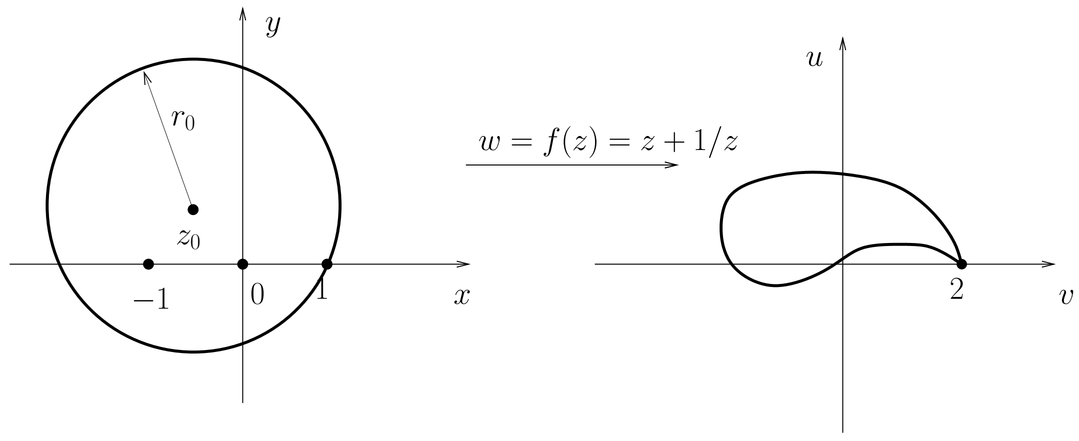


Figure 9: The Joukowski transformation maps the circle to an aerofoil.

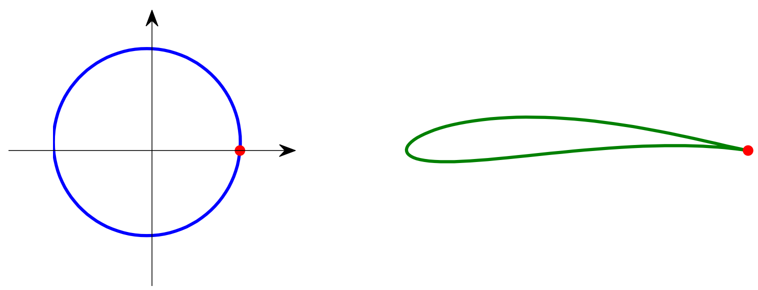


Figure 10: The aerofoil generated from the circle $|z - z_0| = r_0$ by the Joukowski mapping $w = z + 1/z$. Here $z_0 = -0.1 + 0.1i$ and $r_0 = |1 - z_0|$.

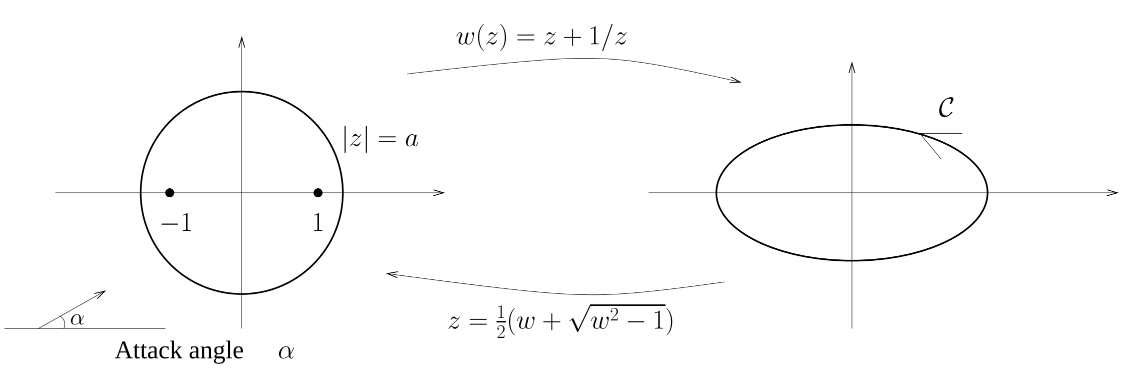


Figure 11: Ideal flow past an elliptic cylinder

Example 7.1 (Application to ideal flows with non-zero attach angle)**.** If we can solve the problem of flow past a circular cylinder, then we can solve the problem of the flow past an elliptic cylinder (or more general aerofoil, but you may need computer to do the calculation).

Consider the flow past a circular cylinder of radius $a > 1$. In the z-plane, the complex potential $U = \phi + i\psi$ is

$$U(z) = ze^{-i\alpha} + \frac{a^2}{z}e^{i\alpha},$$

where α is the attack angle.

The physical problem is defined in the w-plane. Suppose the complex potential is $W(w) = U(z)$, then (ρ is the density of the fluid)

1. Force exerted on the body:

$$F_u - iF_v = \frac{i\rho}{2}\oint_{\mathcal{C}}\left(\frac{dW}{dw}\right)^2 dw$$

2. Moments:

$$M = -\frac{\rho}{2}\mathrm{Re}\left\{\oint_{\mathcal{C}} w\left(\frac{dW}{dw}\right)^2 dw\right\}$$

We can find both the force and the moment, using the Joukowski transformation and the known complex potential U. Can you show the force is zero by some simple argument?

From $U(z) = W(w)$ with $w = z + 1/z$, we get

$$\frac{dW}{dw} = \frac{dU}{dz}\frac{dz}{dw} = \left(e^{-i\alpha} - \frac{a^2}{z^2}e^{i\alpha}\right)\left(\frac{dw}{dz}\right)^{-1} = \left(e^{-i\alpha} - \frac{a^2}{z^2}e^{i\alpha}\right)(1 - 1/z^2)^{-1}.$$

Therefore,

$$\begin{aligned} M &= -\frac{\rho}{2}\mathrm{Re}\int_{|z|=a}\underbrace{\left(z + \frac{1}{z}\right)}_{w}\underbrace{\left(e^{-i\alpha} - \frac{a^2}{z^2}e^{i\alpha}\right)^2(1 - 1/z^2)^{-2}}_{\left(\frac{dW}{dw}\right)^2}\underbrace{\left(1 - \frac{1}{z^2}\right)}_{\frac{dw}{dz}} dz \\ &= -\frac{\rho}{2}\mathrm{Re}\int_{|z|=a}\frac{z(z^2+1)}{z^2-1}\left(e^{-i\alpha} - \frac{a^2}{z^2}e^{i\alpha}\right)^2 dz \end{aligned}$$

The integrand has two simple poles at $z = 1$ and $z = -1$ (remember here $a > 1$) and one triple pole at $z = 0$. Using residual calculus (a complicated calculation), we get

$$M = -\frac{\rho}{2}4\pi\sin 2\alpha = -2\pi\rho\sin 2\alpha.$$

Remark. You can find many applets online to tune the parameters, and in general you have to use a computer to find the optimal set of parameters (like the position of the circle to be mapped from, the amount of circulation, the attack angle...)