AERO III/IV Complementary notes for exam review

Contents

1 Recommended references

The best strategy to get additional references is to get yourself familiar with the **keywords in the syllabus**, check the table of contents of some books in the library and read the relevant chapters. These books usually has the keywords "*(advanced) engineering mathematics*" or "*mathematical methods (for Engineers)*" in the title. For **calculus of variations**, you find it more likely in books about mathematical methods for "*physicists*". Here are a few more references:

- *Mathematical methods in the physical sciences* by Mary L Boas (2nd Edition or 3rd Edition): Chap 9 (Calculus of variations), Chap 14 (Complex variables and conformal mapping) and Chap 15 (Laplace transforms).
- *Advanced modern engineering mathematics* by Glyn James: Chap 8 (Complex variables and conformal mapping) and Chap 9 (Laplace transforms).

There are also books that you only need the relevant chapters:

- *Introduction to complex analysis* by H. A Priestley. The materials about complex variables are explained in greater details in this book (you can skip Chap 2,3,9,15 16 and 22)

2 Suggestions on preparing for the final exam

- The final exam has a similar style and difficulty as the past exams. You also choose four questions out of five.
- There are three types intense computations that you need more practice to succeed: (1) Solving the resulting second order ODEs from calculus of variation (2) Finding the residue (in contour integration and inverse Laplace transform (3) Transforming back and forth using conformal mapping and solving the Laplace equation on simple domains.
- Understand the essential steps and give sufficient details. The penalty on numerical mistakes will be minimized.
- To simplify your life, the parts related to *branches and branch points* is NOT required. Therefore, you will not see complicated contours because of the multivalued functions like $z^{1/2}$ or $\log z$.
- You will be given various hints, like the contours to evaluate integrals, the conformal mapping to solve Laplace equations, and other formulas if needed (beside the standard formula sheet).

3 Additional Comments on the course material

3.1 Calculus of variations

i) You should think of Calculus of variations as a framework to get differential equations from a scalar quantity. In this course, we focus on solving the resulting equation, which in general has to be solved numerically.

ii) There are mainly two types problems arising in calculus of variations:

   (1) Steady state that minimize the energy, for example the catenary minimizes the total gravitational energy or the soap bubble minimizes the total surface area. You get a **boundary value problem**.

   (2) Mechanic or time-dependent systems corresponds to an *action* $\mathcal{A}$, where the action is the integral (w.r.t time) of the *Lagrangian* $\mathcal{L}$ defined to be the **difference** of kinetic energy and potential energy. For example, the trajectory of a particle $\mathbf{x}(t)$ with mass m under the force $-\nabla V(\mathbf{x})$, then the equation can be obtained from $\int_{t_0}^{t_1} \left(\frac{m}{2}|\dot{\mathbf{x}}|^2 - V(\mathbf{x})\right)dt$, or the Newton's equation of the second law. You get a **initial value problem**. In this case, the total energy $\frac{m}{2}|\dot{\mathbf{x}}|^2 + V(\mathbf{x})$ (the **sum** of kinetic and potential energy) is conserved.

iii) Refresh your memory about the different ways to solve second order linear ODEs, and the two new ways for nonlinear ODE for $\int f(x,y,y')dx$, when either x or y is missing from f.

3.2 Complex variables

i) Get used to working with complex numbers or functions directly, instead of always resorting to real and imaginary part. For example, it is very easy to see whether an expression is analytic or not (there should be no dependence on $\bar{z}$ and no singular points), but more difficult to tell with real and imaginary parts represented in x and y (you have to check the Cauchy-Riemann conditions). This is one reason we never check explicitly a function is analytic or not in the second half of the course.

ii) You are free to "deform" the contour in any line integral (the end points should be fixed for open contours), as long as the integrand is analytic **withe respect to the integration variable**, and **on the swept region**.

iii) Besides returning the function value and the derivatives

$$f(z) = \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{f(\xi)}{\xi - z} d\xi, \quad f^{(n)}(z) = \frac{n!}{2\pi i} \int_{\mathcal{C}} \frac{f(\xi)}{(\xi - z)^{n+1}} d\xi,$$

the Cauchy integral formula has some unexpected use, because of the dependence on z is now only on the simple function $\frac{1}{\xi - z}$:

(1) The series expansion of $\frac{1}{\xi - z}$ near z_0 can be easily obtained by geometric series, leading to Taylor expansion and Laurent expansion.

(2) The inverse Laplace transform of $\frac{1}{\xi - s}$ is just $-e^{\xi t}$, from which you can get the inversion formula.

Read the relavent section in the notes for more details.

iv) To evaluate an integral by reducing it to residue calculus, you may have to complete a closed contour and to the appropriate limit. The contour is intimated related to the certain components in the integrand.

(1) You have to choose the upper semi-circle for e^{ix} or lower semi-circle for e^{-ix}, because e^{iz} vanishes exponentially fast as Imz goes to $+\infty$. For example, you can find

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{1 + x^2} = \int_{-\infty}^{\infty} \frac{\cos x}{1 + x^2} = \pi e.$$

(2) From the inversion formula of Laplace transform

$$f(t) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma - i\infty} F(z) e^{zt} dz,$$

we have to choose the left semi-circle if $t > 0$ and right semi-circle if $t < 0$ (eventually give $f(t) = 0$ in this case). The semi-circles are chosen because e^{st} vanishes exponentially fast on these semi-circles.

These integrals on the semicircles vanish in the limit $R \to \infty$, and you are not required to justify this.

v) For integrals of the type $\oint_{\mathcal{C}} P(z)/Q(z) dz$ with polynomial $P(z)$ and $Q(z)$, you can reduce it to residue by "shrinking" the contour $\mathcal{C}$. You can also try to make the contour goes to infinity. For example you get $\int_{|z|=2} \frac{1}{z^2+1} dz = 2\pi i \big(\mathrm{Res}(\frac{1}{z^2+1}, z = -i) + \mathrm{Res}(\frac{1}{z^2+1}, z = i)\big) = 0$ or alternatively by expanding the contour to infinity,

$$\int_{|z|=2} \frac{1}{z^2 + 1} dz = \lim_{R \to \infty} \int_{|z|=R} \frac{1}{z^2 + 1} dz = \lim_{R \to \infty} \int_0^{2\pi} \frac{iRe^{i\theta}}{R^2 e^{2i\theta} + 1} d\theta = 0.$$

For the more complicated example (given that the zeros of $z^6 + z + 1 = 0$ are all inside $|z| < 2$), you can not find the residue because you can not find the zeros of $z^6 + z + 1 = 0$ explicitly, but you can still find the answer by expanding the contour to infinity,

$$\int_{|z|=2} \frac{1}{z^6 + z + 1} dz = \lim_{R \to \infty} \int_{|z|=R} \frac{1}{z^6 + z + 1} dz = 0.$$

vi) There is no wrong contours or wrong ways of deforming contours, but you can get the answer explicitly with the right contours. Moreover, if you can the answers from different contours, the answers should be identical. This principle works for $\int_{|z|=2} \frac{1}{z^2+1} dz$ in the previous example, and also for integrals like $\int_0^\infty \frac{1}{1+x^4} dx$ in **Example 6.2** in **Complex Variables (III)**.

vii) There are two basic methods to calculate the residue of a function $f(z)$ at z_0: (1) series expansion (2) the formula

$$\mathrm{Res}(f; z = z_0) = \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \Big[(z - z_0)^m f(z) \Big] \Big|_{z=z_0},$$

if z_0 is a pole of order m. The derivative and the calculation in the second approach may be complicated if m is larger than 2, for example the residue of $f(z) = \frac{1}{\sin^3 z}$ at the origin.

3.3 Laplace transform

i) The functions $f(t)$ should be identically zero if t is negative, or equivalently f is a **causal** function. As a result, the inverse Laplace transforms in the table should be multiplied by the Heaviside function $H(t)$. This is also consistent with the result from inversion formula.

ii) The Laplace transform $F(s)$ of some function $f(t)$ can be extended to the whole complex, but should be interpreted carefully. For example, $F(s) = 1/(s^2+1)$ is the Laplace transform of $\sin t$ when $s > 0$, and we get $F(-1) = 1/2$. But the integral $\int_0^\infty e^{-(-1)t} \cos t dt$ is divergent.

iii) The two basic methods for inverse Laplace transform (partial fraction or residue calculus) should work for most of the problems, but recognizing some special structures of $F(s)$ can reduce the amount of calculation.

iv) Not all functions have a valid Laplace transform, even for s large enough. This happens usually if $f(t)$ increases too fast as $t \to \infty$, like $f(t) = e^{t^2}$. As a result, you can not solve the ODE

$$y'' - ty' - y = 0, \qquad y(0) = 1,\ y'(0) = 0$$

using Laplace transform (you can verify the solution $y(t) = e^{t^2/2}$). Compare this ODE to **Example 4.3** in the lecture notes.

v) You have to solve another ODE in x, if the original equation is a PDE in x and t. You can treat s as a constant (or let $s = 1$ or other constants) to find the solution procedure. See **Example 4.5** and **4.6** how this is handled (additional details in the updated online version).

vi) It is common that you may have to use some physical intuition (like $F(s)$ is bounded as $s \to \infty$) to determine the coefficient in the transformed equation.

3.4 Conformal Mapping

Conformal Mapping in general: The definition of a conformal mapping $w = f(z)$ (f is analytic and $f' \neq 0$ on the domain D) is in general not checked for every case. The conditions are only required on the interior of D, and may be violated on the boundary for some desired properties (mapping the point to infinity or changing the boundary behaviour drastically).

a) It is common for f to become singular on the boundary, because the corresponding point is transformed to infinity. For example, the unit interior disk $|z| < 1$ is transformed to the upper half plane by $w = f(z) = i\frac{1+z}{1-z}$ (check this!). Here $f(z)$ is singular at $z = 1$, which is transformed to $w = \infty$.

b) It is also common that $f' = 0$ on the boundary, for instance the Joukowski transformation $w = f(z) = z + 1/z$. Here $f'(z) = 1 - 1/z^2$ and $f'(1) = 0$. This is desired because the streamline at the trailing edge can now join smoothly .

Conformal Mapping in solving Laplace equations: The main focus of this chapter is on using *conformal mapping* $w = u+iv = f(z) = f(x+iy)$ *to transform the Laplace equation* $\frac{\partial^2}{\partial x^2}h + \frac{\partial^2}{\partial y^2}h = 0$ *on complicated domain* D *to the Laplace equation* $\frac{\partial^2}{\partial u^2}H + \frac{\partial^2}{\partial v^2}H = 0$ *on a simple domain* $\tilde{D}$, using the crucial fact $h(x, y) = H(u, v)$.

Conformal mapping in linking different ideal flows: The basic idea is: if $U(z)$ is the *complex potential* of a flow on D, and f is a conformal mapping on D, then $\Omega(w) = U(z) = U(z(w))$ is the *complex potential* on the transformed domain $\tilde{D}$.

a) You can get the flow just using velocity potential or stream function by $\tilde{\phi}(u, v) = \phi(x, y)$ or $\tilde{\psi}(u, v) = \psi(x, y)$, but you can not simply equating the corresponding velocity fields.

b) **Example 4.2** in the lecture notes (transforming uniform flow on the upper half plane to a quarter plane) can be generalized to a flow in a corner of angle α (see Figure 1). Since the complex potential is $\Omega(z) = Uz$ and the conformal mapping is $w = f(z) = z^{\frac{\alpha}{\pi}}$ (or equivalently $z = w^{\frac{\pi}{\alpha}}$, the complex potential for the corner flow is

$$\tilde{\Omega}(w) = \Omega(z) = Uz = Uw^{\frac{\pi}{\alpha}}.$$

Using this result, you may generate more realistic flows around a wedge, where the stream function is given by $\psi(r, \theta) = Ar^{\frac{1}{1-\beta}} \sin \frac{\pi-\theta}{1-\beta}$.

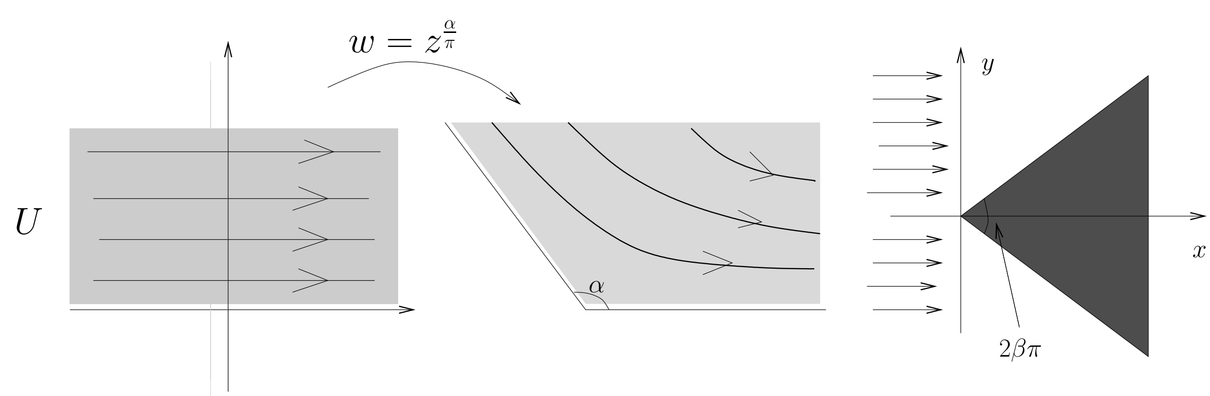


Figure 1: Transformation of a uniform flow to a corner flow by conformal mapping, which can be used to find the flow around a wedge.

c) Joukowski transformation is popular because it is the simplest conformal mapping that preserves the far-field flow pattern. If the far-field flow is preserved, the corresponding mapping must be an identity, i.e., $f(z) = z + g(z)$ with $g(z) \to 0$ as $z \to \infty$, while the simplest function $g(z)$ that vanishes at infinity is $g(z) = 1/z$.

d) The aerofoil is usually generated from the circle $|z - z_0| = r_0$ with $\arg z_0 \in (\pi/2, \pi)$ and $r_0 = |1-z_0| > 1$ to make sure that the point $z = 1$ (where the mapping $w = z+z/1$ is NOT conformal) is on the boundary of the circle. In this way, the point $z = 1$ is mapped to the trailing edge, where the stream lines join smoothly (see Figure 2).

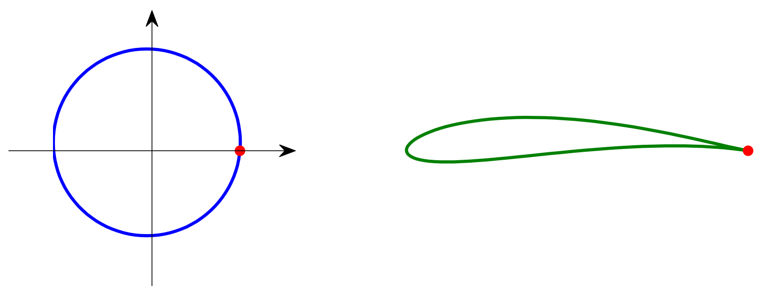


Figure 2: The aerofoil generated from the circle $|z-z_0| = r_0$ by the Joukowski mapping $w = z+1/z$. Here $z_0 = -0.1 + 0.1i$ and $r_0 = |1 - z_0|$.

4 Solutions to selected exercises and additional details on the examples

4.1 Calculus of variations

Exercise (Which method to choose?) Find the extremal curve of the integral $\int (y^2 - y'^2)dx$ (a) by solving the *linear* Euler-Lagrange equation (b) by realizing the fact that $f = y^2 - y'^2$ is independent of x. Which way is easier and faster?
Solutions: (a) Since $\frac{\partial f}{\partial y} = 2y$ and $\frac{\partial f}{\partial y'} = -2y'$, the Euler-Lagrange equation is

$$0 = \frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 2(y + y'').$$

this is a linear equation, whose general solution is $y(x) = c_1 \cos x + c_2 \sin x$.

(b) Since f is independent of x, we have

$$c_1 = f - y'\frac{\partial f}{\partial y'} = y^2 + y'^2$$

a constant. This equation is equivalent to

$$y' = \pm(c_1 - y^2)^{1/2}$$

or

$$x = \int \frac{dy}{(c_1 - y^2)^{1/2}} = \arcsin\frac{y}{\sqrt{c_1}} + c_2.$$

The solution can be written as

$$y = \sqrt{c_1}\sin(x + c_2),$$

which is equivalent to the one in (a). In general, it is more complicated in this approach, because of the integration involved.

4.2 Complex Variables

Exercise. Find the modulus of $\frac{3+i}{1+i}$.
Solution:

$$\left|\frac{3+i}{1+i}\right| = \frac{|3+i|}{|1+i|} = \frac{\sqrt{10}}{\sqrt{2}} = \sqrt{5}.$$

Exercise.[The Circles of Apollonius] The equation $|z-z_1|/|z-z_2| = \lambda(>0)$ for $\lambda \neq 1$ is a circle, called the Circles of Apollonius.

(i) What is the curve corresponding to $\lambda = 1$?

(ii) If $\lambda \neq 1$, substituting $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ into the equation $|z-z_1|/|z-z_2| = \lambda$, we get (with $z = x + iy$)

$$\lambda = \frac{|z-z_1|}{|z-z_2|} = \frac{|(x-x_1)+i(y-y_1)|}{|(x-x_2)+i(y-y_2)|} = \frac{\sqrt{(x-x_1)^2+(y-y_1)^2}}{\sqrt{(x-x_2)^2+(y-y_2)^2}}. \tag{1}$$

Confirm that this is a circle (the expressions for the center and the radius are very complicated).

(iii) Once we are sure it is circle, we can find the center and the radius in other alternative ways, using the geometry.

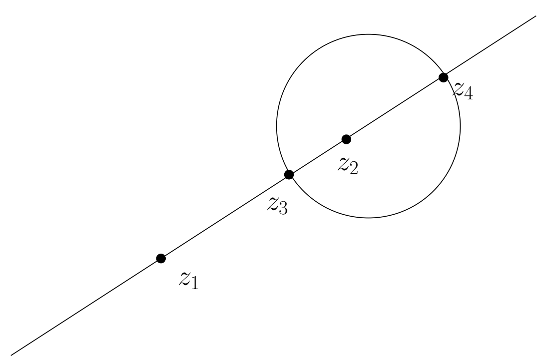


Figure 3: The circle of Apollonius of all the points $|z-z_1|/|z-z_2| = \lambda > 1$, which intersects the straight line (connecting z_1 and z_2) at z_3 and z_4.

Let z_3 and z_4 be the intersection points of the circle with the straight line connecting z_1 and z_2. Without loss of generality, we can take $\lambda > 1$ (the case $0 < \lambda < 1$ works similarly), which looks like in Figure 3. The advantage using this geometric information is that we can get rid of the modulus from the governing equations $|z-z_1|/|z-z_2| = \lambda$. That is

$$\lambda = \frac{z_1 - z_3}{z_3 - z_2} = \frac{z_1 - z_4}{z_2 - z_4},$$

which gives

$$z_3 = \frac{\lambda z_2 + z_1}{\lambda + 1}, \qquad z_4 = \frac{\lambda z_2 - z_1}{\lambda - 1}.$$

By the symmetry, the centre of the radius should be on this straight line, and z_3, z_4 are on the opposite sides of a diameter. Therefore, the centre of the circle is

$$z_0 = \frac{1}{2}(z_3 + z_4) = \frac{\lambda^2 z_2 - z_1}{\lambda^2 - 1}, \tag{2}$$

and the radius is

$$R = \frac{|z_3 - z_4|}{2} = \frac{\lambda}{|\lambda^2 - 1|}|z_1 - z_2|, \tag{3}$$

which should be compared with the centre and radius calculated from the algebraic equation (1).

(iv) Check that when $0 < \lambda < 1$, the centre and the radius are still given by (2) and (3), respectively.

(v) Find the curve on the w-plane, which is transformed from the circle $|z| = 1$ by the map $w = (z + 2)/(z - 2)$.

Solution: (i) When $\lambda = 1$, the curve is the bisector of z_1 and z_2, a straight line.

(ii) If $\lambda \neq 1$, then the equation

$$\lambda = \frac{|z - z_1|}{|z - z_2|} = \frac{|(x - x_1) + i(y - y_1)|}{|(x - x_2) + i(y - y_2)|} = \frac{\sqrt{(x - x_1)^2 + (y - y_1)^2}}{\sqrt{(x - x_2)^2 + (y - y_2)^2}} \tag{4}$$

can be simplified as

$$(\lambda^2 - 1)(x^2 + y^2) + 2(x_1 - \lambda^2 x_2)x + 2(y_1 - \lambda^2 y_2)y + \lambda^2(x_2^2 + y_2^2) - (x_1^2 + y_1^2) = 0.$$

Since the coefficient of the quadratic term x^2 and y^2 are the same, and we can easily find point on the line connecting z_1 and z_2 satisfying the equation, it is a circle. But the expressions for the center and the radius are both very complicated. Alternatively, we can find these geometric quantities using complex numbers.

(iv) If $0 < \lambda < 1$, then the circle looks like the one in Figure 4. Similarly, we have

$$\lambda = \frac{z_1 - z_3}{z_3 - z_2} = \frac{z_1 - z_4}{z_2 - z_4},$$

which is exactly the same as the case when $\lambda > 1$. Therefore, the expressions for the centre and the radius are the same.

(v) If the circle $|z| = 1$ is transformed by the map $w = (z+2)/(z-2)$, the map is equivalent to $z = (2w + 2)/(w - 1)$, implies that the curve satisfies the equation

$$1 = |z| = \left|\frac{2w + 2}{w - 1}\right|$$

or

$$\left|\frac{w + 1}{w - 1}\right| = \frac{1}{2}.$$

Therefore $\lambda = 1/2$, $w_1 = -1$ and $w_2 = 1$, and the curve on the w-plane is a circle with centre $z_0 = \frac{\lambda^2 z_1 - z_2}{\lambda^2 - 1} = -\frac{5}{3}$ and radius $R = \frac{\lambda}{|\lambda^2 - 1|}|z_1 - z_2| = \frac{2}{3}$.

Exercise. Shown that

$$\sin\theta + \sin 2\theta + \cdots + \sin n\theta = \frac{\sin\theta - \sin(n+1)\theta - \sin\theta\cos(n+1)\theta + \cos\theta\sin(n+1)\theta}{2 - 2\cos\theta},$$

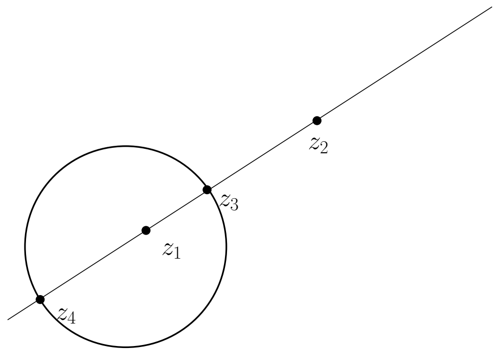


Figure 4: The circle of Apollonius of all the points $|z - z_1|/|z - z_2| = \lambda$ with $0 < \lambda < 1$.

using the relation between Euler equation (The right hand side can be simplified further).
Solutions: Using the fact that $\sin k\theta = \mathrm{Im} e^{ik\theta}$,

$$\sin\theta + \sin 2\theta + \cdots + \sin n\theta = \mathrm{Im}\left(e^{i\theta} + e^{i2\theta} + \cdots + e^{in\theta}\right).$$

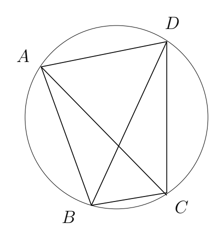
We can find the sum on the right hand side, using the formula for the geometric series, or the following equivalent way. Set $S_n = e^{i\theta} + e^{i2\theta} + \cdots + e^{in\theta}$, then

$$e^{i\theta} S_n = e^{2i\theta} + e^{3i\theta} + \cdots + e^{i(n+1)\theta} = S_n + e^{i(n+1)\theta} - e^{i\theta}$$

or

$$S_n = \frac{e^{i(n+1)\theta} - e^{i\theta}}{e^{i\theta} - 1}.$$

Taking the imaginary part of S_n, we get the desired conclusion.



Exercise. Show that the four points A, B, C, D are on the same circle if and only if the distances satisfies the condition

$$AC \cdot BD = AB \cdot CD + AD \cdot BC.$$

Hint: Notice the condition that A, B, C, D are on the same circle can be characterized by $\angle ADC + \angle ABC = \pi$ (or equivalently $\angle BAD + \angle BCD = \pi$) and the identity $(z_A - z_C)(z_B - z_D) = (z_A - z_B)(z_C - z_D) + (z_A - z_D)(z_B - z_C)$, where z_A, z_B, z_C and z_D are the complex numbers corresponding to the four points.
Solution: If z_A, z_B, z_C, z_D are on the same circle, then $\angle BAD + \angle BCD = \pi$ or equivalently (be careful about how the angle is represented, and which complex numbers are involved):

$$\arg\frac{z_C - z_D}{z_A - z_D} + \arg\frac{z_C - z_B}{z_A - z_B} = \pi.$$

This equation can be written as

$$\arg\frac{z_C - z_D}{z_A - z_D} = \pi - \arg\frac{z_C - z_B}{z_A - z_B} = \arg\frac{z_B - z_C}{z_A - z_B}.$$

and is equivalent to

$$\arg(z_A - z_B)(z_C - z_D) = \arg(z_a - z_D)(z_B - z_C).$$

Since **the triangle inequality** $|z+w| \le |z|+|w|$ **becomes an equality if and only if** $\arg z = \arg w$, we get

$$\begin{aligned} AC\cdot BD = |(z_A - z_C)(z_B - z_D)| &= |(z_A - z_B)(z_C - z_D) + (z_A - z_D)(z_B - z_C)| \\ &= |(z_A - z_B)(z_C - z_D) + (z_A - z_D)(z_B - z_C)| \\ &= AB\cdot CD + AD\cdot BC. \end{aligned}$$

Similarly, we can show that if $AC\cdot BD = AB\cdot CD + AD\cdot BC$ then the four points are on a circle.

Exercise. Shown from the property $|z_1+z_2| \le |z_1|+|z_2|$, we can get

$$|z_1 + \cdots + z_n| \le |z_1| + \cdots + |z_n|.$$

Moreover, for any real number $a_1, \cdots, a_n, b_1, \cdots, b_n$, we have the inequality

$$\sqrt{a_1^2+b_1^2} + \cdots + \sqrt{a_n^2+b_n^2} \le \sqrt{(a_1+\cdots+a_n)^2 + (b_1+\cdots+b_n)^2}.$$

Solution: Using the inequality $|w_1+w_2| \le |w_1|+|w_2|$ recursively,

$$\begin{aligned} |z_1+z_2+\cdots+z_n| &= |z_1 + (z_2+\cdots+z_n)| \\ &\le |z_1| + |z_2+z_3+\cdots+z_n| \\ &= |z_1| + |z_2 + (z_3+\cdots+z_n)| \\ &\le |z_1| + |z_2| + |z_3+\cdots+z_n| \\ &\vdots \\ &\le |z_1| + |z_2| + \cdots + |z_n|. \end{aligned}$$

For any real number $a_1, \cdots, a_n, b_1, \cdots, b_n$, if we define $z_1 = a_1 + b_1 i, \cdots, z_n = a_n + b_n i$, then the above inequality becomes

$$\sqrt{a_1^2+b_1^2} + \cdots + \sqrt{a_n^2+b_n^2} \le \sqrt{(a_1+\cdots+a_n)^2 + (b_1+\cdots+b_n)^2}.$$

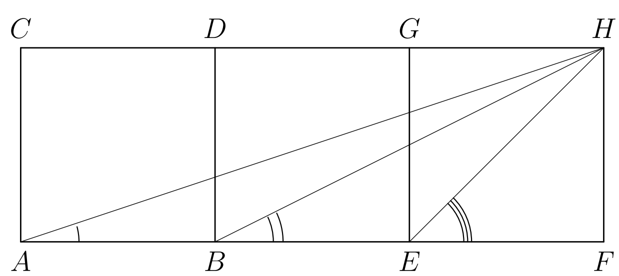


Figure 5: Sum of three angles

Exercise. Three squares placed side by side as shown in Figure 5. Prove that the sum of $\angle HAF$, $\angle HBF$ and $\angle HEF$ is a right angle. Hint: Let the vectors $\overrightarrow{AH}, \overrightarrow{BH}$ and $\overrightarrow{CH}$ be the complex number $3+i, 2+i$ and $1+i$ respectively, the the sum of the three angles is the argument of the product $(3+i)(2+i)(1+i)$.

Solution: The sum of the angles is the argument of the product of $3+i, 2+i$ and $1+i$. Since

$$(3+i)(2+i)(1+i) = (5+5i)(1+i) = 10i,$$

whose argument is $\pi/2$, the sum of the three angles is $\pi/2$.

Exercise. Find the limit of $f(z) = \dfrac{z^2+iz+2}{z-i}$ as $z \to i$.
Solution: The *L'Hospital's Rule* still holds for the limits of complex variables: if both the numerator and denominator vanish in the limit, their ratio is the same as the ratio of their derivatives. Therefore,

$$\lim_{z\to i} \frac{z^2+iz+2}{z-i} = \lim_{z\to i} \frac{2z+i}{1} = 3i.$$

Exercise. Show that the Cauchy-Riemann equations is equivalent to

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \qquad \frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r},$$

when u and v are represented in polar coordinates.
Solution: From the polar coordinates $x = r\cos\theta, y = r\sin\theta$,

$$\frac{\partial x}{\partial r} = \cos\theta = \frac{1}{r}\frac{\partial y}{\partial \theta}, \quad \frac{\partial y}{\partial r} = \sin\theta = -\frac{1}{r}\frac{\partial x}{\partial \theta}.$$

Therefore, using chain rule and the Cauchy-Riemann conditions,

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial v}{\partial y}\frac{\partial x}{\partial r} - \frac{\partial v}{\partial x}\frac{\partial y}{\partial r} = \frac{1}{r}\left(\frac{\partial v}{\partial y}\frac{\partial y}{\partial \theta} + \frac{\partial v}{\partial x}\frac{\partial x}{\partial \theta}\right) = \frac{1}{r}\frac{\partial v}{\partial \theta}.$$

Similarly, we can show that other relation $\frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r}$.

Exercise. Use the Cauchy-Riemann equation to show that $f'(z) = e^z$ for $f(z) = e^z$.
Solution: If $f(z) = e^z = e^x(\cos y + i\sin y)$ with $u = e^x\cos y, v = e^y \sin y$. Then by definition

$$f'(z) = u_x + iv_x = e^x\cos y + ie^x \sin y = e^{x+iy} = e^z.$$

Exercise. Evaluate $\int_{\mathcal{C}} e^z/(z-1)dz$ where $\mathcal{C}$ is the circle $|z-2| = 2$.
Solution: Since the point $z_0 = 1$ is inside the contour, the circle $|z-2| = 2$, using the Cauchy integral formula for the analytic function $f(z) = e^z$,

$$\int_{\mathcal{C}} f(z)/(z-1)dz = 2\pi i f(1) = 2\pi i e^1.$$

4.3 Power series, residue theorem and its applications

Exercise. Find the Laurent expansion of $f(z) = e^{z+\frac{1}{z}}$ on $|z| > 0$.
Solution: If we use the Taylor expansion of the exponential function, then we have to options. The first way is to treat $z + \frac{1}{z}$ as one variable, that is

$$f(z) = e^{z+\frac{1}{z}} = 1 + \left(z+\frac{1}{z}\right) + \frac{1}{2!}\left(z+\frac{1}{z}\right)^2 + \cdots + \cdots + \frac{1}{n!}\left(z+\frac{1}{z}\right)^n + \cdots$$

if we want to get the coefficients, for example, c_0, then we have to add infinite many terms from each of the binomial expansions of $\left(z+\frac{1}{z}\right)^{2n}$ (the odd powers like $\left(z+\frac{1}{z}\right)$ or $\left(z+\frac{1}{z}\right)^3$ have zero constant coefficient), which gives

$$c_n = 1 + \frac{1}{2!}\binom{2}{1} + \frac{1}{4!}\binom{4}{2} + \cdots + \frac{1}{(2n)!}\binom{2n}{n} + \cdots.$$

The second way is to write $f(z) = e^z e^{\frac{1}{z}}$ and to expand the two exponentials first, that is

$$f(z) = e^z e^{\frac{1}{z}} = \left(1 + z + \frac{1}{2!}z^2 + \cdots\right)\left(1 + \frac{1}{z} + \frac{1}{2!}\frac{1}{z^2} + \cdots\right)$$

Once again we have to add infinitely many products of terms from both expansions to get the final coefficients.

In general, we don't expect a compact explicit form for the coefficients, but we can still express them as integrals using the definition, that is

$$c_n = \frac{1}{2\pi i}\oint_{|z|=1}\frac{f(z)}{z^{n+1}}dz = \frac{1}{2\pi i}\oint_{|z|=1} z^{-1-n}\exp\left(z + \frac{1}{z}\right)dz.$$

Using the parametrization $z = e^{i\theta}$, the above integral can be simplified as

$$c_n = \frac{1}{2\pi i}\int_0^{2\pi} e^{-i(1+n)\theta}\exp(e^{i\theta} + e^{-i\theta})e^{i\theta}id\theta = \frac{1}{2\pi}\int_0^{2\pi} e^{-in\theta}\exp(2\cos\theta)d\theta.$$

Exercise. Find the first three terms of the Laurent expansion of $f(z) = \frac{1}{z\sin z}$ on (1) $0 < |z| < \pi$; (2) $\pi < |z| < 2\pi$ (only c_{-1}, c_0 and c_1) and (3) $0 < |z - \pi| < \pi$.

Solution: (1) By Taylor expansion of $\sin z = z - \frac{z^3}{6} + \frac{z^5}{120} + \cdots$,

$$\frac{1}{z\sin z} = \frac{1}{z\left(z + \frac{z^3}{6} + \frac{z^5}{120} + \cdots\right)} = \frac{1}{z^2}\left(1 - \left(\frac{z^2}{6} - \frac{z^4}{120}\right) + \cdots\right)^{-1}.$$

Since $\left(1 - \left(\frac{z^2}{6} - \frac{z^4}{120}\right) + \cdots\right)^{-1} = 1 + \left(\frac{z^2}{6} - \frac{z^4}{120}\right) + \left(\frac{z^2}{6} - \frac{z^4}{120}\right)^2 + \cdots = 1 + \frac{z^2}{6} + \frac{7z^4}{360} + \cdots$, the Laurent expansion is

$$\frac{1}{z\sin z} = \frac{1}{z^2}\left(1 + \frac{z^2}{6} + \frac{7z^4}{360} + \cdots\right) = \frac{1}{z^2} + \frac{1}{6} + \frac{7z^2}{360} + \cdots.$$

(2) In the range $\pi < |z| < 2\pi$, $\frac{1}{z\sin z} = \cdots + \frac{c_{-1}}{z} + c_0 + c_1 z + \cdots$ and we find the three coefficients c_{-1}, c_0 and c_1 from the definition $c_n = \frac{1}{2\pi i}\oint_{\mathcal{C}}\xi^{-n-1}f(\xi)d\xi$ for some contour $\mathcal{C}$ in the annulus $\pi < |z| < 2\pi$.

$$c_{-1} = \frac{1}{2\pi i}\oint_{\mathcal{C}}\frac{1}{\xi\sin\xi}d\xi = \mathrm{Res}\left(\frac{1}{\xi\sin\xi}, \xi = 0\right) + \mathrm{Res}\left(\frac{1}{\xi\sin\xi}, \xi = \pi\right) + \mathrm{Res}\left(\frac{1}{\xi\sin\xi}, \xi = -\pi\right).$$

From the computation in (1), $\mathrm{Res}\left(\frac{1}{\xi\sin\xi}, \xi = 0\right) = 0$ since the coefficient of ξ^{-1} is zero.

$$\mathrm{Res}\left(\frac{1}{\xi\sin\xi}, \xi = \pi\right) = \lim_{\xi\to\pi}\frac{\xi - \pi}{\xi\sin\xi} = \lim_{\xi\to\pi}\frac{1}{\xi\cos\xi} = -\pi^{-1} = \mathrm{Res}\left(\frac{1}{\xi\sin\xi}, \xi = -\pi\right).$$

Therefor, $c_{-1} = -2/\pi$. Similarly,

$$c_0 = \frac{1}{2\pi i}\oint_{\mathcal{C}} \frac{1}{\xi^2 \sin\xi} d\xi = \operatorname{Res}\left(\frac{1}{\xi^2\sin\xi}, \xi = 0\right) + \operatorname{Res}\left(\frac{1}{\xi^2\sin\xi}, \xi = \pi\right) + \operatorname{Res}\left(\frac{1}{\xi^2\sin\xi}, \xi = -\pi\right).$$

The residue $\operatorname{Res}\left(\frac{1}{\xi^2\sin\xi}, \xi = 0\right) = 1/6$ is exactly the constant from (1) and

$$\operatorname{Res}\left(\frac{1}{\xi^2\sin\xi}, \xi = \pi\right) = \lim_{\xi\to\pi} \frac{\xi-\pi}{\xi^2\sin\xi} = \lim_{\xi\to\pi}\frac{1}{\xi^2\cos\xi} = -\pi^{-2} = \operatorname{Res}\left(\frac{1}{\xi^2\sin\xi}, \xi = -\pi\right).$$

Therefore, $c_0 = \frac{1}{6} - \frac{2}{\pi^2}$. Finally,

$$c_0 = \frac{1}{2\pi i}\oint_{\mathcal{C}} \frac{1}{\xi^3 \sin\xi} d\xi = \operatorname{Res}\left(\frac{1}{\xi^3\sin\xi}, \xi = 0\right) + \operatorname{Res}\left(\frac{1}{\xi^3\sin\xi}, \xi = \pi\right) + \operatorname{Res}\left(\frac{1}{\xi^3\sin\xi}, \xi = -\pi\right).$$

Similarly, $\operatorname{Res}\left(\frac{1}{\xi^3\sin\xi}, \xi = 0\right) = 0$ and $\operatorname{Res}\left(\frac{1}{\xi^3\sin\xi}, \xi = \pi\right) = \operatorname{Res}\left(\frac{1}{\xi^3\sin\xi}, \xi = -\pi\right) = -\pi^{-3}$ and $c_1 = -\frac{2}{\pi^3}$.

(3) Using the substitution $w = z - \pi$, we get $\frac{1}{z\sin z} = -\frac{1}{(w+\pi)\sin w}$. When $0 < |w| < \pi$, $\frac{1}{w+\pi} = \frac{1}{\pi}\frac{1}{1+w/\pi} = 1/\pi - w/\pi^2 + w^2/\pi^3 + \cdots$ and

$$\frac{1}{\sin w} = \frac{1}{w - w^3/6 + \cdots} = \frac{1}{w}\left(1 - \frac{w^2}{6}\right)^{-1} = \frac{1}{w}\left(1 + \frac{w^2}{6} + \cdots\right).$$

Therefore,

$$\begin{aligned} -\frac{1}{(w+\pi)\sin w} &= -\left(\frac{1}{\pi} - \frac{w}{\pi^2} + \frac{w^2}{\pi^3} + \cdots\right)\frac{1}{w}\left(1 + \frac{w^2}{6} + \cdots\right) \\ &= -\frac{1}{w}\left(\frac{1}{\pi} - \frac{w}{\pi^2} + \left(\frac{1}{6\pi} + \frac{1}{\pi^3}\right) w^2\right) \end{aligned}$$

or equivalently

$$\frac{1}{z\sin z} = -\frac{1}{z-\pi} + \frac{1}{\pi^2} + \left(\frac{1}{6\pi} + \frac{1}{\pi^3}\right)(z-\pi) + \cdots, \qquad 0 < |z-\pi| < \pi.$$

Exercise. Find $\operatorname{Res}(f; z = 1)$ with $f(z) = \frac{1}{z(z-1)^2}$.
Solution: If we use the Laurent expansion at $z = 1$, then

$$\begin{aligned} f(z) = \frac{1}{(z-1)^2}\frac{1}{1+(z-1)} &= \frac{1}{(z-1)^2}\left(1 - \frac{1}{z-1} + \frac{1}{(z-1)^2} + \cdots\right) \\ &= \frac{1}{(z-1)^2} - \frac{\mathbf{1}}{\mathbf{z-1}} + 1 - (z-1) + \cdots. \end{aligned}$$

The residue of $f(z)$ at $z = 1$ is the coefficient c_{-1} of the term $(z-1)^{-1}$, which is -1.

Alternatively we can use the formula

$$c_{-1} = \lim_{z\to z_0} \frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}(z-z_0)^m f(z)$$

for a pole of order m at $z = z_0$, then

$$c_{-1} = \lim_{z\to 1} \frac{d}{dz}(z-1)^2 f(z) = \lim_{z\to 1} \frac{d}{dz}\frac{1}{z} = \lim_{z\to 1}\left(-\frac{1}{z^2}\right) = -1.$$

Exercise. Find $I = \dfrac{1}{2\pi i}\displaystyle\int_{\mathcal{C}} \cot z dz$, where $\mathcal{C}$ is the unit circle.
Solution: The only pole of $f(z) = \cot z = \frac{\cos z}{\sin z}$ inside the unit cicle is $z = 0$, which is a simple pole. Therefore,

$$I = \frac{1}{2\pi i}\int_{\mathcal{C}} \cot z dz = \mathrm{Res}(\cot z, z=0) = \lim_{z\to 0} z\cot z = 1.$$

4.4 Laplace Transform

Exercise. Find the Laplace transform of $tf'(t)$ and $tf''(t)$.
Solution: Using the definition and integrating by parts,

$$\begin{aligned}\mathcal{F}[tf'(t)](s) &= \int_0^\infty tf'(t)e^{-st}dt = \int_0^\infty te^{-st}df(t) = te^{-st}f(t)\Big|_{t=0}^{t=\infty} - \int_0^\infty f(t)d(te^{-st}) \\ &= -\int_0^\infty f(t)e^{-st}dt + s\int_0^\infty tf(t)e^{-st}dt \\ &= -F(s) + s\mathcal{F}[tf(t)](s).\end{aligned}$$

Taking the derivative of the definition of the Laplace transform,

$$F'(s) = \frac{d}{ds}\int_0^\infty f(t)e^{-st}dt = \int_0^\infty f(t)\frac{d}{ds}e^{-st}dt = -\int_0^\infty tf(t)e^{-st}dt.$$

Therefore, the Laplace transform of $tf'(t)$ is $-F(s) - sF'(s)$.

Similarly, we have

$$\begin{aligned}\mathcal{F}[tf''(t)](s) &= \int_0^\infty tf''(t)e^{-st}dt = \int_0^\infty te^{-st}df'(t) = -\int_0^\infty f'(t)d(te^{-st}) \\ &= -\int_0^\infty f'(t)e^{-st} + s\int_0^\infty tf'(t)e^{-st}dt.\end{aligned}$$

Using the Laplace transform of $f'(t)$ and $tf'(t)$, we get

$$\mathcal{F}[tf''(t)](s) = -(sF(s) - f(0)) + s(-F(s) - sF'(s)) = f(0) - 2sF(s) - s^2F'(s).$$

Exercise. Verify the inverse transform of $n!/s^{n+1}, 1/(s+a), s/(s^2+\omega^2), \omega/(s^2+\omega^2), e^{-sT}/s$ in the table of transforms.
Solution: We have the series expansion,

$$e^{st}n!/s^{n+1} = \left(1 + st + \frac{(st)^2}{2!} + \cdots + \frac{(st)^n}{n!}\right)\frac{n!}{s^{n+1}} = \frac{n!}{s^{n+1}} + \frac{n!t}{s^n} + \frac{n!t^2}{2!s^{n-1}} + \cdots + \frac{t^n}{s} + \cdots.$$

Therefore, $\mathcal{L}^{-1}[n!/s^{n+1}](t) = \mathrm{Res}(e^{st}n!/s^{n+1}, s=0)$ is the coefficient of $\frac{1}{s}$, that is t^n. For the rest functions,

$$\mathcal{L}^{-1}[1/(s+a)](t) = \mathrm{Res}(e^{st}/(s+a), s=-a) = \lim_{s\to -a} e^{st} = e^{-at};$$

$$
\begin{aligned}
\mathcal{L}^{-1}[s/(s^2+\omega^2)](t) &= \mathrm{Res}(e^{st}s/(s^2+\omega^2), s=i\omega) + \mathrm{Res}(e^{st}s/(s^2+\omega^2), s=-i\omega) \\
&= \lim_{s\to i\omega}\frac{se^{st}}{s+i\omega} + \lim_{s\to -i\omega}\frac{se^{st}}{s-i\omega} \\
&= \frac{1}{2}(e^{i\omega t}+e^{-i\omega t}) = \cos\omega t.
\end{aligned}
$$

Similarly, $\mathcal{L}^{-1}[\omega/(s^2+\omega^2)](t) = \mathrm{Res}(e^{st}\omega/(s^2+\omega^2), s=i\omega)+\mathrm{Res}(e^{st}\omega/(s^2+\omega^2), s=-i\omega) = \sin\omega t$.

The inverse Laplace transform for the last function e^{-sT}/s is trickier than it looks, and is related to the subtle left and right semi-circle in the definition. You get either 0 or 1, depending on $t<T$ or $t>T$. The situation is similar to the fact the inverse Laplace transform $f(t)=\mathcal{L}^{-1}[F](t)$ is always chosen to be zero for $t<0$ (or $f(t)$ is a **causal** function).

Exercise. Show that the solution of the integral equation

$$
g(x) = 1 - \int_0^x (x-y)g(y)dy
$$

is $g(x)=\cos x$ using Laplace transform.
Solution: Let $G(s)$ be the Laplace transform of $g(x)$, then the integral equation becomes

$$
G(s) = \frac{1}{s} - \frac{1}{s^2}G(s).
$$

Then $G(s)=\frac{s}{1+s^2}$, which gives $g(x)=\cos x$.

4.5 Conformal Mapping

Exercise. Find the bilinear transformation that carries the points $-1,\infty,i$ on the z-plane to the following points on the w-plane:

$$
(a) i,\ 1,\ 1+i; \qquad\qquad (b) \infty,\ i,\ 1.
$$

Solution: (a) Let the bilinear mapping to be $w=\frac{az+b}{cz+d}$, then the points $-1,\infty$ and i mapped to i, 1 and $1+i$ implies that

$$
i = \frac{b-a}{d-c}, \qquad 1 = \frac{a}{c}, \qquad 1+i = \frac{ai+b}{ci+d}.
$$

There are only three equations for four parameters (a,b, c and d). We can take a as a known parameter and solve the equations for b, c and d, which gives

$$
b=(2+i)a, \quad c=a, \quad d=(2-i)a.
$$

Therefore, the bilinear mapping is

$$
w = \frac{az+(2+i)a}{az+(2-i)a} = \frac{z+2+i}{z+2-i}.
$$

(b) The points $-1,\infty$ and i mapped to ∞, i and 1 implies that

$$
0=-c+d, \quad i=\frac{a}{c}, \quad 1=\frac{b+ai}{d+ci}.
$$

We can solve the equation for a, b and d (in terms of c), then

$$
a=ci, \quad b=(2+i)c, \quad d=c.
$$

Therefore, the bilinear map is

$$w = \frac{ciz + (2+i)c}{cz + c} = \frac{iz + 2 + i}{z + 1}.$$

Exercise. Consider the following bilinear transformation

$$w = f(z) = \frac{2iz - 2}{2z - i}.$$

(a) Determine the invariant points of the transformation (those points such that $z = f(z)$).

(b) Find the point ξ for which the equation $f(z) = \xi$ has no solution for z in the finite complex plane.

(c) Show that the imaginary axis is mapped onto itself.

(d) Determine the image of the disc $|z| < 1$.

Solution: (a) The invariant points satisfies

$$z = f(z) = \frac{2iz - 2}{2z - i}$$

or $2z^3 - 3iz + 2 = 0$. This quadratic equation can be solved by the same way as for real coefficients, i.e.,

$$z_\pm = \frac{3i \pm \sqrt{(-3i)^2 - 4 * 2 * 2}}{4} = \frac{3i \pm 5i}{4}.$$

Therefore, the two invariant points are $2i$ and $-i/2$.

(b) We can write the equation $f(z) = \xi$ as $z(2\xi - 2i) = i\xi - 2$. This equation has no solution if and only if $2\xi - 2i = 0$ and $i\xi - 2 \neq 0$, which is exactly when $\xi = i$.

(c) The imagianary axis is parameterized by $z = it$ for real number t. Then

$$w = f(it) = \frac{2i(it) - 2}{2(it) - i} = \frac{-2t - 2}{2it - i} = \frac{2t + 2}{2t + 1} i,$$

which is purely imaginary. Therefore, the imaginary axis is mapped onto itself.

(d)

Exercise. Show that the circle $|z| = a (\neq 1)$ in the z-plane is mapped into the ellipse in the w-plane:

$$\frac{u^2}{A^2} + \frac{v^2}{B^2} = 1, \qquad A = a + 1/a, \; B = a - 1/a.$$

Show also that the circle $|z| = 1$ is mapped to the line segment from -2 to 2.
Solution: The circle $|z| = a$ can be parameterized by $z = ae^{i\theta}$ for $0 \leq \theta \leq 2\pi$. Then

$$w = u + iv = z + \frac{1}{z} = ae^{i\theta} + \frac{1}{a} e^{-i\theta} = \left(a + \frac{1}{a}\right) \cos\theta + i \left(a - \frac{1}{a}\right) \sin\theta.$$

Therefore,

$$u = \left(a + \frac{1}{a}\right) \cos\theta, \quad v = \left(a - \frac{1}{a}\right) \sin\theta,$$

which is an ellipse with semi-axis $A = a + 1/a$ and $B = |a - 1/a|$.

If $a = 1$, then $B \equiv 0$ and the ellipse becomes the line segment connecting -2 and 2.

More details on Example 7.1: Since there is no circulation generated by vorticity, we expect the force is zero. In fact,

$$F_u - iF_v = \frac{i\rho}{2} \oint_{\mathcal{C}} \left(\frac{dW}{dw}\right)^2 dw = \frac{i\rho}{2} \oint_{|z|=a} F(z)dz, \qquad F(z) = \frac{z^2}{z^2-1}\left(e^{-i\alpha} - \frac{a^2}{z^2}e^{i\alpha}\right)^2.$$

There are three singular points inside the circle $|z| = a$: $z = 1$ and $z = -1$ are both simple poles, and $z = 0$ is a double pole. The total force becomes

$$F_u - iF_v = (-\pi\rho)\frac{1}{2\pi i} \oint_{|z|=a} F(z)dz = (-\pi\rho)\Big[\mathrm{Res}(F; z = -1) + \mathrm{Res}(F; z = -1) + \mathrm{Res}(F; z = 0)\Big].$$

The residue of F at $z = -1$ is

$$\mathrm{Res}(F, z = -1) = \lim_{z\to-1}(z+1)F(z) = \lim_{z\to-1} \frac{z^2}{z-1}\left(e^{-i\alpha} - \frac{a^2}{z^2}e^{i\alpha}\right)^2 = -\frac{1}{2}\left(e^{-i\alpha} - a^2e^{i\alpha}\right)^2.$$

Similarly, the residue of F at $z = 1$ is

$$\mathrm{Res}(F, z = 1) = \lim_{z\to1}(z-1)F(z) = \lim_{z\to1} \frac{z^2}{z+1}\left(e^{-i\alpha} - \frac{a^2}{z^2}e^{i\alpha}\right)^2 = \frac{1}{2}\left(e^{-i\alpha} - a^2e^{i\alpha}\right)^2 = -\mathrm{Res}(F, z = -1).$$

Finally the residue of F at $z = 0$ is calculated most conveniently using series expansions at the origin, i.e.,

$$F(z) = -z^2(1-z^2)^{-1}\left(e^{-2i\alpha} - \frac{2a^2}{z^2} + \frac{a^4}{z^4}e^{2i\alpha}\right) = -z^2(1 + z^2 + \cdots)\left(e^{-2i\alpha} - \frac{2a^2}{z^2} + \frac{a^4}{z^4}e^{2i\alpha}\right).$$

The coefficient of z^{-1} should be zero (actually all coefficients of odd powers) and the total force is zero.

Similarly, we can calculate the moments

$$M = -\frac{\rho}{2}\mathrm{Re}\int_{|z|=a} G(z)dz, \qquad G(z) = \frac{z(z^2+1)}{z^2-1}\left(e^{-i\alpha} - \frac{a^2}{z^2}e^{i\alpha}\right)^2,$$

from the residue calculus

$$\frac{1}{2\pi i}\int_{|z|=a} G(z)dz = \mathrm{Res}(G; z = -1) + \mathrm{Res}(G; z = -1) + \mathrm{Res}(G; z = 0).$$

$$\mathrm{Res}(G; z = -1) = \lim_{z\to-1}(z+1)G(z) = \lim_{z\to-1} \frac{z(z^2+1)}{z-1}\left(e^{-i\alpha} - \frac{a^2}{z^2}e^{i\alpha}\right)^2 = \left(e^{-i\alpha} - a^2e^{i\alpha}\right)^2,$$

$$\mathrm{Res}(G; z = 1) = \lim_{z\to1}(z-1)G(z) = \lim_{z\to1} \frac{z(z^2+1)}{z+1}\left(e^{-i\alpha} - \frac{a^2}{z^2}e^{i\alpha}\right)^2 = \left(e^{-i\alpha} - a^2e^{i\alpha}\right)^2.$$

Finally, we use series expansion to calculate $\mathrm{Res}(G; z=0)$ (it is much more complicated to use the formula $\mathrm{Res}(G; z=0) = \lim_{z\to 0} \frac{1}{2}\frac{d^2}{dz^2}\left[z^3 G(z)\right]$.

$$\begin{aligned} G(z) &= -z(1+z^2)(1-z^2)^{-1}\left(e^{-2i\alpha} - \frac{2a^2}{z^2} + \frac{a^4}{z^4}e^{2i\alpha}\right) \\ &= -z(1+z^2)(1+z^2+\cdots)\left(e^{-2i\alpha} - \frac{2a^2}{z^2} + \frac{a^4}{z^4}e^{2i\alpha}\right) \\ &= -z(1+2z^2+\cdots)\left(e^{-2i\alpha} - \frac{2a^2}{z^2} + \frac{a^4}{z^4}e^{2i\alpha}\right). \end{aligned}$$

Notice here that we can ignore the terms z^4 or higher. Collecting the coefficient of z^{-1} from $G(z)$, we get

$$\mathrm{Res}(G; z=0) = 2a^2 - 2a^4 e^{2i\alpha}.$$

Therefore,

$$\frac{1}{2\pi i}\int_{|z|=a} G(z)dz = \mathrm{Res}(G; z=-1) + \mathrm{Res}(G; z=-1) + \mathrm{Res}(G; z=0) = 2e^{-i\alpha} - 2a^2$$

and

$$M = -\frac{\rho}{2}\mathrm{Re}\int_{|z|=a} G(z)dz = -\frac{\rho}{2}\mathrm{Re}\big\{2\pi i(2e^{-i\alpha} - 2a^2)\big\} = -\frac{\rho}{2}\mathrm{Re}\big\{2\pi i(-2\sin 2\alpha)i\big\} = -2\pi\rho\sin 2\alpha.$$