Aero III/IV Complex power series, residue theorem and its applications

1 Series and convergence

Series: $\sum_{j=1}^{\infty} f_j(z) = f_1(z) + f_2(z) + \cdots$ with $f_j(z)$ usually monomials like z^j

Partial sum: $S_n(z) = \sum_{j=1}^{n} f_j(z)$

Convergence: $\sum_{j=1}^{\infty} f_j(z)$ is said to be *convergent* to $f(z)$ if for any z (possibly limited to some domain), $\lim_{n\to\infty} S_n(z) = f(z)$.

More precisely: $\sum_{j=1}^{\infty} f_j(z)$ converges to $f(z)$ if for any z and $\epsilon > 0$, there exists an integer N (which may depend on z and ϵ), such that $|S_n(z) - f(z)| < \epsilon$, for all $n > N(z, \epsilon)$.

Uniform convergence: $\sum_{j=1}^{\infty} f_j(z)$ converges to $f(z)$ *uniformly* if N is independent of z.

Example 1.1. The geometric series $1 + z + z^2 + \cdots$ converges to $1/(1-z)$ for any $|z| < 1$, but *not* uniformly.

Uniformly convergent series are preferred because of the following good properties:

i) Preserves continuity: if $\sum f_j(z)$ converges to $f(z)$ uniformly, and each term $f_j(z)$ is *continuous*, then $f(z)$ is continuous

ii) Integration term-by-term:

$$\int_{\mathcal{C}} f(z)dz = \sum_{j}^{\infty} \int_{\mathcal{C}} f_j(z)dz,$$

where $\mathcal{C}$ lies entirely in the region of uniform convergence.

iii) Differentiate term-by-term: $\sum f_j(z)$ converges to $f(z)$ uniformly, and each term of f_j is analytic, then $f(z)$ is analytic and

$$f'(z) = \sum f_j'(z).$$

Recall that for the convergence of the series $\sum a_n$ we have the following general criteria :

(a) *Comparison test*: If $\sum |b_n|$ converges and $|a_n| \le |b_n|$, then $\sum a_n$ converges.

(b) *Ratio test*: Let

$$l := \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

exists. If $l < 1$ then the series $\sum a_n$ converges, otherwise if $l > 1$, the series diverges.

(c) *Root test*: Let

$$l := \lim_{n\to\infty} |a_n|^{\frac{1}{n}}$$

exists. If $l < 1$ then the series $\sum a_n$ converges, otherwise if $l > 1$, the series diverges.

Remark. All the complications happen at $l = 1$. But in this course, we only consider the series *strictly inside* the radius of convergence (or $l < 1$).

When $a_n = c_n(z - z_0)^n$, we are interested in the domain where the series $\sum_{n=1}^{\infty} c_n(z - z_0)^n$ converges, which leads to the concept of *radius of convergence* R:

$$R = \max\left\{ |z - a| : \sum_{n=1}^{\infty} |c_n(z - a)^n| \text{ converges} \right\} = \lim_{n\to\infty} \left| \frac{c_n}{c_{n+1}} \right|.$$

Be careful that in this definition the ratio is $|c_n/c_{n+1}|$ instead of $|c_{n+1}/c_n|$.

Remark. Since the series $f(z) = \sum c_n(z - z_0)^n$ diverges when $|z - z_0| > R$ and is complicated when $|z - z_0| = R$, we only consider the case $|z - z_0| < R$, in which we can manipulate the series as an analytic function on the disk $|z - z_0| < R_1$ for any $R_1 < R$:

(a) $c_n(z - z_0)^n \to 0$ as n goes to infinity

(b) $\sum |c_n(z - z_0)^n|$ converges, or the series $\sum c_n(z - z_0)^n$ converges absolutely

(c) $\sum c_n(z - z_0)^n$ is continuous and analytic on the disk

(d) The derivative of the series $\sum c_n(z - z_0)^n$ is given by

$$\sum_{n=1}^{\infty} n c_n (z - z_0)^{n-1},$$

which is convergent and have the **same** radius of convergence R.

(e) We can differentiate the series more times and the higher order derivative at z_0 is $f^{(n)}(z_0) = n! c_n$.

Example 1.2. Expand the function $f(z) = \frac{1}{1+z^2}$ at $z = 0$ and show that the radius of convergence if $R = 1$. Even though $f(x)$ is a well-behaved function, decays to zero at infinity and has derivatives of any order on the real line, it has a finite radius of convergence. The reason is that there are actually singularities at $z = \pm i$ and we have to extend the scope to the whole complex plane to understand the function.

2 Taylor Series

The Taylor series is a straightforward generalization of the result for real variable.

Theorem 2.1. *If $f(z)$ is analytic within $|z - z_0| \le R$, then $f(z)$ can be expanded as*

$$\begin{aligned} f(z) &= f(z_0) + f'(z_0)(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + \cdots + \frac{f^{(n)}}{n!}(z - z_0)^n + \cdots \\ &= \sum_{n=0}^{\infty} c_n (z - z_0)^n \end{aligned}$$

with $c_n = f^{(n)}(z_0)/n!$, and this series is convergent in $|z - z_0| < R$.

Remark. The coefficient c_n can also be obtained from contour integration

$$c_n = f^{(n)}(z_0)/n! = \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{f(\xi)}{(\xi - z_0)^{n+1}} d\xi.$$

Example 2.1. Some well-known Taylor series:

$$\begin{aligned} e^z &= 1 + z + \frac{1}{2!}z^2 + \cdots + \frac{1}{n!}z^n + \cdots, \\ \sin z &= z - \frac{1}{3!}z^3 + \cdots + \frac{(-1)^n}{(2n+1)!}z^{2n+1} + \cdots, \\ \cos z &= 1 - \frac{1}{2!}z^2 + \cdots + \frac{(-1)^n}{(2n)!}z^{2n} + \cdots, \\ \frac{1}{1-z} &= 1 + z + z^2 + \cdots, \\ \log(1+z) &= z - \frac{1}{2}z^2 + \frac{1}{3}z^3 + \cdots + \frac{(-1)^{n+1}}{n}z^n + \cdots. \end{aligned}$$

Example 2.2. Expand $f(z) = (1 - z)^{-K}$ about $z = 0$ for positive integer K. What's the radius of convergence?
Solution:

$$\begin{aligned} f(0) &= 1, \\ f'(z) &= K(1-z)^{-K-1}, &&\Longrightarrow && f'(0) = K, \\ f''(z) &= K(K+1)(1-z)^{-K-2}, &&\Longrightarrow && f''(0) = K(K+1), \\ f^{(n)}(z) &= K(K+1)\cdots(K+n-1)(1-z)^{-K-n}, &&\Longrightarrow && f^{(n)}(0) = K(K+1)\cdots(K+n-1). \end{aligned}$$

Therefore,

$$\begin{aligned} f(z) &= 1 + Kz + \frac{1}{2}K(K+1)z^2 + \cdots + \frac{1}{n!}K(K+1)\cdots(K+n-1)z^n + \cdots \\ &= 1 + Kz + \frac{1}{2}K(K+1)z^2 + \cdots + \frac{(K+n-1)!}{n!(K-1)!}z^n + \cdots \end{aligned}$$

The radius of convergence is determined by

$$1 > \lim_{n\to\infty}\left|\frac{(K+n)!}{(n+1)!(K-1)!}z^n\right| \Big/ \left|\frac{(K+n-1)!}{n!(K-1)!}z^n\right| = \lim_{n\to\infty}\left|\frac{(K+n)nz}{(n+1)(K+n-1)}\right| = |z|.$$

Therefore, the radius of convergence is $R = 1$, which is the distance between $z = 0$ (the point the expansion is based on) and $z = 1$ (the nearest singular point).

Example 2.3 (Taylor expansion by decomposing into simple fractions)**.** For example, expand $f(z) = \frac{5}{4+3z-z^2}$ about $z = 1$.
Solution: It is *not* convenient to find the coefficients by differentiating the function as above. We can find it by decompose the function into simpler components,

$$f(z) = \frac{1}{z+1} + \frac{1}{4-z} = \frac{1}{2+(z-1)} + \frac{1}{3-(z-1)}.$$

We can expand both $\frac{1}{2+(z-1)}$ and $\frac{1}{3-(z-1)}$ in geometric series,

$$\frac{1}{2+(z-1)} = \frac{1}{2}\frac{1}{1+\frac{1}{2}(z-1)} = \frac{1}{2}\left\{1 - \frac{1}{2}(z-1) + \frac{1}{4}(z-1)^2 + \cdots + \frac{(-1)^n}{2^n}(z-1)^n + \cdots\right\},$$
$$\frac{1}{3-(z-1)} = \frac{1}{3}\frac{1}{1-\frac{1}{3}(z-1)} = \frac{1}{3}\left\{1 + \frac{1}{3}(z-1) + \frac{1}{9}(z-1)^2 + \cdots + \frac{1}{3^n}(z-1)^n + \cdots\right\}.$$

Therefore,

$$f(z) = \frac{1}{2} + \frac{1}{3} + \left(\frac{1}{9} - \frac{1}{4}\right)(z-1) + \cdots + \left(\frac{1}{3^{n+1}} + \frac{(-1)^n}{2^{n+1}}\right)(z-1)^n + \cdots.$$

In this way it is easy to determine the radius of convergence $R = 2$, which is the distance between $z = 1$ and the nearest singular point $z = -1$.

3 Laurent series

If $f(z)$ is not analytic at z_0, then it is impossible to expand $f(z)$ to Taylor series at z_0.

But even in such cases, it is possible to represent $f(z)$ by a power series expansion which consists of both positive and negative powers of $z - z_0$. Such a series is known as *Laurent series*.

Laurent expansion If $f(z)$ is analytic in the annulus $R_1 \le |z - z_0| \le R_2$, then $f(z)$ can be represented by the series

$$f(z) = \sum_{n=-\infty}^{\infty} c_n(z-z_0)^n, \qquad R_1 < |z-z_0| < R_2,$$

and the coefficients are given by

$$c_n = \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{f(\xi)}{(\xi - z)^{n+1}}d\xi,$$

where $\mathcal{C}$ is any simple closed curve within the annulus.

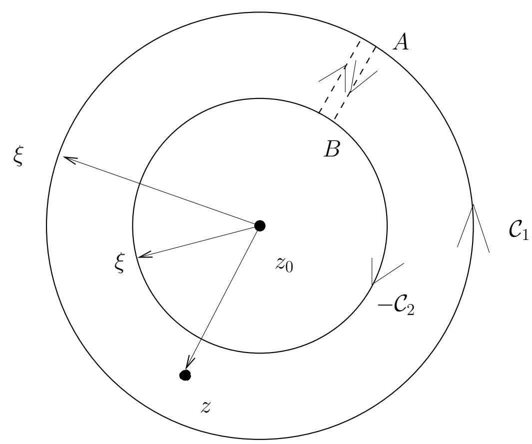


Figure 1: The closed contour $\mathcal{C} = \mathcal{C}_2 + AB - \mathcal{C}_1 - AB$ used to derive Laurent expansion.

Derivation of Laurent expansion from contour integration: The derivation involves two steps: during the first step, we get the coefficient c_n from *special contours* to allow a convergent expansion; during the second step, the contours can be deformed to the desired one $\mathcal{C}$.

Choose the *closed* contour $\tilde{\mathcal{C}} = \mathcal{C}_1 + AB - \mathcal{C}_2 - AB$ as in Figure 1, then since f is analytic on the region bounded by the contour, by Cauchy's integral theorem

$$f(z) = \frac{1}{2\pi i}\oint_{\tilde{\mathcal{C}}} \frac{f(\xi)}{\xi - z} d\xi = \frac{1}{2\pi i}\oint_{\mathcal{C}_1} \frac{f(\xi)}{\xi - z} d\xi - \frac{1}{2\pi i}\oint_{\mathcal{C}_2} \frac{f(\xi)}{\xi - z} d\xi.$$

For the first integral, we have

$$\frac{1}{\xi - z} = \frac{1}{(\xi - z_0) - (z - z_0)} = \frac{1}{\xi - z_0} - \frac{1}{1 - \frac{z - z_0}{\xi - z_0}} = \frac{1}{\xi - z_0}\sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(\xi - z_0)^n}.$$

Here the series converges because $\left|\frac{z-z_0}{\xi-z_0}\right| < 1$ for any ξ on $\mathcal{C}_1$.

Similarly, for the second integral,

$$-\frac{1}{\xi - z} = \frac{1}{(z - z_0) - (\xi - z_0)} = \frac{1}{z - z_0}\frac{1}{1 - \frac{\xi - z_0}{z - z_0}} = \frac{1}{z - z_0}\sum_{m=0}^{\infty} \frac{(\xi - z_0)^m}{(z - z_0)^m},$$

which is convergent as $|\xi - z_0|/|z - z_0| < 1$ for any ξ on $\mathcal{C}_2$.

Putting these together, we get

$$f(z) = \sum_{n=0}^{\infty} A_n (z - z_0)^n + \sum_{m=0} B_m (z - z_0)^{-(m+1)}$$

where

$$A_n = \frac{1}{2\pi i}\int_{\mathcal{C}_1} \frac{f(\xi)}{(\xi - z_0)^{n+1}} d\xi, \quad B_m = \frac{1}{2\pi i}\int_{\mathcal{C}_2} f(\xi)(\xi - z_0)^m d\xi.$$

Since f is analytic in the annulus, the two integrands above defining A_n and B_m are analytic too, and hence $\mathcal{C}_1$ and $\mathcal{C}_2$ can be deformed into $\mathcal{C}$. Now let $c_n = A_n$ when $n \neq 0$ and $c_n = B_{-n-1}$ when $n \leq -1$, then we get the desired Laurent expansion.

Remark. (1) If f is analytic inside $|z - z_0| = R_1$, then $c_n = 0$ for $n \leq -1$ (because $f(z)/(z - z_0)^{n+1}$ is analytic). We have just Taylor series as expected.

(2) If f is NOT analytic inside $|z - z_0| < R_1$ but still differentiable at x_0, then in general $c_n = f^{(n)}(z_0)/n!$ for $n \geq 0$. Check **Example 2.3** with $f(z) = \frac{5}{4+3z-z^2}$ with the annulus $2 < |z| < 3$.

How do you find Laurent series for a given f on an annulus? There are basically two ways:

a) Using the formula for c_n:

$$c_n = \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{f(\xi)}{(\xi - z)^{n+1}} d\xi.$$

You may have to choose a special contour $\mathcal{C}$.

b) Using other (simpler) series expansion: geometric series, Taylor expansions, ...

In many cases, the second approach is much faster.

Example 3.1. Using the above two approaches to find the Laurent expansion of $f(z) = e^{1/z}$ on $|z| > 0$.

Solution: a) We choose $\mathcal{C}$ to be the unit circle. Then ($\xi = e^{i\theta}$)

$$c_n = \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{e^{1/\xi}}{\xi^{n+1}} d\xi = \frac{1}{2\pi i}\int_0^{2\pi} e^{e^{-i\theta}} e^{-i(n+1)\theta} e^{i\theta} i d\theta = \frac{1}{2\pi}\int_0^{2\pi} e^{e^{-i\theta}} e^{-in\theta} d\theta.$$

Using the change of variable, $\phi = -\theta$, then

$$\frac{1}{2\pi}\int_0^{2\pi} e^{e^{-i\theta}} e^{-in\theta} d\theta = -\frac{1}{2\pi}\int_0^{-2\pi} e^{e^{i\phi}} e^{in\phi} d\phi = \frac{1}{2\pi}\int_0^{2\pi} e^{e^{i\phi}} e^{in\phi} d\phi.$$

Now change back to complex integration (because we want to use Cauchy integral theorem) with $z = e^{i\phi}$,

$$c_n = \frac{1}{2\pi}\int_0^{2\pi} e^{e^{i\phi}} e^{in\phi} d\phi = \frac{1}{2\pi i}\int_{|z|=1} e^z z^{n-1} dz.$$

If $n \geq 1$, we get $c_n = 0$, because $e^z z^{n-1}$ is analytic on the unit disk. When $n = -m \leq 0$,

$$c_n = \frac{1}{2\pi i}\int_{|z|=1} e^z z^{n-1} dz = \frac{1}{2\pi i}\int_{|z|=1} \frac{e^z}{z^{m+1}} dz = \frac{1}{m!}\frac{d^m}{dz^m} e^z\Big|_{z=0} = \frac{1}{m!} = \frac{1}{(-n)!}.$$

b) From Taylor expansion,

$$f(z) = e^{1/z} = 1 + 1/z + \frac{1}{2!}(1/z)^2 + \cdots .$$

Therefore, $c_n = 0$ if $n \geq 1$ and $c_n = \frac{1}{(-n)!}$ if $n \leq 0$.

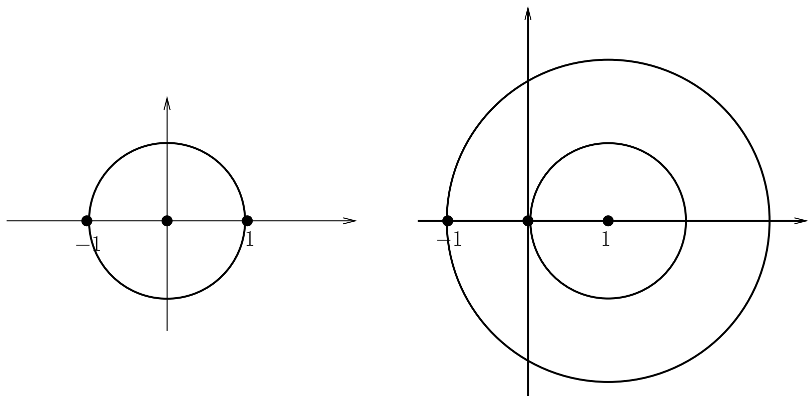


Figure 2: The Laurent expansion at $z = 0$ and $z = 1$ for **Example 3.2** and **Example 3.3**.

Exercise. Find the Laurent expansion of $f(z) = e^{z+\frac{1}{z}}$ on $|z| > 0$.

Example 3.2. Find the Laurent expansion of $f(z) = \frac{1}{z(1-z^2)}$ in the region (a) $0 < |z| < 1$, (b) $|z| > 1$.
Solution: When $0 < |z| < 1$,

$$f(z) = \frac{1}{z}\frac{1}{1-z^2} = \frac{1}{z}\big(1 + z^2 + z^4 + z^6 + \cdots\big) = \frac{1}{z} + z + z^3 + \cdots .$$

When $|z| > 1$,

$$f(z) = -\frac{1}{z^3}\frac{1}{1-z^{-2}} = -\frac{1}{z^3}\big(1 + z^{-2} + z^{-4} + \cdots\big) = -z^{-3} - z^{-5} - z^{-7} - \cdots .$$

Example 3.3. Find the Laurent expansion of $f(z) = \frac{1}{z(1-z^2)}$ in the region (a) $0 < |z-1| < 1$, (b) $1 < |z-1| < 2$, (c) $2 < |z-1|$.
Solution: Using the change of variable $w = z - 1$, then

$$f(z) = f(w+1) = \frac{1}{(1+w)(1-(w+1)^2)} = -\frac{1}{w(1+w)(2+w)} := \tilde{f}(w).$$

When $0 < |w| < 1$,

$$\begin{aligned}\tilde{f}(w) = -\frac{1}{2w}\frac{1}{1+w}\frac{1}{1+w/2} &= -\frac{1}{2w}\big(1 - w + w^2 - \cdots\big)\big(1 - w/2 + w^2/4 - w^3/6 + \cdots\big)\\ &= -\frac{1}{2w}\Big(1 - \frac{3}{2}w + \frac{7}{4}w^2 + \cdots\Big)\\ &= -\frac{1}{2w} + \frac{3}{4} - \frac{7}{8w} + \cdots .\end{aligned}$$

Alternatively, we can decompose $\tilde{f}$ into partial fraction

$$\begin{aligned}\tilde{f}(w) = -\frac{1}{w}\Big(\frac{1}{1+w} - \frac{1}{2+w}\Big) &= -\frac{1}{w}\left(\frac{1}{1+w} - \frac{1}{2}\frac{1}{1+w/2}\right) \\ &= -\frac{1}{w}\left((1-w+w^2-\cdots) - \frac{1}{2}(1-w/2+w^2/4-\cdots)\right) \\ &= -\frac{1}{2w} + \frac{3}{4} - \frac{7w}{8} + \cdots .\end{aligned}$$

In this way we can get the general formula for all the coefficients.

When $1 < |w| < 2$, we expect infinite many terms for both positive and negative powers of w and it is better to use partial fraction.

$$\begin{aligned}\tilde{f}(w) &= -\frac{1}{w}\left(\frac{1}{w}\frac{1}{1+1/w} - \frac{1}{2}\frac{1}{1+w/2}\right) \\ &= -\frac{1}{w}\left(\frac{1}{w}\big(1-1/w+1/w^2+\cdots\big) - \frac{1}{2}(1-w/2+w^2/4+\cdots)\right) \\ &= -\frac{1}{w}\left(\cdots + \frac{1}{w^3} - \frac{1}{w^2} + \frac{1}{w} - \frac{1}{2} + \frac{w}{4} - \frac{w^2}{8} + \cdots\right) \\ &= \cdots - \frac{1}{w^2} + \frac{1}{2w} - \frac{1}{4} + \frac{w}{8} - \cdots\end{aligned}$$

When $|w| > 2$,

$$\begin{aligned}\tilde{f}(w) &= -\frac{1}{w}\left(\frac{1}{w}\frac{1}{1+1/w} - \frac{1}{w}\frac{1}{1+2/w}\right) \\ &= -\frac{1}{w^2}\Big(\big(1-1/w+1/w^2+\cdots\big) - \big(1-2/w+4/w^2+\cdots\big)\Big) \\ &= -\frac{1}{w^3} + \frac{3}{w^4} - \frac{7}{w^5} + \cdots .\end{aligned}$$

Put all these together,

$$f(z) = \begin{cases} -\frac{1}{2(z-1)} + \frac{3}{4} - \frac{7}{8}(z-1) + \cdots , & 0 < |z-1| < 1, \\ \cdots - \frac{1}{(z-1)^2} + \frac{1}{2(z-1)} - \frac{1}{4} + \frac{z-1}{8} - \cdots , & 1 < |z-1| < 2, \\ -\frac{1}{(z-1)^3} + \frac{3}{(z-1)^4} - \frac{7}{(z-1)^5} + \cdots , & |z-1| > 2. \end{cases}$$

Remark. The specific Laurent expansion is valid only on one annulus $R_1 < |z-z_0| < R_2$. The function is analytic on this annulus and has singularity on the circles $|z-z_0| = R_1$ and $|z-z_0| = R_2$. In fact, the numbers $0, 1, 2$ determining the annulus are exactly the distance of the singular points to $z_0 = 1$, which is true for general function with more complicated expressions.

Exercise. Find the first three terms of the Laurent expansion of $f(z) = \frac{1}{z\sin z}$ on (1) $0 < |z| < \pi$; (2) $\pi < |z| < 2\pi$ and (3) $0 < |z-\pi| < \pi$.

Comparison between Taylor series and Laurent series: Even though, Laurent series looks very similar to Taylor series (it is a generalization of Taylor series), the presence of negative powers of $z - z_0$ makes some fundamental differences:

(a) For the Taylor series, the coefficient c_n given by

$$c_n = \frac{1}{2\pi i} \int_\gamma \frac{f(\xi)}{(\xi - z_0)^{n+1}} d\xi \tag{$*$}$$

is related to *the derivatives of f at z_0*, i.e., $c_n = f^{(n)}(z_0)/n!$, but in general, the function expanded by a Laurent series is not defined at z_0 (hence the derivatives are not defined there either).

(b) Another difference is that the contour $\mathcal{C}$ in $(*)$ for Taylor series can be deformed into an arbitrary small circle around z_0 (f is analytic around z_0), but it is not true for Laurent series (the function is defined only on $R_1 < |z - z_0| < R_2$ for some $R_1 > 0$.

4 Singularity of complex function

Definition 4.1. A point z_0 is a **singular point** of $f(z)$ if $f(z)$ is not analytic at z. It is a **isolated singular point** if there is no other singular point in the neighbourhood of z_0, i.e., there is a region $0 < |z - z_0| < r$ in which $f(z)$ is analytic for r small.

Remark. The function $f(z)$ can be represented as a Laurent series at an isolated singular point z_0, otherwise it is not. One example of non-isolated singular point is $z = 0$ for $f(z) = \frac{1}{\sin(1/z)}$.

In general, the isolated singular point z_0 of a function $f(z)$ can be classified in terms of its Laurent series $f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n$ as follow:

(1) Regular point: if $c_n = 0$ for all $n < 0$, and $c_0 = f(z_0)$.

(2) Removable singularity: if $c_n = 0$ for all $n < 0$ and $c_0 \neq f(z_0)$. For example,

$$f(z) = \begin{cases} \frac{1-\cos z}{z^2}, & z \neq 0, \\ a, & z = 0, \end{cases}$$

when $a \neq 1/2$. Since

$$\frac{1 - \cos z}{z^2} = \frac{1}{z^2} \left(1 - \left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots \right) \right) = \frac{1}{2} - \frac{z^2}{24} + \cdots,$$

we can **redefine** $f(0) = 1/2$, then $f(z)$ is analytic on the whole complex plane.

(3) Pole of order $m (m \geq 1)$: If $c_{-m} \neq 0$ and $c_n = 0$ for all $n < -m$. For example $f(z) = \frac{\cos z}{z}$ has a **simple pole** at $z = 0$, and $f(z) = \frac{z}{\sin^3 z}$ has a **double pole** at $z = 0$.

(4) Essential singularity at z_0: if there are infinite number of inverse powers in the Laurent series. For example, $f(z) = \cos\frac{1}{z}$.

There are two other types of less common singularities:
Singularity at infinity: $f(z)$ is said to be singular at ∞ if $\tilde{f}(z)$ is singular at the origin, where $\tilde{f}(z) = f(z^{-1})$. For example $f(z) = \frac{1}{z-1}$ is NOT singular at ∞, while $f(z) = \frac{z^2}{z-1}$ is singular at ∞.
Branch singularity: This type of singularity is related to multi-valueness of some complex function, for example $f(z) = \log z$, and $f(z) = z^{1/2}$ at $z = 0$. Writing $z = re^{i\theta} = re^{i(\theta+2k\pi)}$ for any integer k, then

$$\log z = \log(re^{i(\theta+2k\pi)}) = \log r + i(\theta + 2k\pi), \quad k = 0, \pm 1, \pm 2, \cdots$$

and

$$z^{1/2} = r^{1/2}e^{i(\theta/2+k\pi)} = \pm r^{1/2}e^{i\theta/2}.$$

Example 4.1. Find the type of singularity of $f(z) = \frac{1}{z(1-z)^2}$ and the Laurent expansion around them.
Solution: The singularities are $z = 0$ and $z = 1$.
(a) When $0 < |z| < 1$,

$$f(z) = \frac{1}{z}\big(1 + z + z^2 + \cdots\big)^2 = \frac{1}{z} + 2 + 3z + \cdots .$$

Therefore, $z = 0$ is a simple pole.
(b) When $0 < |z-1| < 1$,

$$f(z) = \frac{1}{(z-1)^2}\frac{1}{1+(z-1)} = \frac{1}{(z-1)^2}\big(1 - (z-1) + (z-1)^2 - \cdots\big)$$
$$= \frac{1}{(z-1)^2} - \frac{1}{z-1} + 1 - \cdots .$$

Therefore $z = 1$ is a double pole.

Example 4.2. Find the type of singularity of $f(z) = \sin\frac{1}{z}$ and the Laurent expansion around them.
Solution When $z \neq 0$, $f(z) = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\frac{1}{z^{n+1}}$. Therefore $z = 0$ is an essential singularity of $f(z)$.

5 Blasius laws and Kutta-Joukowski's lifting force

If the origin is inside the obstacle, then the complex velocity

$$V(z) = \Omega'(z) = \overline{U_\infty} + \frac{c_1}{z} + \frac{c_2}{z^2} + \cdots$$

for some complex number $U_\infty, c_1, \cdots$. The far field flow is matched for $U_\infty = u_\infty + iv_\infty$.

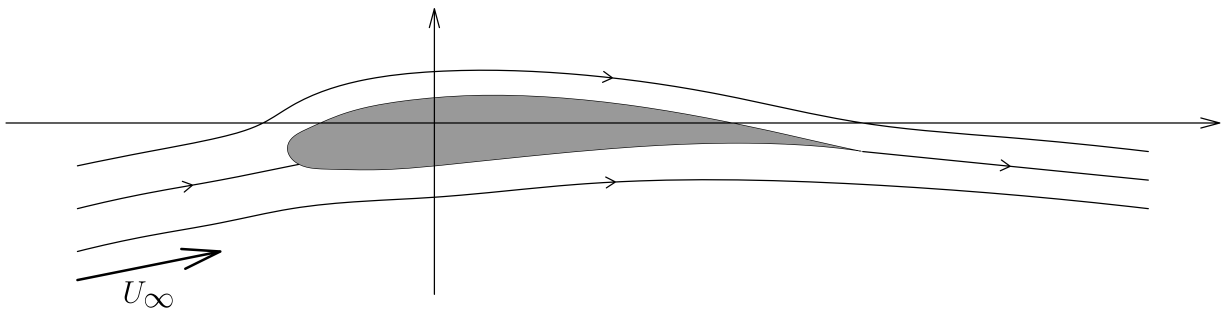


Figure 3: Streamlines representing the potential flow with niform upstream velocity U_∞ past an obstacle

Tangent and normal on the boundary :

$$\vec{t} = \frac{d\vec{r}}{ds} = \frac{dx}{ds}\vec{e}_x + \frac{dy}{ds}\vec{e}_y, \quad \vec{n} = \frac{dy}{ds}\vec{e}_x - \frac{dx}{ds}\vec{e}_y$$

The *flux* across a curve γ is defined by

$$\mathcal{F}_\gamma = \int_\gamma \vec{v}\cdot\vec{n}ds = \int_\gamma udy - vdx = \iint \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) dxdy = \iint \nabla\cdot\vec{v}\; dxdy$$

and the *circulation* along a curve γ is

$$\mathcal{C}_\gamma = \int_\gamma \vec{v}\cdot d\vec{r} = \int_\gamma udx + vdy = \iint \omega dxdy.$$

The contour integral of the complex velocity around the body defined by the closed curve γ is equal to $\mathcal{C}_\gamma + i\mathcal{F}_\gamma$, where $\mathcal{C}_\gamma$ and $\mathcal{F}_\gamma$ are the circulation and flux around the body, respectively. This can be checked easily as

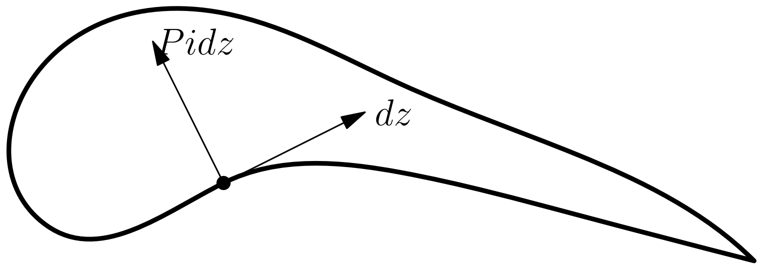
$$\begin{aligned}\oint_\gamma V(z)dz &= \oint_\gamma (u - iv)(dx + idy)\\ &= \oint_\gamma (udx + vdy) + i\oint_\gamma udy - vdx\\ &= \mathcal{C}_\gamma + i\mathcal{F}_\gamma\end{aligned}$$

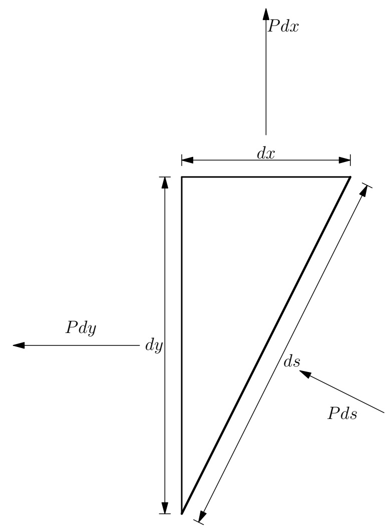
Blasius laws of hydrodynamic force and moment: First, the pressure is

$$P = p_0 - \frac{\rho}{2}(u^2 + v^2) = p_0 - \frac{\rho}{2}\left|\frac{d\Omega(z)}{dz}\right|^2$$

and the force and moment are defined as

$$F_x = -\oint_\gamma Pdy, \quad F_y = \oint_\gamma Pdx, \quad M = \oint_\gamma P(xdx + ydy).$$





Now we can define the complex force as

$$F = F_x - iF_y = -\oint_\gamma P(dy + idx) = -i\oint_\gamma Pd\bar{z} = -ip_0\oint_\gamma d\bar{z} + \frac{\rho}{2}i\oint_\gamma \left|\frac{d\Omega(z)}{dz}\right|^2 d\bar{z}.$$

The first integral $-ip_0\oint_\gamma d\bar{z}$ vanishes identically (check this!). The second integral is not ready for use to use yet, because of the presence of the $|\Omega'(z)|^2$ and $d\bar{z}$.

But we can convert the integral in the complex force F into a "regular" one, using the special boundary condition on γ (which can be deformed in the "regular" complex integral you see before).

Because γ is the fixed boundary of the body, the flow can not penetrate the boundary, leading to $\vec{v}\cdot\vec{n} = 0$ or the fact that ψ is a constant on γ. Therefore, $d\psi \equiv 0$ and

$$d\Omega(z) = d\phi - id\psi = d\phi = d\overline{\Omega(z)}$$

Therefore,

$$|\Omega'(z)|^2 d\bar{z} = \frac{d\Omega(z)}{dz}\frac{\overline{\partial\Omega(z)}}{dz}dz = \frac{d\Omega(z)}{dz}\frac{d\Omega(\bar{z})}{d\bar{z}}d\bar{z} = \frac{d\Omega(z)}{dz}d\Omega(z) = (\Omega'(z))^2 dz.$$

Alternatively, we can use $\Omega'(z) = u - iv$ to show that

$$|\Omega'(z)|^2 d\bar{z} = (u^2 + v^2)(dx - idy) = (u^2 - v^2 - 2uvi)(dx + idy) = \Omega'(z)^2 dz$$

where the identity $0 = d\psi = -vdx + udy$ is essential.

As a result we get the *Blasius law for hydrodynamic force*,

$$F = F_x - iF_y = \frac{\rho}{2}i\oint_\gamma \Omega'(z)^2 dz$$

In the tutorial sheet, you can show that

$$M = -\frac{\rho}{2}\oint |\Omega'(z)|^2(xdx + ydy) = \mathrm{Re}\left(-\frac{\rho}{2}\oint |\Omega'(z)|^2 zd\bar{z}\right) = \mathrm{Re}\left(-\frac{\rho}{2}\oint \Omega'(z)^2 zdz\right).$$

Kutta-Joukowski's lifting force: Let the complex potential be

$$\Omega(z) = \overline{U_\infty} z + \frac{\Gamma}{2\pi i}\log z + \frac{b_1}{z} + \frac{b_2}{z} + \cdots$$

with far-field velocity (u_∞, v_∞) and $U_\infty = u_\infty - iv_\infty$.

Then the circulation $\mathrm{Re}\oint_\gamma \Omega'(z)dz$ is exactly Γ and the total force on the obstacle is

$$\begin{aligned}
F_x - iF_y &= \frac{\rho}{2} i \oint_\gamma \Omega'(z)^2 dz \\
&= \frac{\rho}{2} i \oint_\gamma \left[\overline{U_\infty} + \frac{\Gamma}{2\pi i}\frac{1}{z} - \frac{b_1}{z^2} + \cdots\right]^2 dz \\
&= \frac{\rho}{2} i \oint_\gamma \left[\overline{U_\infty}^2 + \frac{\Gamma\overline{U}_\infty}{\pi i}\frac{1}{z} + O\left(\frac{1}{z^2}\right)\right] dz \\
&= i\rho\Gamma\overline{U_\infty}.
\end{aligned}$$

which is the celebrated *Kutta-Joukowski theorem* representing the lifting force as a function of the circulation and far field velocity.

6 Cauchy Residue Theorem

If a function $f(z)$ has a Laurent expansion $\sum_{n=-\infty}^{\infty} c_n(z-z_n)^n$ on the annulus $R_1 < |z-z_0| < R_2$, the coefficient c_{-1} is special because it is *the only one* that appears in the integral $\int_\gamma f(z)dz$ for any contour γ on the annulus, and is call the **residue of** f **at** z_0, and is denoted as $\texttt{Res}(f; z_0)$.

Exercise. Find $\texttt{Res}(f; z_0 = 1)$ with $f(z) = \frac{1}{z(z-1)^2}$.

Calculation of residue depending on the form of the singularity

a) Essential singularity like $f(z) = \sin\frac{1}{z}$ at $z = 0$: Laurent expansion is the only way.

b) Functions of the special form $f(z) = \frac{\Phi(z)}{(z-z_0)^m}$ with $m \geq 1$, $\Phi(z)$ is analytic at z_0 and $\Phi(z_0) \neq 0$, then

$$\texttt{Res}(f; z_0) = \frac{1}{(m-1)!}\Phi^{(m-1)}(z_0) = \frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\Big[(z-z_0)^m f(z)\Big]\Bigg|_{z=z_0}$$

In particular for $m = 1$ (simple pole), $\texttt{Res}(f; z_0) = \Phi(z_0)$.

c) If $f(z) = p(z)/q(z)$, $p(z)$, $q(z)$ are analytic at z_0, and $q(z_0) = 0$, $q'(z_0) \neq 0$m then

$$\texttt{Res}(f; z_0) = \frac{p(z_0)}{q'(z_0)}.$$

Example 6.1. Find the residue of $f(z) = \frac{1}{(z-1)^2(z+1)^3}$ at $z = 1$ and $z = -1$.
Solution: Since $f(z)$ satisfies the form in b) above,

$$\texttt{Res}(f; z = 1) = \frac{d}{dz}\frac{1}{(z+1)^3}\bigg|_{z=1} = -\frac{3}{(1+z)^4}\bigg|_{z=1} = -\frac{3}{16},$$

$$\texttt{Res}(f; z = -1) = \frac{1}{2!}\frac{d^2}{dz^2}\frac{1}{(z-1)^2}\bigg|_{z=-1} = \frac{1}{2}\frac{6}{(z-1)^4}\bigg|_{z=1} = \frac{3}{16}.$$

Example 6.2. Find the residue of $f(z) = \frac{e^z}{e^{2z}-1}$ at $z = k\pi i$.
Solution: Let $p(z) = e^z, q(z) = e^{2z} - 1$ as in c) above, then

$$\texttt{Res}(f; z = k\pi i) = \frac{p(k\pi i)}{q'(k\pi i)} = \frac{e^{k\pi i}}{2e^{2k\pi i}} = \frac{(-1)^k}{2}.$$

Theorem 6.1 (Cauchy' residue theorem)**.** *Let f be analytic inside and on a contour γ except a finite number of poles, $z_1, z_2, \cdots, z_N$. Then*

$$\int_\gamma f(z)dz = 2\pi i \sum_{i=1}^{N} \texttt{Res}(f; z_i).$$

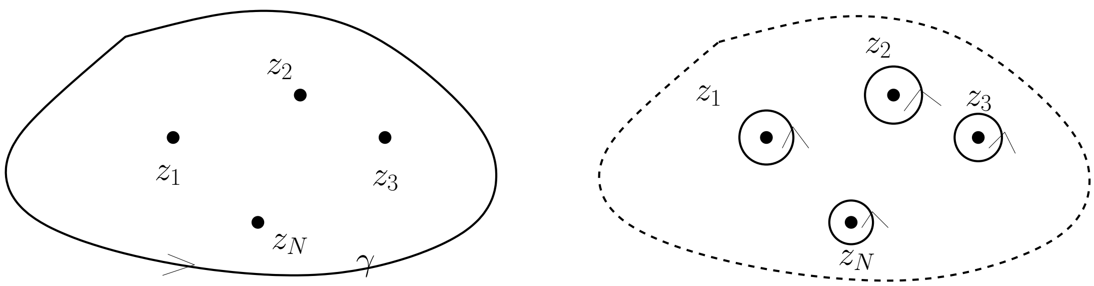


Figure 4: Cauchy Residue Theorem: The integral on the contour γ is reduced to integrals on small circles around each poles.

Example 6.3. Find the contour integral $\int_\gamma \frac{1}{z(1+z^2)}dz$, where γ is the circle $|z| = 2$, with counter-clockwise orientation.
Solution: Since there are three singular points $0, \pm i$ inside the contour,

$$\int_\gamma \frac{1}{z(1+z^2)}dz = 2\pi i\left(\text{Res}(\frac{1}{z(1+z^2)}, 0) + \text{Res}(\frac{1}{z(1+z^2)}, i) + \text{Res}(\frac{1}{z(1+z^2)}, -i)\right)$$
$$= 2\pi i\left(1 + \frac{1}{2i^2} + \frac{1}{2(-i)^2}\right) = 0.$$

Is this result consistent with the case when γ is deformed to the circle $|z| = R$ with radius R goes to infinity?

Exercise. Find $I = \dfrac{1}{2\pi i}\displaystyle\int_{\mathcal{C}} \cot z dz$ where $\mathcal{C}$ is the unit circle.

7 Applications of Residue Theorem

In this section, several examples are given for the application of residue theorem to evaluate certain integrals.

Example 7.1 (Integrals involving trigonometric functions)**.** Find $\displaystyle\int_0^{2\pi} \frac{1}{1+a^2-2a\cos\theta}d\theta$ for $0 < a < 1$ and $a > 1$ using contour integration and residue theorem.

Solution: For integrals with trigonometric on $[0, 2\pi]$, we can use the fact that on the unit circle $z = e^{i\theta}$ and $\cos\theta = (e^{i\theta} + e^{-i\theta})/2 = (z + z^{-1})/2$ and $\sin\theta = (z - z^{-1})/2i$. This change of variable implies that $d\theta = \frac{1}{iz}dz$ and for this problem

$$\int_0^{2\pi} \frac{1}{1+a^2-2a\cos\theta}d\theta = \frac{1}{i}\int_{\mathcal{C}} \frac{1}{1+a^2-a(z+z^{-1})}\frac{dz}{z} = \frac{1}{i}\int_{\mathcal{C}} \frac{1}{(1+a^2)z-a(z^2+1)}dz,$$

where $\mathcal{C}$ is the unit circle with counter-clockwise orientation.

When $a > 1$, the only pole inside the unit circle is $z = 1/a$ and is a simple pole. Therefore by residue theorem

$$\begin{aligned}\int_0^{2\pi} \frac{1}{1+a^2-2a\cos\theta}d\theta &= 2\pi\text{Res}\left(\frac{1}{(1+a^2)z-a(z^2+1)}, \frac{1}{a}\right)\\ &= 2\pi \lim_{z\to 1/a}\frac{z-1/a}{(1+a^2)z-a(z^2+1)}\\ &= 2\pi \lim_{z\to 1/a}\frac{1}{1+a^2-2az} = \frac{2\pi}{a^2-1}.\end{aligned}$$

Similarly, when $0 < a < 1$, we can get

$$\int_0^{2\pi} \frac{1}{1+a^2-2a\cos\theta}d\theta = \frac{2\pi}{1-a^2}.$$

Remark. For integrals involving trigonometric functions, we usually convert it to an integral on a circle and then use Residue theorem. But not for all such integral, for instance the following integral

$$I = \int_{-\infty}^{\infty} \frac{\cos ax}{(x^2+b^2)^2}dx.$$

For integrals with unbounded limit, we usually have to complete the contour which is a sector of a circle of radius R and then let R goes to infinity.

Example 7.2. Find $\displaystyle\int_0^{\infty} \frac{1}{1+x^4}dx$

Solution: There are different ways to complete a closed contour on the complex plane, which includes the integral we want to evaluate. The motivation comes from the fact that we want to related $\frac{1}{1+z^4}$ to our integral if it does not vanish or $\frac{1}{1+z^4}$ is a real number. We can use any of these and for simplicity, we choose the first one.

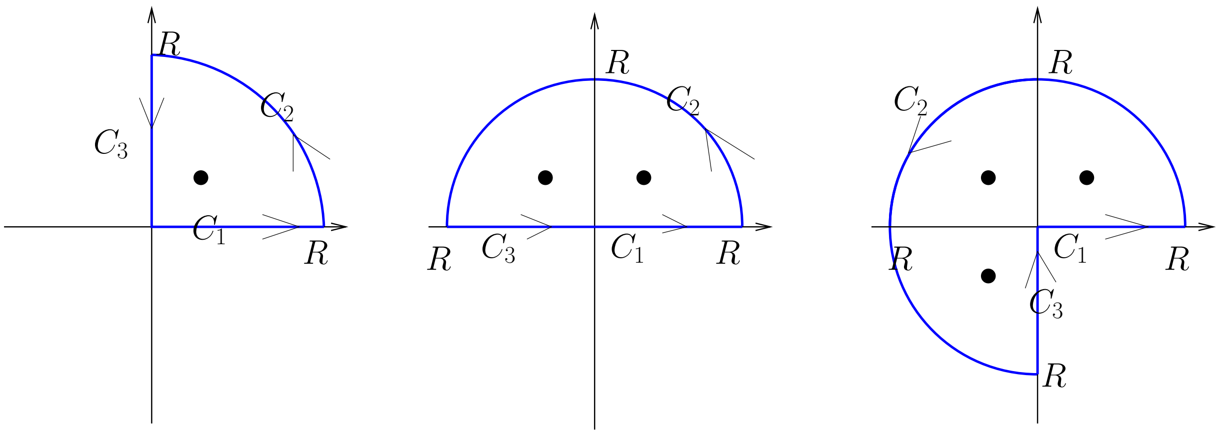


Figure 5: Three possible contours for example 7.2.

By residue theorem, since $z = e^{i\pi/4}$ is the only singularity inside the contour and is a simple pole,

$$\int_{C_1+C_2+C_3} \frac{1}{1+z^4}dz = 2\pi i \text{Res}(\frac{1}{1+z^4}, e^{i\pi/4}) = 2\pi i \lim_{z\to e^{i\pi/4}} \frac{z-e^{i\pi/4}}{1+z^4} = \frac{\pi i}{2}e^{-3\pi i/4} = \frac{\pi}{2}e^{-i\pi/4}$$

The contour on C_2 vanishes, because after the parametrization $z = Re^{i\theta}$,

$$\left|\int_{C_2} \frac{1}{1+z^4}dz\right| = \left|\int_0^R \frac{Re^{i\theta}}{1+R^4e^{i\theta}} id\theta\right| \leq \int_0^R \frac{R}{R^4-1}d\theta = \frac{2\pi R}{R^4-1} \to 0$$

as $R \to \infty$. The integral on C_3 is related that on C_1. Using the parametrization $z = ix$,

$$\int_{C_3} \frac{1}{1+z^4}dz = -i\int_0^R \frac{1}{1+x^4}dx.$$

Taking the limit $R \to \infty$, we have

$$(1-i)\int_0^\infty \frac{1}{1+x^4}dx = \frac{\pi}{2}e^{-i\pi/4}$$

or

$$\int_0^\infty \frac{1}{1+x^4} = \frac{\pi}{2}\frac{e^{-i\pi/4}}{1-i} = \frac{\pi}{2\sqrt{2}}.$$

Example 7.3 (Indented contours)**.** Find the integral $\int_0^\infty \frac{\sin x}{x}dx$.

Solution: As usual, we can to make the integrand $\frac{\sin x}{x}$ complex. But in general $\sin z$ is exponentially large when $|z| \to \infty$. More precisely, if we write $\sin z = (e^{iz} + e^{-iz})/2$, then e^{iz} decays on when z goes to infinity on the upper half plane, and becomes exponentially large on the lower half plane. The case for e^{-iz} is the opposite. Therefore, we have to separate e^{iz} and e^{-iz} and consider them on the upper and lower semicircle.

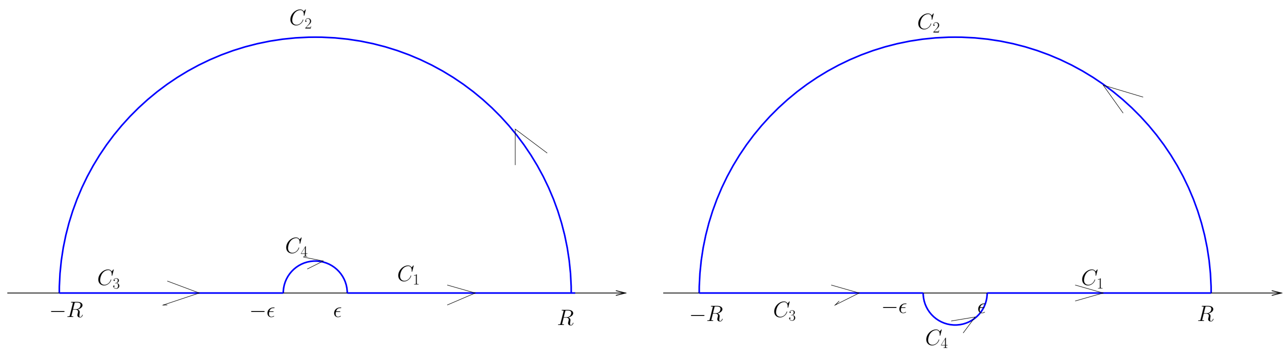


Figure 6: Two possible contours for example 7.3.

One alternative way is the fact that

$$\int_0^\infty \frac{\sin x}{x}dx = \mathrm{Im}\left(\int_0^\infty \frac{e^{ix}}{x}dx\right),$$

which enable us to focus on e^{iz} and the upper semicircle (Figure 6).

Another comment is that because now $z=0$ is a singularity, we have to use a dented contour around zero. There are two choices, both giving the same final answer. To reduce the complex, we can consider the one where $z=0$ is outside the contour. Therefore, we use the contour on the left of Figure 6, and (forget about the imaginary part for the moment)

$$\int_{C_1+C_2+C_3+C_4} \frac{e^{iz}}{z}dz = 0.$$

It is easy to see that $\int_{C_3} e^{iz}/z dz = \int_{C_1} e^{iz}/z dz$. On C_2, $z = Re^{i\theta}$ and

$$\int_{C_2} \frac{e^{iz}}{z}dz = \int_0^\pi e^{iRe^{i\theta}} i d\theta = i\int_0^\pi e^{iR(\cos\theta + i\sin\theta)}d\theta.$$

We can show that this integral goes to zero. Taking the modulus,

$$\left|\int_{C_2} \frac{e^{iz}}{z}dz\right| \le \int_0^\pi e^{-R\sin\theta}d\theta.$$

Using the inequality $\sin\theta = \frac{2}{\pi}\theta$, we get

$$\left|\int_{C_2} \frac{e^{iz}}{z}dz\right| \le \int_0^\pi e^{-R\sin\theta}d\theta \le \int_0^\pi e^{-\frac{2R}{\pi}\theta}d\theta = \frac{\pi}{2R}\left(1-e^{-2R}\right) \to 0,$$

as $R\to\infty$.

Finally, $z = \epsilon e^{i\theta}$ on C_4 and

$$\int_{C_4} \frac{e^{iz}}{z}dz = -i\int_0^\pi e^{i\epsilon e^{i\theta}}d\theta \to -i\pi,$$

as $\epsilon \to 0$.

Therefore,

$$0 = \lim_{\epsilon\to 0, R\to\infty} \text{Im}\left(\int_{C_1+C_2+C_3+C_4} \frac{e^{iz}}{z} dz\right) = 2\int_0^\infty \frac{\sin x}{x} dx - \pi i$$

or $\int_0^\infty \frac{\sin x}{x} dx = \pi/2$.

We don't focus too much on multi-valued functions like $\log z$ and $z^{1/2}$ in this course. In the context of contour integration, the extra difficulty with these multivalued functions comes from the fact that the contour in general **can not across the branch cut** (which we introduce to get a unique function value). Here is one example with $\log z$.

Example 7.4 (Contours with branch cut)**.** Find the integral $\int_0^\infty \frac{\log x}{1+x^2} dx$ using contour integration.

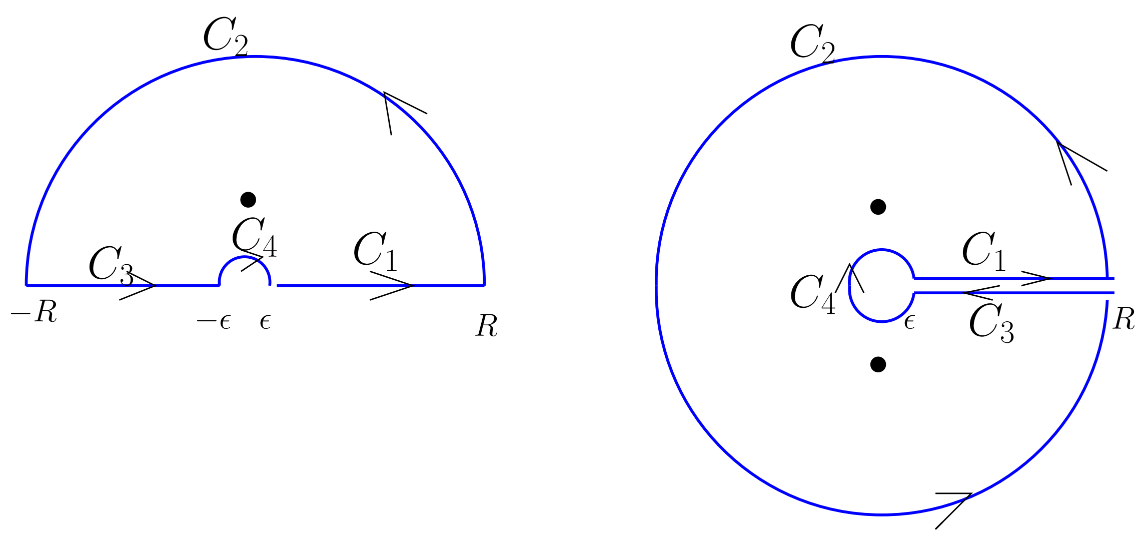


Figure 7: Two choices of the contours for example 7.4. Both contours give you the same answer.

Solution: There are two contours that keep the denominator $1 + z^2$ invariant, as shown in Figure 7.

If we are using the contour on the left, the only singularity is $z = i$ and

$$\int_{C_1+C_2+C_3+C_4} \frac{\log z}{1+z^2} dz = 2\pi i \text{Res}\left(\frac{\log z}{1+z^2}, i\right) = 2\pi i \lim_{z\to i} \frac{(z-i)\log z}{1+z^2} = \frac{\pi^2}{2} i.$$

When $R \to \infty$, the integral on C_2 vanishes, and the integral on C_3 is related to the desired integral. Using the parametrization, $z = (-1)t$ for $t \in [\epsilon, R]$ then $\log z = i\pi + \log t$ (Here is the place you choose the right branch of $\log z$, since $\log z$ can be $-i\pi + \log t$ or other values. Therefore

$$\int_{C_3} \frac{\log z}{1+z^2} dz = \int_\epsilon^R \frac{i\pi + \log t}{1+t^2} dt = i\pi \int_\epsilon^R \frac{1}{1+t^2} dt + \int_\epsilon^R \frac{\log t}{1+t^2} dt.$$

In the limit $\epsilon \to 0$ and $R \to \infty$,

$$\lim_{\epsilon\to 0, R\to\infty} \int_{C_3} \frac{\log z}{1+z^2} dz = i\pi \int_0^\infty \frac{1}{1+t^2} dt + \int_0^\infty \frac{\log t}{1+t^2} dt = \frac{\pi^2}{2} i + \int_0^\infty \frac{\log t}{1+t^2} dt.$$

On C_4, using the parametrization $z = \epsilon e^{i\theta}, \theta \in [0, \pi]$, then

$$\int_{C_4} \frac{\log z}{1+z^2} dz = -\int_0^\pi \frac{\log(\epsilon e^{i\theta})}{1+\epsilon^2 e^{2i\theta}} \epsilon e^{i\theta} i d\theta$$

The limit of the integral when $\epsilon \to 0$ is zero. Therefore,

$$\frac{\pi^2}{2} i = \lim_{\epsilon\to 0, R\to\infty} \int_{C_1+C_2+C_3+C_4} \frac{\log z}{1+z^2} dz = \frac{\pi^2}{2} i + 2\int_0^\infty \frac{\log x}{1+x^2} dx.$$

You get the same answer using the contour on the right in Figure 7. For these multivalued function, the integral of two contours C_1 and C_3 along the positive real axis but different orientations **do not** cancel each other. In factor, the positive real axis is a branch cut. (Try the calculation for this contour).

Example 7.5 (Summation of infinite series)**.** Show that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ by applying the residue theorem to the function $f(z) = \dfrac{\pi}{z^2} \cot \pi z$.

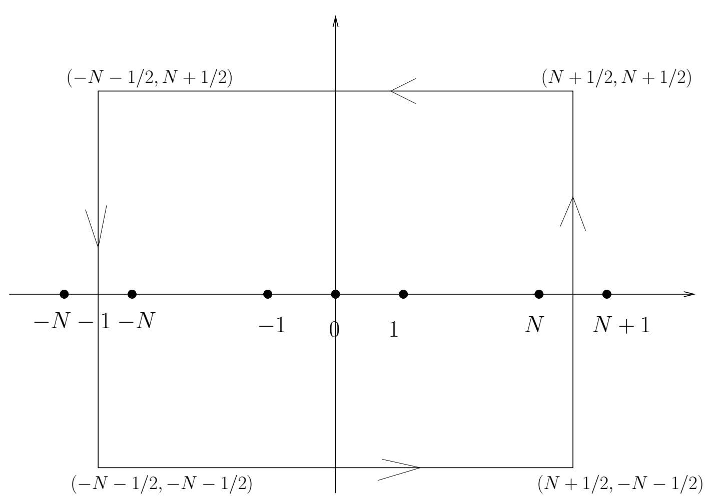


Figure 8: The contour for example 7.5.

Solution: The contour C_N is a square with corners $\big(\pm(N+1/2), \pm(N+1/2)\big)$, where N is an integer. This contour has to be choose in this way such that as N becomes infinity, the integral $\int_C f(z)dz$ vanishes because $\cot z$ is bounded on C. Therefore,

$$\left| \int_{C_N} f(z) dz \right| \le \max_{z\in C_N} \left| \frac{\pi \cot \pi z}{z^2} \right| \text{Length}(C_N) = O(1/N),$$

since Length(C_N) $= O(N)$ and $|\frac{\pi}{z^2}\cot\pi z| = O(1/N^2)$.

On the other hand, the only singularity of f inside C_N is $0, \pm1, \cdots, \pm N$. By residue theorem,

$$\frac{1}{2\pi i}\int_{C_N} f(z)dz = \sum_{n=-N} \mathrm{Res}(f, n).$$

If $n \neq 0$, then $z = n$ is a simple pole and

$$\mathrm{Res}(f,n) = \lim_{z\to n}(z-n)f(z) = \lim_{z\to n}\frac{\pi}{z^2}\cos\pi z\frac{(z-n)}{\sin\pi z} = \frac{\pi}{n^2}\cos n\pi \lim_{z\to n}\frac{(z-n)}{\sin\pi z} = \frac{1}{n^2}.$$

When $z = 0$, it is a triple pole and we can use power series expansion to find it (may be more complicated to use $\lim_{z\to 0}\frac{1}{2}\frac{d^2}{dz^2}z^3 f(z)$).

$$f(z) = \frac{\pi}{z^2}\frac{\cos\pi z}{\sin\pi z} = \frac{\pi}{z^2}\frac{1-\pi^2z^2/2+\cdots}{\pi z - \pi^3z^3/6+\cdots} = \frac{1}{z^3} - \frac{\pi^2}{3z} + \cdots.$$

Therefore $\mathrm{Res}(f,0) = -\pi^2/3$. Put all these together,

$$\int_{C_N} f(z)dz = -\frac{\pi^2}{3} + \sum_{n=-N,n\neq 0}^{N}\frac{1}{n^2} = -\frac{\pi^2}{3} + 2\sum_{n=1}^{N}\frac{1}{n^2}.$$

Let $N \to \infty$, then we have the identity $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$.

General questions for integration using residue theorem:

1. What kind of contours you can guess from the integrand?
2. How do you deal with integrand involving trigonometric functions $\sin x$ and $\cos x$, on finite intervals (say $[0, 2\pi]$ or infinite intervals (say $[0, \infty)$)?
3. How to keep things simple (the minimal number of curves to integrate)?
4. Can you taking advantage of the fact that the integral is real?

Destiny of different parts of the contours:

a) vanish in the limit ($R \to \infty$ or zero);

b) connect to what you want to evaluate;

c) something you know or you are given.